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THE SOLUTION FOR MIXED BOUNDARY VALUE PROBLEMS OF TWO-DIMENSIONAL POTENTIAL THEORY

FUQIAN YANG[1] AND RONG YAO[2]

University of Rochester, Rochester, NY 14627, *U.S.A.*

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Using Green's functions, the Dirichlet mixed boundary value problems for two dimensional potential theory are transformed to singular integral equations, and solved exactly.

1. INTRODUCTION

Mixed boundary value problems for computing potential and diffusion rates arise in mathematical physics, mechanics and engineering[1, 2]. These problems have been studied extensively : Mills *et al.*[3], Mushkelishili[4] and Sneddon[1] have reviewed the available solution procedures. These techniques were classified as follows : integral transforms; integral equation techniques; complex variables and numerical calculation.

For the mixed boundary value problems of two dimensional potential theory, several works have recently been published. Ranger[5] considered the two dimensional potential problem of a plate charged to an arbitrary potential situated symmetrically between and parallel to earthed planes. Srivastav had earlier considered the case where the plate is perpendicular to the planes. The problem was reduced to dual series equations and then solved[1]. The stated problem in Ranger[5] was reduced, by successive integral transformations, to a two part boundary value problem for a harmonic function and solved. Rose and De Hoog[6] and Tait and Moodie[7] using the complex variables, and Ejike[8], Singh *et al.*[9], Yang and Li[10] and Davidson[11] using Fourier transforms and integral equation methods solved a set of mixed boundary value problems for the two dimensional potential problem.

As shown in Fig. 1, the problem to be considered here is that of finding the potential distribution associated with two parallel planes; one plane has a coplanar infinite strip $0 \le |X| \le a$, $Y = 0$ with potential $g(X)$, and $|X| \ge a$, $Y = 0$ is insulated, and $Y = L$ the other boundary either is earthed of insulated. The problems may be

[1]Department of Mechanical Engineering.
[2]Department of Electrical Engineering.

reduced using Green's function to the integral equations with logarithmic kernels, and solved exactly.

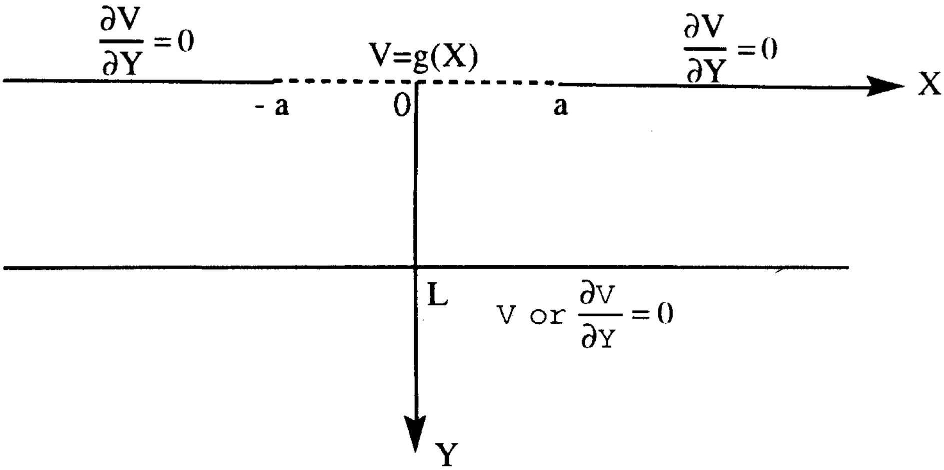

FIG. 1. Physical configuration.

2. THE MIXED BOUNDARY VALUE PROBLEM I

2.1. *Formulation of the Mixed Boundary Value Problem I*

Consider two parallel planes, one has a coplanar infinite strip, $0 \le |X| \le a$, $Y = 0$ with potential $g(X)$ and $|X| \ge a$, $Y = 0$ is insulated, and the other boundary $Y = L$ is earthed. We want to find the potential V at any point between the planes.

