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ON THE LINEAR STABILITY OF INVISCID INCOMPRESSIBLE SWIRLING FLOWS

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Recently, Barston has studied the linear stability of plane, parallel flows of an inviscid, incompressible homogeneous fluid to two dimensional disturbances which are more general than normal mode disturbances. In this paper, we use the method of Barston to study the linear stability of swirling flows of an inviscid, incompressible, homogeneous fluid confined between two concentric cylinders. We show that the flow is stable if the gradient of the vorticity either vanishes nowhere in the flow domain or vanishes throughout the flow domain. Also, we obtain sufficient conditions for the stability of flows whose vorticity gradients have a simple zero or two distinct simple zeros. Lastly, we show that any swirling flow whose vorticity gradient has a simple zero or two distinct simple zeros can be made stable by the addition of a background swirl to it.

1. INTRODUCTION

The normal mode method has been extensively used to analyse the linear stability of many fluid flows[2, 3, 4]. Recently, Barston[1] has introduced a new method for analysing the linear stability of plane, parallel shear flows of an inviscid, incompressible homogeneous fluid. The significance of Barston's method lies in the fact that it proves stability to disturbances of the form (function of (y, t)). e^{ikx} which obviously include normal mode disturbances. He has shown that the parallel shear flow is stable if the velocity profile has no inflexion point. Also, he has obtained sufficient conditions for the stability of flows with finite number of inflexion points. In fact, he has shown that any velocity profile with finite number of inflexion points can be made stable by the addition of a background shear flow to it. Barston's proof of these stability results depends on the construction of positive definite constants of motion for the linearized equations of motion. However in contrast to the wide applicability of the normal mode method, Barston's method has been used only in the linear stability analysis of plane, parallel shear flows of an inviscid, incompressible homogeneous fluid.

In this paper, we analyse the linear stability of swirling flows of an inviscid, incompressible homogeneous fluid confined between two concentric cylinders to two-dimensional disturbances using Barston's approach. Earlier, this problem has been studied to two-dimensional normal mode disturbances, and a necessary condition for instability, namely that the gradient of the basic flow vorticity must change sign atleast once in the flow domain, has been obtained[3]. In this paper, we prove the generalization of the above criterion which states that the swirling flow is stable if the gradient of the vorticity vanishes nowhere in the flow domain. Also we prove that the swirling flow is stable if the gradient of the vorticity vanishes throughout the flow domain. Further we obtain sufficient conditions for stability of flows whose vorticity gradients have a simple zero or two distinct simple zeros in the flow domain. Then, we show that any swirling flow, whose vorticity gradient has a simple zero or two distinct simple zeros in the flow domain can be made stable by the addition of a background swirl to it.

2. Formulation of the Problem

Consider the motion of an inviscid, incompressible homogeneous fluid confined between two rigid coaxial cylinders of radii R_1 and R_2 with $R_2 > R_1 > 0$. It is natural to use cylindrical polar coordinates (r, θ, z).

Consider the basic flow vairables given by $\vec{U} = (0, V(r), 0)$, $p_0 = p_0(r)$ with constant density ρ_0, where the pressure $p_0(r)$ is related to the velocity $V(r)$ through the relation

$$\frac{dp_0}{dr} = \frac{\rho_0 V^2}{r}. \qquad \text{... (2.1)}$$

Here $V(r)$ and $p_0(r)$ are twice continuously differentiable functions of r in the flow domain. Then the equation governing the linear stability of this basic flow to two-dimensional disturbances is [Drazin and Reid[3], equation (15.55), p. 80],

$$\frac{\partial \Delta \psi}{\partial t} + \frac{V}{r}\frac{\partial \Delta \psi}{\partial \theta} - \frac{1}{r}\left(V' + \frac{V}{r}\right)' \frac{\partial \psi}{\partial \theta} = 0 \qquad \text{... (2.2)}$$

where the Laplacian Δ is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \qquad \text{... (2.3)}$$

ψ is the Stokes stream function and a prime denotes differentiation with respect to r.

