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VIEW-OBSTRUCTION AND A CONJECTURE OF SCHOENBERG

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Here a conjecture of Schoenberg regarding the billiard ball problem for spheres is proved in the Euclidean space $\mathbb{R}^n$ for $n = 3, 4$. Markoff type chains of related isolated extreme values are also obtained. This is achieved by using the theory of view-obstruction problems developed by the authors earlier and applying known results about covering radii of lattices in the plane and in $\mathbb{R}^3$. Analogous results for $(n-2)$-dimensional trajectories in $\mathbb{R}^n$, for all $n \geq 3$, are also obtained.

1. INTRODUCTION

The view-obstruction problem was originally formulated by Cusick[3], though it had been studied earlier in another formulation by Wills[17]. It was later generalised by the authors[4, 5] where rays were replaced by flats. In Dumir *et al.*[5], we observed that the problem of obstructing the view through lines is related to the billiard ball motion problem considered by Schoenberg[11-13] (see also König and Szücs[8] and Hardy and Wright[9], p. 378). Here we shall use some results obtained in Dumir *et al.*[5] to prove a conjecture of Schoenberg for spheres in three and four dimensional spaces. We also obtain Markoff type chains of related isolated extreme values and some analogous results for $(n-2)$-dimensional trajectories in $\mathbb{R}^n$ for all $n \geq 3$. A different analogous problem has been studied by various authors, see e.g. Bambah[1].

Let $\mathbb{R}^n$ be n-dimensional Euclidean space; $\mathbb{Z}^n$, the integral lattice, $\frac{1}{2}$, the point $\left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right) \in \mathbb{R}^n$; and Λ, the shifted lattice $\frac{1}{2} + \mathbb{Z}^n$. Let C be a closed, convex body with centre $\mathbf{o}$, and $d_C(\mathbf{K}, \mathbf{K}')$, the C-norm distance between subsets $\mathbf{K}, \mathbf{K}' \subset \mathbb{R}^n$. For each flat $\mathbf{F} \subset \mathbb{R}^n$ and for each subspace $\mathbf{U} \subset \mathbb{R}^n$, we define, as in Dumir *et al.*[5]

$$\nu(C, \mathbf{F}) = d_C(\Lambda, \mathbf{F})$$

$$= \inf \{\alpha > 0 : (\alpha \mathbf{C} + \Lambda) \bigcap \mathbf{F} \neq \phi\},$$

$$\bar{\nu}(\mathbf{C}, \mathbf{U}) = \sup \{\nu(\mathbf{C}, \mathbf{F}) : \mathbf{F} \text{ is a translate of } \mathbf{U}\}$$

and, for each dimension $d \geq 1$,

$$\bar{\nu}(\mathbf{C}, \mathbf{d}) = \sup \{\bar{\nu}(\mathbf{C}, \mathbf{U}) : \dim \mathbf{U} = d;\ \mathbf{U} \text{ not contained in a coordinate hyperplane}\}$$

$$= \sup \{\nu(\mathbf{C}, \mathbf{F}) : \dim \mathbf{F} = d;\ \mathbf{F} \text{ not contained in a hyperplane } x_i = \text{constant}\}.$$

In Dumir *et al.*[5] we showed that $\bar{\nu}(\mathbf{C}, \mathbf{U})$ can be obtained by computing $\bar{\nu}(\mathbf{C}, \mathbf{V})$ for a suitable "rational" subspace $\mathbf{V}$ (see section 2 for more details). Using this we determined $\bar{\nu}(\mathbf{C}, n-1)$ and, in fact, the complete spectrum $\bar{\nu}(\mathbf{C}, \mathbf{U})$ for $(n-1)$-dimensional subspaces $\mathbf{U}$. It is easily seen that for a rational subspace $\mathbf{U}$ of dimension d, $\bar{\nu}(\mathbf{C}, \mathbf{U})$ is equal to the covering radius of an $(n-d)$-dimensional lattice with respect to a suitable convex body. Here we shall determine a formula for $\bar{\nu}(\mathbf{B}, \mathbf{S})$, where $\mathbf{B}$ is the Euclidean ball of diameter 1 and $\mathbf{S}$ is a rational subspace of dimension $n-2$. This leads to the value of $\bar{\nu}(\mathbf{B}, n-2)$ and related isolation results for each dimension $n \geq 3$ (see Theorem 3 and Corollaries 3 and 4). For dimension $n = 4$, we use a method developed in Dumir *et al.*[5] to find upper bounds for $\bar{\nu}(\mathbf{B}, \mathbf{U})$ for rational subspaces $\mathbf{U}$ and obtain $\bar{\nu}(\mathbf{B}, 1) = \sqrt{5/4}$ and related isolations (see Theorems 5-8 and Corollary 5). In a later paper we shall show that $\bar{\nu}(\mathbf{B}, n-3) = \sqrt{3/2}$ for $n \geq 6$.