This is equivalent to finding the solution $V(X, Y)$ of Laplace's equation

$$\nabla^2 V(X,Y) = 0 \qquad \text{... (1)}$$

in the domain $0 < Y < L$, subject to the boundary conditions

$$\frac{\partial V}{\partial Y} = 0, \qquad \text{for } |X| \ge a,\ Y = 0 \qquad \text{... (2)}$$

$$V(X, 0) = g(X), \quad \text{for } |X| \le a,\ Y = 0 \qquad \text{... (3)}$$

$$V(X, 0) = 0, \qquad \text{at } Y = L. \qquad \text{... (4)}$$

Further $V(X, Y)$ should tend to zero as $|X| \to \infty$.

Before solving the above boundary value problem, we first introduce the following dimensional parameters :

$$x = X/a, \ y = Y/a. \qquad \text{... (5)}$$

Substituting the above dimensionless parameters into eqns. (1)-(4), we obtain

$$\nabla^2 V(x,y) = 0 \qquad \text{... (6)}$$

in the domain $0 < y < L/a$, subject to the boundary conditions

$$\frac{\partial V}{\partial y} = 0, \qquad \text{for } |x| \geq 1,\ y = 0 \qquad \text{... (7)}$$

$$V(x, y) = g(x), \quad \text{for } |x| \leq 1,\ y = 0 \qquad \text{... (8)}$$

$$V(x, y) = 0, \qquad \text{at } y = L/a \qquad \text{... (9)}$$

$$|V(x, y)| < \infty, \quad \text{for } |x| \to \infty . \qquad \text{... (10)}$$

The solution for $V(x, y)$ in terms of the Green's function can be expressed as

$$V(x, y) = \int_{-1}^{1} G(x, y, x_0, 0) \left.\frac{\partial V}{\partial y_0}\right|_{y_0 = 0} dx_0 \qquad \text{... (11)}$$

where $G(x, y, x_0, y_0)$ is the Green's function and is defined below as

$$\nabla^2 G(x, y, x_0, y_0) = \delta(x_0, y_0) \qquad \text{... (12)}$$

in the domain $0 < y < L/a$, subject to the boundary conditions

$$\frac{\partial G(x, y, x_0, y_0)}{\partial y} = 0, \qquad \text{for } y = 0 \qquad \text{... (13)}$$

$$G(x, y, x_0, y_0) = 0, \qquad \text{for } y = L/a. \qquad \text{... (14)}$$

2.2. *Solution for Green's Function* $G(x, y, x_0, y_0)$

Using the Fourier series, the Green's function $G(x, y, x_0, y_0)$ can be expressed as

$$G(x, y, x_0, y_0) = \sum_{n=1}^{\infty} g_n \cos\left(\left(n + \frac{1}{2}\right)\frac{a\pi y}{L}\right), \qquad \text{... (15)}$$

satisfying the boundary conditions (13) and (14). Substituting eqn. (15) into eqn. (12), we obtain

$$\sum_{n=1}^{\infty} \left\{ \frac{\partial^2 g_n}{\partial x^2} - \left(\left(n + \frac{1}{2}\right)\frac{a\pi}{L}\right)^2 g_n \right\} \cos\left(\left(n + \frac{1}{2}\right)\frac{a\pi y}{L}\right) = \delta(x_0, y_0),$$

... (16)

which in turn gives

$$g_n = \frac{2a}{L} \cos\left(\left(n + \frac{1}{2}\right)\frac{a\pi y_0}{L}\right) h_n , \qquad \text{... (17)}$$

where

$$\frac{\partial^2 h_n}{\partial x^2} - \left(\left(n + \frac{1}{2} \right) \frac{a\pi}{L} \right)^2 h_n = \delta(x_0). \qquad \text{... (18)}$$

Using eqn. (18) and the condition

$$| h_n | < \infty \quad \text{for} \quad | x | \to \infty, \qquad \text{... (19)}$$

we obtain

$$h_n = - \frac{L}{2a\pi\,(n + 1/2)} \exp\left(- \left(n + \frac{1}{2} \right) \frac{a\pi}{L} | x - x_0 | \right) \qquad \text{... (20)}$$

which in turn gives

$$G(x, y, x_0, y_0) = - \sum_{n=1}^{\infty} \frac{1}{(n + 1/2)\pi} \exp\left(- \left(n + \frac{1}{2} \right) \frac{a\pi}{L} | x - x_0 | \right)$$

$$\cos\left(\left(n + \frac{1}{2} \right) \frac{a\pi y}{L} \right) \cos\left(\left(n + \frac{1}{2} \right) \frac{a\pi y_0}{L} \right). \qquad \text{... (21)}$$