Barston's stability analysis of Rayleigh problem[1] is based on an equation for the Fourier transform of the disturbance stream function ψ with respect to the space variable. But for our problem, we introduce a function $\phi(r, m, t)$ defined by

$$\phi(r, m, t) = \int_0^{2\pi} \psi(r, \theta, t)\, e^{-im\theta}\, d\theta, \qquad \text{... (2.4)}$$

and do the stability analysis based on an equation for ϕ. The equation for ϕ, obtained from (2.2), is

$$P\dot{\phi} + im\left[\frac{V}{r}P + \left(V' + \frac{V}{r}\right)'\right]\phi = 0,\ R_1 < r < R_2, t > 0, \quad \text{... (2.5)}$$

where $P \equiv -\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\right] + \frac{m^2}{r}$, and $(\dot{\cdot}) \equiv \frac{\partial}{\partial t}()$.

The associated boundary conditions are

$$\phi = 0,\ r = R_1, R_2. \quad \text{... (2.6)}$$

If $\phi(r, m, t) = \phi_1(r)e^{-ikct}$, then eqn. (2.5) becomes

$$\left(\frac{mV}{r} - kc\right)\left[\phi_1'' + \frac{\phi_1'}{r} - \frac{m^2}{r}\phi_1\right] - \frac{m}{r}\left(V' + \frac{V}{r}\right)'\phi_1 = 0, \quad \text{... (2.7)}$$

which is the stability equation for two-dimensional normal mode disturbances [Drazin and Reid[3], (15.57), p. 81].

3. Properties of the Operator P

Consider the solutions ϕ of (2.5) that are, for each fixed $t > 0$, twice continuously differentiable functions of r on $[R_1, R_2]$ and satisfy the boundary conditions (2.6). Then, with respect to the complex inner product $\langle f, g\rangle = \int_{R_1}^{R_2} f^* g\,dr$, where f^* is the complex conjugate of f, the operator P is self-adjoint. Also, the operator P is positive-definite and admits an inverse P^{-1} (an integral operator) which is also positive-definite, compact and Hermitian. The kernel of P^{-1} is the Green's function for P. That is, if

$$P\phi(r, m, t) \equiv \xi(r, m, t),\ R_1 \le r \le R_2,\ t \ge 0, \quad \text{... (3.1)}$$

then ϕ can be expressed in terms of ξ by mens of a suitable function $G(r, s)$ as

$$\phi(r, m, t) = \int_{R_1}^{R_2} G(r, s)\,\xi(s, m, t)\,ds. \quad \text{... (3.2)}$$

Hence, eqn. (2.5) can be rewritten as

$$\dot{\xi} = -\text{im}\,W\xi,\ t > 0, \quad \text{... (3.3)}$$

where $W = \frac{V}{r} + \left(V' + \frac{V}{r}\right)' P^{-1}$.

The Green's function for P can be easily computed and is given by

$$G(r,\ s) = \begin{cases} \dfrac{(r^m - R_1^{2m} r^{-m})(s^m - R_2^{2m} s^{-m})}{2m(R_1^{2m} - R_2^{2m})} & (R_1 \le r < s \le R_2) \\[2ex] \dfrac{(r^m - R_2^{2m} r^{-m})(s^m - R_1^{2m} s^{-m})}{2m(R_1^{2m} - R_2^{2m})} & (R_1 \le s < r \le R_2). \end{cases} \quad \dots (3.4)$$

Moreover, the system governed by the equation

$$P\phi = \lambda\phi, \quad \dots (3.5)$$

together with the boundary conditions (2.6), has non-trivial solutions only for a denumerable number of real values $\lambda_1, \lambda_2, \lambda_3, \dots$ of λ: the eigen values of the symmetric kernel $G(r,\ s)$ (Tricomi[6]).