Let $\mathbf{F} = \mathbf{U} + \mathbf{p}$ and $\mathbf{F}' = \mathbf{U} + \mathbf{p} - \frac{1}{2}$ be two translates of the subspace U. Then, since the metric d_{C} is translation invariant

$$d_{\mathrm{C}}(\Lambda, \mathbf{F}) = d_{\mathrm{C}}\left(\frac{1}{2} + \mathbb{Z}^n,\ \mathbf{U} + \mathbf{p}\right)$$

$$= d_{\mathrm{C}}\left(\mathbb{Z}^n, \mathbf{U} + \mathbf{p} - \frac{1}{2}\right)$$

$$= d_{\mathrm{C}}(\mathbb{Z}^n, \mathbf{F}').$$

It follows that the quantities $\bar{\nu}(\mathbf{C}, \mathbf{U})$ and $\bar{\nu}(\mathbf{C}, d)$ remain unchanged if we modify their definitions by writing $d_{\mathrm{C}}(\mathbb{Z}^n, \mathbf{F})$ at each appearance of $\nu(\mathbf{C}, \mathbf{F})$. This observation permits us to link view-obstruction problems with Schoenberg's problem of billiard ball motion.

Schoenberg[11-13] (see also König and Szücs[8] and Hardy and Wright[9], p. 378) considered billiard ball motion within the unit cube : $|x_i| \leq \frac{1}{2}$, $i = 1, 2, \ldots, n$ in $\mathbb{R}^n$. A point $\mathbf{p} = \mathbf{p}(t)$ moves with uniform rectilinear motion within the cube and is reflected in the usual way on striking a boundary hyperplane $x_i = \pm\frac{1}{2}$. The resulting trajectory Γ_n is called 'non-trivial' if it is not contained in a hyperplane x_i = constant.

If **C** is an arbitrary closed convex body with centre **o**, we can follow Schoenberg's definitions for l_p-balls and set

$$d_{\mathbf{C}}(\Gamma_n) = d_{\mathbf{C}}(\{\mathbf{o}\}, \Gamma_n)$$

and

$$\rho_n^{\mathbf{C}} = \sup d_{\mathbf{C}}(\Gamma_n),$$

where the supremum is taken over all non-trivial trajectories Γ_n.

Schoenberg[13] determined $\rho_n^{\mathbf{C}}$ when **C** is the unit box with centre **o**. This also follows from an earlier result of Wills[16]. The initial segment of a trajectory Γ_n determines a line **L** in $\mathbb{R}^n$. If the convex body **C** is symmetric by reflection in the coordinate hyperplanes, this line satisfies

$$d_{\mathbf{C}}(\{\mathbf{o}\}, \Gamma_n) = d_{\mathbf{C}}(\mathbb{Z}^n, \mathbf{L}).$$

Moreover, since non-trivial trajectories correspond to lines not contained in hyperplanes x_i = constant, it follows that

$$\rho_n^{\mathbf{C}} = \bar{\nu}(\mathbf{C}, 1).$$

In a later paper, Schoenberg[14] considered higher dimensional trajectories and introduced, at least in the l_∞ case, quantities similar to our $\bar{\nu}(\mathbf{C}, d)$.

Schoenberg[12] conjectured that the quantity related to the l_2-ball of radius 1 satisfies

$$\rho_n^{(2)} = \sqrt{\frac{n}{12} - \frac{1}{12n}}$$

and that the supremum is attained essentially for the 'lucky shot' Γ_n^* obtained by sending the particle along the line with direction ratios (1, 1, ..., 1) and through a point in the unit cube which is a translate of $\left(0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\right)$ by a point of $\mathbb{Z}^n$. He proved this conjecture for $n = 2$ in Schoenberg[11] and announced a proof for $n = 3$ in Schoenberg[12].

The results about $\bar{\nu}(\mathbf{B}, \mathbf{U})$ that we prove here give, in particular, a proof of Schoenberg's conjecture for $n = 3, 4$. Because we follow the view-obstruction tradition and take **B** to be the l_2-ball of diameter 1 rather than radius 1,

$$\rho_n^{(2)} = \frac{1}{2}\bar{\nu}(\mathbf{B}, 1).$$

In the next section, we give a precise statement of the results which will be proved here.