Using eqn. (21), we obtain

$$G(x, y, x_0, 0) = \frac{1}{\pi} \operatorname{Re}\left[\ln\left(\frac{\sqrt{r} - 1}{\sqrt{r} + 1} \right) \right], \qquad \text{... (22)}$$

where r is

$$r = \exp\left(- \frac{a\pi}{L} (| x - x_0 | - iy) \right). \qquad \text{... (23)}$$

2.3. *Solution for the Potential V(x, y)*

Simplifying eqn. (22) and using eqns. (8) and (11), we obtain

$$g(x) = \frac{1}{\pi} \int_{-1}^{1} f(x_0) \ln | \tanh (a\pi(x - x_0)/4L) | \, dx_0, \qquad \text{... (24)}$$

which is a singular integral equation, where $f(x_0)$ is defined as

$$f(x_0) = \frac{\partial V}{\partial y_0} \bigg|_{y_0 = 0} \qquad \text{... (25)}$$

To solve this equation, we consider the following two cases.

(a) *g(x) is an even function in x* — Since $g(x)$ is even in x and $f(x_0)$ is even in x_0, we obtain

$$\pi\, g(x) = \int_{0}^{1} f(x_0) \ln | \tanh (a\pi(x - x_0)/4L) \tanh (a\pi(x + x_0)/4L) | \, dx_0,$$

$$\text{for } 0 \le x_0 \le 1. \qquad \text{... (26)}$$

Taking the derivative in x, we obtain

$$\frac{2a}{L}\int_0^1 f(x_0)\frac{\sinh(a\pi x/2L)\cosh(a\pi x_0/2L)}{\cosh(a\pi x/L)-\cosh(a\pi x_0/L)}\,dx_0 = g'(x), \text{ for } 0 \le x_0 \le 1, \qquad \text{... (27)}$$

which is identical to

$$\frac{a}{L}\int_0^1 f(x_0)\frac{\cosh(a\pi x_0/2L)}{\sinh^2(a\pi x/2L)-\sinh^2(a\pi x_0/2L)}\,dx_0$$

$$= \frac{g'(x)}{\sinh(a\pi x/2L)}. \qquad \text{... (28)}$$

Using the solution given by Cooke[12], the solution of eqn. (28) can be expressed as

$$f(x) = \frac{a}{2L\sinh(a\pi x/L)}\frac{d}{dx}\int_x^1\left(\frac{\sinh(a\pi x_0/L)\,dx_0}{[\sinh^2(a\pi x_0/2L)-\sinh^2(a\pi x/2L)]^{1/2}}\right.$$

$$\left.\times\int_0^{x_0}\frac{g'(\xi)\sinh(a\pi\xi/L)\,d\xi}{[\sinh^2(a\pi x_0/2L)-\sinh^2(a\pi\xi/2L)]^{1/2}}\right)$$

$$+\frac{2A}{[\sinh^2(a\pi/2L)-\sinh^2(a\pi x/2L)]^{1/2}} \qquad \text{... (29)}$$

where A is a constant to be determined by $f(0^+)$, which is the charge density on the strip $0^+ \le x \le 1$ for a symmetric potential distribution. Substituting eqns. (29) and (22) into eqn. (11), we find that the potential $V(x, y)$ is given by

$$V(x,y) = \frac{1}{\pi}\int_0^1\left(\ln\left[\frac{\sqrt{\sinh^2(a\pi|x-x_0|/2L)+\sin^2(a\pi y/2L)}}{\cosh(a\pi|x-x_0|/2L)+\cos(a\pi y/2L)}\right]\right.$$

$$\left.+\ln\left[\frac{\sqrt{\sinh^2(a\pi|x+x_0|/2L)+\sin^2(a\pi y/2L)}}{\cosh(a\pi|x+x_0|/2L)+\cos(a\pi y/2L)}\right]\right)f(x_0)\,dx_0. \qquad \text{... (30)}$$