Now, multiplying the equation (3.5) by ϕ^* (the complex conjugate of ϕ) and then integrating the resulting equation over (R_1, R_2) using (2.6), we obtain

$$\lambda \int_{R_1}^{R_2} |\phi|^2\, dr = \int_{R_1}^{R_2} [r\,|\phi'|^2 + \frac{m^2}{r}\,|\phi|^2]\, dr. \quad \dots (3.6)$$

Using the Rayleigh-Ritz inequality in (3.6), we get

$$\lambda \ge \frac{R_1 \pi^2}{(R_2 - R_1)^2} + \frac{m^2}{R_2} = \Lambda \text{ (say)}.$$

Since P^{-1} is a compact, Hermitian operator, $\|P^{-1}\|$ is an eigenvalue of P^{-1} (Rudin[5]). Hence, we have

$$\|P^{-1}\| \le \Lambda^{-1}. \quad \dots (3.7)$$

4. Stability Analysis

The swirling flow with velocity $(0,\ V(r),\ 0)$ is said to be stable if every solution ϕ of eqns. (2.5) and (2.6) is bounded uniformly in t; that is, for every solution ϕ of (2.5) and (2.6), there exists a non-negative constant b such that

$$\|\phi\| \le b; \quad \forall \quad t \ge 0. \quad \dots (4.1)$$

Now, consider

$$|\phi(r, m, t)|^2 = \left| \int_{R_1}^{r} \phi'(x, m, t)\, dx \right|^2 = \left| \int_{R_1}^{r} \phi'(x, m, t)\, x^{1/2} x^{-1/2}\, dx \right|^2.$$

This, with the help of Cauchy-Schwarz inequality yields

$$|\phi(r,m,t)|^2 \le \left(\int_{R_1}^{r} x\,|\phi'|^2\,dx\right)\left(\int_{R_1}^{r}\frac{dx}{x}\right)$$

$$\le \left(\int_{R_1}^{R_2} r\,|\phi'|^2\,dr\right)\log\left(\frac{R_2}{R_1}\right). \qquad \text{... (4.2)}$$

Since, $\langle P^{-1}\xi,\xi\rangle = \langle \phi, P\phi\rangle = \int_{R_1}^{R_2}\left[r\,|\phi'|^2 + \frac{m^2}{r}|\phi|^2\right]dr,$

we have

$$\langle P^{-1}\xi,\xi\rangle \ge \int_{R_1}^{R_2} r\,|\phi'|^2\,dr. \qquad \text{... (4.3)}$$

Inequality (4.2) together with the help of (4.3) and (3.7) yields

$$|\phi(r,m,t)|^2 \le \log\left(\frac{R_2}{R_1}\right)\Lambda^{-1}\,\|\xi\|^2. \qquad \text{... (4.4)}$$

If we are able to construct positive, Hermitian, time-independent operators G such that $\langle \xi, G\xi\rangle = \langle \xi_0, G\xi_0\rangle$, $t \ge 0$, for every solution ξ of (3.3), where $\xi_0 = \xi(r,m,0)$, then we have

$$\langle \xi_0, G\xi_0\rangle = \langle \xi, G\xi\rangle \ge \delta\,\|\xi\|^2, \quad t \ge 0, \qquad \text{... (4.5)}$$

where $\delta > 0$ and δ is independent of ξ.
Hence, inequality (4.4) gives

$$|\phi(r,m,t)|^2 \le \log\left(\frac{R_2}{R_1}\right)\Lambda^{-1}\,\delta^{-1}\langle \xi_0, G\xi_0\rangle \equiv R(r,m). \qquad \text{... (4.6)}$$

Inequality (4.6) demonstrates the pointwise boundedness of ϕ independent of t, which proves stability for perturbations of wave number m.

Now we shall prove the following theorems.

Theorem 4.1 — If the basic flow profile $V(r)$ is such that $\left(V' + \frac{V}{r}\right)' \ne 0$ for any $r \in [R_1, R_2]$, then the flow is stable.

PROOF : If $\left(V' + \frac{V}{r}\right)' \ne 0$ for any $r \in [R_1, R_2]$, then the operator $\left[r\left(V' + \frac{V}{r}\right)'\right]^{-1}$ is well-defined and is also Hermitian. This implies that the operator $\left[r\left(V' + \frac{V}{r}\right)'\right]^{-1} W$ is also Hermitian. Therefore, for every solution ξ of (3.3) we have

$$\frac{d}{dt}\langle \xi, \left[r\left(V' + \frac{V}{r} \right)' \right]^{-1} \xi \rangle = \langle \dot{\xi}, \left[r\left(V' + \frac{V}{r} \right)' \right]^{-1} \xi \rangle$$

$$+ \langle \xi, \left[r\left(V' + \frac{V}{r} \right)' \right]^{-1} \dot{\xi} \rangle$$

$$= \langle -\mathrm{im}\, W\xi, \left[r\left(V' + \frac{V}{r} \right)' \right]^{-1} \dot{\xi} \rangle + \langle \xi, -\left[r\left(V' + \frac{V}{r} \right)' \right]^{-1} \mathrm{im}\, W\xi \rangle$$

$$= \mathrm{im} \langle W\xi, \left[r\left(V' + \frac{V}{r} \right)' \right]^{-1} \dot{\xi} \rangle - \mathrm{im} \langle \xi, \left[r\left(V' + \frac{V}{r} \right)' \right]^{-1} W\xi \rangle$$

$$= \mathrm{im} \langle \left[r\left(V' + \frac{V}{r} \right)' \right]^{-1} W\xi, \xi \rangle - \mathrm{im} \langle \left[r\left(V' + \frac{V}{r} \right)' \right]^{-1} W\xi, \xi \rangle$$

$$= 0.$$

Hence the operator G defined by $G = \left| r\left(V' + \frac{V}{r} \right)' \right|^{-1}$ is a positive, Hermitian operator with

$$\langle \xi_0, G\xi_0 \rangle = \langle \xi, G\xi \rangle \geq \delta \, \| \xi \|^2, \qquad \ldots (4.7)$$

where $$\delta = \min_r \left| r\left(V' + \frac{V}{r} \right)' \right|^{-1}.$$

Therefore, inequality (4.4) yields

$$| \phi(r, m, t) |^2 \leq \log\left(\frac{R_2}{R_1} \right) \Lambda^{-1} \delta^{-1} \langle \xi_0, G\xi_0 \rangle = R(r, m).$$

Thus the theorem is proved.

Theorem 4.2 — If the basic flow profile $V(r)$ is such that $\left(V' + \frac{V}{r} \right)' = 0$ in $[R_1, R_2]$, then the flow is stable.

PROOF : If $\left(V' + \frac{V}{r} \right)' = 0$, then eqn. (3.3) becomes

$$\dot{\xi} = -\mathrm{im} \frac{V}{r} \xi. \qquad \ldots (4.8)$$

Therefore, we have

$$\frac{d}{dt}\langle \xi, \xi \rangle = \langle \dot{\xi}, \xi \rangle + \langle \xi, \dot{\xi} \rangle$$

$$= \langle -\mathrm{im} \frac{V}{r} \xi, \xi \rangle + \langle \xi, -\mathrm{im} \frac{V}{r} \xi \rangle$$

$$= \text{im}\left\langle \frac{V}{r}\xi, \xi \right\rangle - \text{im}\left\langle \xi, \frac{V}{r}\xi \right\rangle = 0.$$

Thus, $\| \xi \|^2 = \| \xi_0 \|^2$. Hence, inequality (4.4) yields

$$| \phi (r, m, t) |^2 \le \log\left(\frac{R_2}{R_1}\right) \Lambda^{-1} \| \xi_0 \|^2. \quad \text{... (4.9)}$$

This proves the theorem.