(b) *g(x) is an odd function in x* — Since $g(x)$ is odd in x and $f(x_0)$ is odd in x_0, eqn. (26) is changed to

$$\pi g(x) = \int_0^1 f(x_0)\ln\left|\frac{\tanh(a\pi(x-x_0)/4L)}{\tanh(a\pi(x+x_0)/4L)}\right|dx_0, \quad \text{for} \quad 0 \le x_0 \le 1, \qquad \text{... (31)}$$

which is identical to

$$\pi\, g(x) = -\int_0^1 f(x_0) \ln \left| \frac{\sinh(a\pi x/2L) + \sinh(a\pi x_0/2L)}{\sinh(a\pi x/2L) - \sinh(a\pi x_0/2L)} \right| dx_0. \qquad \ldots (32)$$

Using the solution given by Cooke[12], we obtain

$$f(x) = \frac{a}{2L}\frac{d}{dx}\int_x^1 \left(\frac{\sinh(a\pi x_0/L)\, dx_0}{[\sinh^2(a\pi x_0/2L) - \sinh^2(a\pi x/2L)]^{1/2}} \right.$$

$$\left. \times \int_0^{x_0} \frac{g'(\xi)\, d\xi}{[\sinh^2(a\pi x_0/2L) - \sinh^2(a\pi \xi/2L)]^{1/2}} \right)$$

$$- g(0)\frac{a}{L}\frac{\sinh(a\pi/2L)\, c\tanh(a\pi x/2L)}{[\sinh^2(a\pi/2L) - \sinh^2(a\pi x/2L)]^{1/2}} \qquad \ldots (33)$$

which is the charge density on the strip $0^+ \le x \le 1$ for the antisymmetric case. Substituting eqns. (33) and (22) into eqn. (11), we find that the potential $V(x, y)$ for $y > 0$ is given by

$$V(x, y) = \frac{1}{\pi}\int_0^1 \left(\ln\left[\frac{\sqrt{\sinh^2(a\pi\,|x - x_0|/2L) + \sin^2(a\pi y/2L)}}{\cosh(a\pi\,|x - x_0|/2L) + \cos(a\pi y/2L)} \right] \right.$$

$$\left. - \ln\left[\frac{\sqrt{\sinh^2(a\pi\,|x + x_0|/2L) + \sin^2(a\pi y/2L)}}{\cosh(a\pi\,|x + x_0|/2L) + \cos(a\pi y/2L)} \right] \right) f(x_0)\, dx_0.$$

$$\ldots (34)$$

3. The Mixed Boundary Value Problem II

3.1. *Formulation of the Mixed Boundary Value Problem II*

In the above section, a mixed boundary value problem leading to a singular integral equation is solved. Here we consider another mixed boundary value problem: two parallel infinite planes, one plane has a infinite stripe on it, $0 \le |X| \le a$, $Y = 0$ with potential $g(X)$, the other, $|X| \ge a$, $Y = 0$ and the other plane $Y = L$ is insulated. We want to find the potential V at any point between the planes.

This is equivalent to finding the solution $V(X, Y)$ of Laplaces's equation

$$\nabla^2 V(X, Y) = 0, \qquad \ldots (35)$$

in the domain $0 < Y < L$, subject to the boundary conditions

$$\frac{\partial V}{\partial Y} = 0, \qquad \text{for } |X| \ge a,\ Y = 0, \qquad \ldots (36)$$

$$V(X, 0) = g(X), \qquad \text{for } |X| \le a,\ Y = 0, \qquad \ldots (37)$$

$$\frac{\partial V}{\partial Y} = 0, \qquad \text{at } Y = L. \qquad \dots (38)$$

Using the dimensionless parameters given in eqn. (5), we obtain eqns. (6)-(8), (10) and

$$\frac{\partial V}{\partial Y} = 0, \qquad \text{at } y = L/a, \qquad \dots (39)$$

$$| V(x, y) | < \infty, \qquad \text{for } | x | \to \infty. \qquad \dots (40)$$

Similarly, the solution for $V(x, y)$ by Green's function can be expressed as eqn. (11). The Green's function $G(x, y, x_0, y_0)$ is defined below

$$\nabla^2 G(x, y, x_0, y_0) = \delta(x_0, y_0), \qquad \dots (41)$$

in the domain $0 < y < L/a$, subject to the boundary conditions

$$\frac{\partial G(x, y, x_0, y_0)}{\partial y} = 0, \qquad \text{at } y = 0, \qquad \dots (42)$$

$$\frac{\partial G(x, y, x_0, y_0)}{\partial y} = 0, \qquad \text{at } y = L/a. \qquad \dots (43)$$

3.2. *Solution for Green's Function* $G(x, y, x_0, y_0)$

Using the same method as in the previous section, the Green's function $G(x, y, x_0, y_0)$ can be expressed as

$$G(x, y, x_0, y_0) = \frac{a}{2L} | x - x_0 | - \sum_{n=1}^{\infty} \frac{1}{n\pi} \exp\left(- \frac{an\pi}{L} | x - x_0 | \right) \times \cos\left(\frac{an\pi}{L} y \right) \cos\left(\frac{an\pi}{L} y_0 \right). \qquad \dots (44)$$

Using eqn. (44), we obtain

$$G(x, y, x_0, 0) = \frac{a}{2L} | x - x_0 | + \frac{1}{\pi} \operatorname{Re} [\ln (1 - r)], \qquad \dots (45)$$

where r is defined in eqn. (23).

3.3. *Solution for the Potential* $V(x, y)$

Simplifying eqn. (45) and using eqns. (8) and (11), we obtain

$$g(x) = \frac{1}{\pi} \int_{-1}^{1} f(x_0) \ln | 2 \sinh (a\pi (x - x_0)/2L) | \, dx_0, \qquad \dots (46)$$

which is a singular integral equation, and where $f(x_0)$ is defined in eqn. (25). To solve this equation, we consider the following two cases.

(a) *g(x) is an even function in x* — Since $g(x)$ is even in x and $f(x_0)$ is even in x_0, we obtain

$$\pi\, g(x) = \int_0^1 f(x_0) \ln |\, 4 \sinh (a\pi\,(x - x_0)/2L)$$

$$\sinh (a\pi\,(x + x_0)/2L)\,|\, dx_0, \qquad \text{for } 0 \le x_0 \le 1, \qquad \text{... (47)}$$

which is identical to

$$\pi g(x) = \int_0^1 f(x_0) \ln |\, 2 \cosh (a\pi x/L) - 2 \cosh (a\pi x_0/L)\,|\, dx_0. \qquad \text{... (48)}$$

Using the solution given by Cooke[12], the solution of eqn. (48) is

$$f(x) = \frac{a}{L} \frac{1}{(\cosh (a\pi x/L) - 1)^{1/2}} \frac{d}{dx}$$

$$\int_x^1 \left(\frac{\sinh (a\pi x_0/L)\, dx_0}{[2 \cosh (a\pi x_0/L) - 2 \cosh (a\pi x/L)]^{1/2}} \right.$$

$$\left. \times \int_0^{x_0} \left[\frac{\cosh (a\pi\xi/L) - 1}{\cosh (a\pi x_0/L) - \cosh (a\pi\xi/L)} \right]^{1/2} g'(\xi)\, d\xi \right)$$

$$+ \frac{\pi a^2}{L^2} \frac{\sinh (a\pi x/L)}{[(\cosh (a\pi x/L) - 1)(\cosh (a\pi/L) - \cosh (a\pi x/L))]^{1/2}}$$

$$\times \frac{1}{\ln [\cosh (a\pi/L) - 1] - \ln 2}$$

$$\times \int_0^1 \frac{g(\xi) \sinh (a\pi\xi/L)\, d\xi}{[(\cosh (a\pi\xi/L) - 1)(\cosh (a\pi/L) - \cosh (a\pi\xi/L))]^{1/2}}, \qquad \text{... (49)}$$

which is the charge density on the strip $0^+ \le x \le 1$ for a symmetric potential distribution. Substituting eqns. (49) and (45) into eqn. (11), we find that the potential $V(x, y)$ is given by

$$V(x, y) = \frac{1}{2\pi} \int_0^1 \Big(\ln 4 + \ln [\cosh (a\pi\,|\,x - x_0\,|/L) - \cosh (a\pi y/L)]$$

$$+ \ln\, [\cosh (a\pi\,|\,x + x_0\,|/L) - \cos (a\pi y/L)] \Big) f(x_0)\, dx_0. \qquad \text{... (50)}$$