Theorem 4.3 — If the basic flow profile $V(r)$ is such that $\left(V' + \frac{V}{r}\right)'$ has finitely many simple zeros in $[R_1, R_2]$ and that V at every zero $r = s$ of $\left(V' + \frac{V}{r}\right)'$ has the common value C_1, then the flow is stable if either $\dfrac{V - C_1}{r\left(V' + \frac{V}{r}\right)'} \ge 0$ for all $r \in [R_1, R_2]$ or $M = \max\limits_r \dfrac{V - C_1}{r\left(V' + \frac{V}{r}\right)'} < -\dfrac{(R_2 - R_1)^2}{R_1 \pi^2}$.

PROOF : Obviously $\lim\limits_{r \to s} \dfrac{V - C_1}{r\left(V' + \frac{V}{r}\right)'}$ exists and is finite for every simple zero of $\left(V' + \frac{V}{r}\right)'$ in $[R_1, R_2]$. Define

$$G_1 = \frac{(rW - C_1)}{r\left(V' + \frac{V}{r}\right)'} = \frac{V - C_1}{r\left(V' + \frac{V}{r}\right)'} + P^{-1}. \quad \text{... (4.10)}$$

It is clear that G_1 is a Hermitian operator. Also, for every solution ξ of (3.3), we have

$$\langle \xi, G_1 \xi \rangle = \langle \xi_0, G_1 \xi_0 \rangle, \; t \ge 0. \quad \text{... (4.11)}$$

Now, if $\dfrac{V - C_1}{r\left(V' + \frac{V}{r}\right)'} \ge 0$ for all $r \in [R_1, R_2]$, then G_1 is positive definite and we have

$$\langle \xi, G_1 \xi \rangle \ge \langle \xi, P^{-1} \xi \rangle. \quad \text{... (4.12)}$$

Using (4.3), (4.12) and (4.11) in (4.2), we get

$$| \phi (r, m, t) |^2 \le \log\left(\frac{R_2}{R_1}\right) \langle \xi_0, G_1 \xi_0 \rangle. \quad \text{... (4.13)}$$

This proves the stability.

On the otherhand, if $M = \max_r \dfrac{V-C_1}{r\left(V'+\frac{V}{r}\right)'} < -\Lambda^{-1}$, then $\langle \xi, G_1 \xi \rangle \le (M+\Lambda^{-1}) \|\xi\|^2 < 0$, so that $-G_1$ is positive definite and hence we have stability for disturbances with wave number m satisfying $m^2 > R_2 \left[-M^{-1} - \dfrac{R_1 \pi^2}{(R_2-R_1)^2} \right]$. Thus, if $M < -\dfrac{(R_2-R_1)^2}{R_1 \pi^2}$ then also we have stability for every m.

Hence the theorem is proved.

Theorem 4.4 — Let the basic flow profile $V(r)$ be such that $\left(V'+\frac{V}{r}\right)'$ has two distinct simple zeros at r_1 and r_2 with $r_1 < r_2$ in $[R_1, R_2]$. Let $m_2 = \min_r \dfrac{(V-C_1)(V-C_2)}{r\left(V'+\frac{V}{r}\right)'}$ and $M_2 = \max_r \dfrac{(V-C_1)(V-C_2)}{r\left(V'+\frac{V}{r}\right)'}$ where $C_j = V(r_j)$ $(j = 1, 2)$.

Then the flow is stable if either $m_2 > \dfrac{2R_2 (R_2-R_1)^2 (\hat{C}_2 - \min_r V)}{R_1 \pi^2}$ or $M_2 < \dfrac{-2R_2 (R_2-R_1)^2 (\max_r V - \hat{C}_2)}{R_1 \pi^2}$, where $\hat{C}_2 = \dfrac{C_1 + C_2}{2}$.

PROOF : Let $\left(V'+\frac{V}{r}\right)' = (r-r_1)(r-r_2)\,h(r)$. Define

$$G_2 = \frac{(rW-C_1)(rW-C_2)}{r\left(V'+\frac{V}{r}\right)'} + P^{-1}\frac{V}{r}P^{-1} = \frac{(V-C_1)(V-C_2)}{r\left(V'+\frac{V}{r}\right)'} + H_2 \quad \text{... (4.14)}$$

where $$H_2 = P^{-1}(V-\hat{C}_2) + (V-\hat{C}_2)P^{-1} + P^{-1} r \left(V'+\frac{V}{r}\right)' P^{-1} + P^{-1}\frac{V}{r}P^{-1}.$$

Then we have

$$\langle \xi, H_2 \xi \rangle = 2 \int_{R_1}^{R_2} (V-\hat{C}_2)\left[r\,|\phi'|^2 + \frac{m^2}{r}|\phi|^2 \right] dr \quad \text{... (4.15)}$$

with $\min V < \hat{C}_2 < \max_r V$.

This implies that

$$2\left(\min_r V - \hat{C}_2\right)\langle \xi, P^{-1}\xi \rangle \le \langle \xi, H_2 \xi \rangle$$

$$\le 2\left(\max_r V - \hat{C}_2\right)\langle \xi, P^{-1}\xi \rangle. \quad \text{... (4.16)}$$

By the definition of G_2, we have

$$\langle \xi, G_2 \xi \rangle = \langle \xi, \frac{(V-C_1)(V-C_2)}{r\left(V' + \frac{V}{r}\right)'} \xi \rangle + \langle \xi, H_2 \xi \rangle.$$

Hence, using (4.16), we get

$$\left[m_2 + 2\left(\min_r V - \hat{C}_2 \right) \Lambda^{-1} \right] \| \xi \|^2 \le \langle \xi, G_2 \xi \rangle$$

$$\le \left[M_2 + 2\left(\max_r V - \hat{C}_2 \right) \Lambda^{-1} \right] \| \xi \|^2. \quad \text{... (4.17)}$$

So, if $m_2 > 0$, G_2 will be positive definite for disturbances with wave number m satisfying

$$m^2 > R_2 \left[2m_2^{-1} \left(\hat{C}_2 - \min_r V \right) - \frac{R_1 \pi^2}{(R_2 - R_1)^2} \right],$$

while if $M_2 < 0$, $-G_2$ will be positive definite for disturbances with wave number m satisfying

$$m^2 > R_2 \left[-2M_2^{-1} (\max V - \hat{C}_2) - \frac{R_1 \pi^2}{(R_2 - R_1)^2} \right].$$

Thus if either

$$m_2 > \frac{2R_2 (R_2 - R_1)^2 (\hat{C}_2 - \min_r V)}{R_1 \pi^2} \quad \text{... (4.18)}$$

or

$$M_2 < \frac{-2R_2 (R_2 - R_1)^2 (\max_r V - \hat{C}_2)}{R_1 \pi^2} \quad \text{... (4.19)}$$

then the flow is stable for every m.

Theorem 4.5 — If the basic flow profile $V(r)$ is such that $\left(V' + \frac{V}{r}\right)'$ has a simple zero or two distinct simple zeros in $[R_1, R_2]$ then the flow can be made stable by the addition of a background swirl to it.

PROOF : Suppose that $\left(V' + \frac{V}{r}\right)' = 0$ at a single point $r = r_1$ with $\left(V' + \frac{V}{r}\right)' = (r - r_1)\, h(r)$, $h(r) > 0$ in $[R_1, R_2]$. Consider the one parameter family of flows with velocities $(0, \tilde{V}(r), 0)$, where $\tilde{V}(r) = V(r) + Ar$ (A real constant), with $\left(\tilde{V}' + \frac{\tilde{V}}{r}\right)' = \left(V' + \frac{V}{r}\right)'$.