(b) *g(x) is an odd function in x* — Since $g(x)$ is odd in x and $f(x_0)$ is odd in x_0, eqn. (46) is changed to

$$\pi g(x) = \int_0^1 f(x_0) \ln \left| \frac{\sinh (a\pi (x - x_0)/2L)}{\sinh (a\pi (x + x_0)/2L)} \right| dx_0, \quad \text{for} \quad 0 \le x_0 \le 1, \quad \dots (51)$$

which is identical to

$$\pi g(x) = -\int_0^1 f(x_0) \ln \left| \frac{\tanh (a\pi x/2L) + \tanh (a\pi x_0/2L)}{\tanh (a\pi x/2L) - \tanh (a\pi x_0/2L)} \right| dx_0. \quad \dots (52)$$

Using the solution given by Cooke[12], we obtain

$$f(x) = \frac{a}{L}\frac{d}{dx}\int_x^1 \left(\frac{\tanh (a\pi x_0/2L)\, dx_0}{\cosh^2 (a\pi x_0/2L)\, [\tanh^2 (a\pi x_0/2L) - \tanh^2 (a\pi x/2L)]^{1/2}} \right.$$

$$\left. \times \int_0^{x_0} \frac{g'(\xi)\, d\xi}{[\tanh^2 (a\pi x_0/2L) - \tanh^2 (a\pi \xi/2L)]^{1/2}} \right)$$

$$- g(0)\frac{2a}{L}\frac{\tanh (a\pi/2L) \sinh^{-1} (a\pi x/L)}{[\tanh^2 (a\pi/2L) - \tanh^2 (a\pi x/2L)]^{1/2}}, \quad \dots (53)$$

which is the charge density on the strip $0^+ \le x \le 1$ for antisymmetric case. Substituting eqns. (53) and (45) into eqn. (11), we find that the potential $V(x, y)$ for $y > 0$ is given by

$$V(x, y) = \frac{1}{2\pi}\int_0^1 \ln \left| \frac{\cosh (a\pi |x - x_0|/L) - \cos (a\pi y/L)}{\cosh (a\pi |x + x_0|/L) - \cos (a\pi y/L)} \right| f(x_0)\, dx_0. \quad \dots (54)$$

4. Special Cases

We first consider the symmetric case of a half space (i.e. $L \to \infty$) due to an infinite strip $0 \le |X| \le a$, $Y = 0$ being charged to a unifrom voltage V_0. Since $L \to \infty$, we have

$$\tanh\left(\frac{a\pi x_0}{2L}\right) = \frac{a\pi x_0}{2L} \quad \text{and} \quad \sinh\left(\frac{a\pi x_0}{2L}\right) = \frac{a\pi x_0}{2L}. \quad \dots (55)$$

For problem I, substituting eqn. (29) into eqn. (26) , eqn. (29) becomes

$$f(x) = -\frac{1}{\ln 2}\frac{V_0}{\sqrt{1 - x^2}}, \quad \dots (56)$$

which is the potential distribution on the strip $|x| < 1$, $y = 0$, and is the same as that given by Ejike[8]. Substitution of eqn. (56) into eqn. (30) yields the potential distribution $v(x, y)$ for $y > 0$.

For problem II, since $\cosh\left(\frac{a\pi x_0}{2L}\right) = 1$ for $\frac{a}{L} \to \infty$, eqn. (49) gives

$$f(x) = 0 \quad \text{... (57)}$$

which indicates that there is an uniform field inside the half space.
For the antisymmetric case of a half space due to an infinite strip $0 \le |X| \le a$, $Y = 0$ being charged to a uniform voltage $V_0 \operatorname{sgn}(x)$, we, combining eqns. (55), (33) and (53) obtain

$$f(x) = -\frac{2}{\pi} \frac{V_0}{x\sqrt{1-x^2}} \quad \text{... (58)}$$

which is the same as that obtained by dual integral equation method[8]. Substitution of eqn. (58) into eqn. (54) yields the potential distribution $v(x, y)$ for $y > 0$.

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