Let $V_1(r) = \dfrac{V(r) - V(r_1)}{r - r_1}$ for $r \neq r_1$ and $V_1(r_1) = V'(r_1)$. Then $V_1(r)$ is continuous on $[R_1, R_2]$ and the choice $\tilde{C}_1 = \tilde{V}(r_1)$ gives

$$\tilde{G}_1 = \frac{\tilde{V} - \tilde{C}_1}{r\left(V' + \frac{V}{r}\right)'} + P^{-1} = \frac{V_1(r) + A}{rh(r)} + P^{-1}.$$

Hence $\tilde{G}_1 > 0$ if $A > A_2 \equiv - \min_r V_1(r)$, while $-\tilde{G}_1 > 0$ if

$$A < A_1 \equiv - \max\left[\frac{R_2 (R_2 - R_1)^2 h(r)}{R_1 \pi^2} + V_1(r)\right].$$

Thus the addition of a background swirl Ar to $V(r)$ results in a stable flow provided that the swirl coefficient A lies outside the interval $[A_1, A_2]$.

Similarly, if $V(r)$ is such that $\left(V' + \frac{V}{r}\right)' = 0$ at $r = r_1, r_2$ with $\left(V' + \frac{V}{r}\right)' = (r - r_1)(r - r_2)h(r), h(r) > 0$ in $[R_1, R_2]$, then consider the family of flows $\tilde{V}(r) = V(r) + Ar$ where A is a constant.

Define $V_j(r) = \dfrac{V(r) - V(r_j)}{r - r_j}$, $r \neq r_j$, and $V_j(r_j) = V'(r_j)$ $(j = 1, 2)$.

Then V_j is continuous on $[R_1, R_2]$. For the choice $C_j = \tilde{V}(r_j)$ $(j = 1, 2)$ define $\tilde{G}_2 = \dfrac{(\tilde{V} - C_1)(\tilde{V} - C_2)}{r\left(V' + \frac{V}{r}\right)'} + \tilde{H}_2$, where

$$\tilde{H}_2 = P^{-1}(\tilde{V} - \hat{C}_2) + (\tilde{V} - \hat{C}_2)P^{-1} + P^{-1} r\left(V' + \frac{V}{r}\right)' P^{-1} + P^{-1}\frac{\tilde{V}}{r}P^{-1}$$

with $$\hat{C}_2 = \frac{C_1 + C_2}{2}.$$

But $$\frac{(\tilde{V} - C_1)(\tilde{V} - C_2)}{r\left(V' + \frac{V}{r}\right)'} = \frac{(V_1 + A)(V_2 + A)}{rh(r)}$$ and

$$\langle \xi, \tilde{H}_2 \xi \rangle = 2 \int_{R_1}^{R_2} (\tilde{V} - \hat{C}_2)\left[r|\phi'|^2 + \frac{m^2}{r}|\phi|^2\right] dr$$

$$= 2 \int_{R_1}^{R_2} [(V - \hat{V}) + A(r - \hat{r})]\left[r|\phi'|^2 + \frac{m^2}{r}|\phi|^2\right] dr,$$

where $\hat{V} = \dfrac{V(r_1) + V(r_2)}{2}$ and $\hat{r} = \dfrac{r_1 + r_2}{2}$.

Now, if $A > A_+ \equiv -\min\left\{\min_r V_1(r), \min_r V_2(r)\right\}$, then

$$\left\langle \xi, \frac{(\tilde{V}-C_1)(\tilde{V}-C_2)}{r\left(V' + \frac{V}{r}\right)'} \xi \right\rangle \geq \frac{(A-A_+)^2}{R_2\bar{h}} \|\xi\|^2, \qquad \dots (4.20)$$

and

$$\langle \xi, \tilde{H}_2 \xi \rangle \geq 2 \{(A_+ - A)(\hat{r} - R_1) - (\hat{V} + A_+ \hat{r})$$

$$- \min (V + A_+ r)\} \Lambda^{-1} \|\xi\|^2, \qquad \dots (4.21)$$

where $\bar{h} = \max_r h(r)$.

Therefore using (4.20) and (4.21), we obtain

$$\langle \xi, \tilde{G}_2 \xi \rangle \geq [(A-A_+)^2 - (A-A_+)B - C] \frac{\|\xi\|^2}{R_2\bar{h}}, \qquad \dots (4.22)$$

where

$$B = 2R_2 \Lambda^{-1} \tilde{h} (r - R_1) > 0$$

and $C = 2R_2 \Lambda^{-1} \tilde{h} [(\tilde{V} + A_+ \tilde{r}) - \min (V + A_+ r)] > 0.$

Hence, $\tilde{G}_2 > 0$ if $A > A_+ + \frac{1}{2}[B + (B^2 + 4C)^{1/2}]$.

Similarly if $A < A_- \equiv -\max\left\{\max_r V_1(r), \max_r V_2(r)\right\}$, then

$$-\tilde{G}_2 > 0 \text{ if } A < A_- - \frac{1}{2}[D + (D^2 + 4E)^{1/2}],$$

where

$$D = 2R_2 \Lambda^{-1} \bar{h} (R_2 - \hat{r}) \text{@PARA} = \text{ and}$$
$$E = 2R_2 \Lambda^{-1} \tilde{h} [(\hat{V} + A_- \hat{r}) - \min (V + A_- r)].$$

Thus $\tilde{G}_2$ is positive if A lies outside the interval $[A_-, A_+]$. Therefore, the flow can be made stable by the addition of a background swirl Ar to it where A lies outside the interval $[A_-, A_+]$. This completes the proof.

Now we shall explain the above stability theorems. For the basic flow (0, $V(r)$, 0) the (scalar) vorticity is $\left(V' + \frac{V}{r}\right)'$ which is a function of r alone. From the above theorems we can see that the stability or otherwise of a swirling flow of an inviscid incompressible homogeneous fluid to two-dimensional disturbances depends on the distribution of the vorticity of the basic flow. This is natural since the stability equation (2.2) can also be derived from the vorticity equation satisfied by the stream function in cylindrical polar coordinates (Drazin and Reid[3], p.103).

Theorem 4.2 states that the basic swirling flow is stable if $\left(V' + \frac{V}{r}\right)' = 0$ throughout the flow domain. This means that constant vorticity swirling flows are stable. Hence, for a swirling flow to be unstable it is necessary that its vorticity varies with r. But, if the vorticity of a swirling flow is strictly increasing (or decreasing) as r varies from $r = R_1$ (inner cylinder) to $r = R_2$ (outer cylinder) then such a flow is stable by Theorem 4.1. Another explanation of Theorem 4.1 is that for a basic swirling flow to be unstable it is necessary that the gradient of the basic vorticity $\left(V' + \frac{V}{r}\right)'$ must change its sign atleast once in the flow domain and consequently $\left(V' + \frac{V}{r}\right)'$ should become zero atleast once in $[R_1, R_2]$. Swirling flows for which the gradient of the vorticity has a simple zero or two distinct simple zeros will be stable if they satisfy the conditions of Theorems 4.3 and 4.4. However, even if a swirling flow is unstable it can be made stable by the addition of a background swirling flow to it if our unstable flow is such that its gradient of the vorticity has only a simple zero or two simple zeros as shown in Theorem 4.5.

5. Conclusions

In this paper, we have analysed the linear stability of swirling flows of an inviscid, incompressible homogeneous fluid confined between two concentric cylinders to two-dimensional disturbances using Barston's method. We have shown that the swirling flow is stable if the gradient of the vorticity either vanishes nowhere in the flow domain or vanishes throughout the flow domain. Also, we have obtained sufficient conditions for the stability of those flows whose vorticity gradients have a simple zero or two distinct simple zeros in the flow domain. In fact, if the basic swirling flow is such that the vorticity gradient has a simple zero or two distinct simple zeros in the flow domain we have shown that the addition of a background swirl to it results in a stable flow.

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