2. Statement of Results

A d-dimensional subspace $\mathbf{U}$ of $\mathbb{R}^n$ is called rational if it is $\{\mathbf{o}\}$ or if it has a basis consisting of vectors in $\mathbf{Q}^n$. Equivalently, a rational subspace of dimension d is an intersection of $n-d$ independent hyperplanes with normals in $\mathbf{Q}^n$. If a subspace is not rational then it is called irrational. We showed in Dumir *et al.*[4, 5] that every irrational subspace $\mathbf{U}$ is contained in a unique rational subspace $\mathbf{M}(\mathbf{U})$ of least dimension. Also for $\mathbf{p} \in \mathbb{R}^n$, $\nu(\mathbf{B}, \mathbf{U}+\mathbf{p}) = \nu(\mathbf{B}, \mathbf{M}(\mathbf{U})+\mathbf{p})$ and $\bar{\nu}(\mathbf{B}, \mathbf{U}) = \bar{\nu}(\mathbf{B}, \mathbf{M}(\mathbf{U}))$ (see section 2 of Dumir *et al.*[5]).

Interchange of co-ordinates and reflection in a co-ordinate hyperplane are automorphisms of both $\mathbf{B}$ and Λ. A subspace $\mathbf{U}$ in $\mathbb{R}^n$ is called equivalent to a subspace $\mathbf{U}'$ ($\mathbf{U} \sim \mathbf{U}'$) if $\mathbf{U}$ is obtained from $\mathbf{U}'$ by applying such automorphisms. Clearly, if $\mathbf{U} \sim \mathbf{U}'$ then $\nu(\mathbf{B}, \mathbf{U}) = \nu(\mathbf{B}, \mathbf{U}')$ and $\bar{\nu}(\mathbf{B}, \mathbf{U}) = \bar{\nu}(\mathbf{B}, \mathbf{U}')$. The equivalence of flats (and in particular of points) is defined in an analogous manner.

It is easy to determine $\nu(\mathbf{B}, \mathbf{F})$ for hyperplanes $\mathbf{F}$ and hence to determine $\bar{\nu}(\mathbf{B}, n-1)$ (see Section 3 of Dumir *et al.*[5]). Here we shall consider lower dimensional subspaces. Let $n \geq 3$ and let $\mathbf{S}$ be an $(n-2)$-dimensional subspace of $\mathbb{R}^n$ not lying in a coordinate hyperplane. Henceforth we reserve the symbol $\mathbf{S}$ for this special role and denote the Euclidean norm by $|\mathbf{x}|$. We shall prove

Theorem 1 — If $\mathbf{S}$ is an irrational $(n-2)$-dimensional subspace of $\mathbb{R}^n$ which does not lie in a co-ordinate hyperplane then $\bar{\nu}(\mathbf{B}, \mathbf{S}) \leq 1/\sqrt{2}$ and strict inequality holds except when $\mathbf{S}$ is contained in a hyperplane $\mathbf{c}\cdot\mathbf{x} = 0$, $\mathbf{c} \in \mathbb{Z}^n$, $|\mathbf{c}|^2 = 2$.

When the subspace $\mathbf{S}$ is rational, $\mathbf{S}^{\perp} \cap \mathbb{Z}^n$ is a 2-dimensional lattice of determinant Δ (say). It follows from a result of Smith[15] (see also McMullen[10])) that $\det(\mathbf{S} \cap \mathbb{Z}^n) = \det(\mathbf{S}^{\perp} \cap \mathbb{Z}^n) = \Delta$. Let $\mathbf{c}_1, \mathbf{c}_2$ be a basis of $\mathbf{S}^{\perp} \cap \mathbb{Z}^n$ with $\mathbf{c}_1$ a nonzero lattice point nearest to the origin and $0 \leq \mathbf{c}_1\cdot\mathbf{c}_2 \leq \frac{1}{2}|\mathbf{c}_1|^2$. We have

Theorem 2 — Let $\mathbf{S}$ be a rational $(n-2)$-dimensional subspace of $\mathbb{R}^n$, which does not lie in a co-ordinate hyperplane. If a basis $\mathbf{c}_1, \mathbf{c}_2$ of $\mathbf{S}^{\perp} \cap \mathbb{Z}^n$ is chosen as above then
$$\bar{\nu}^2(\mathbf{B}, \mathbf{S}) = \frac{1}{|\mathbf{c}_1|^2}\left(1 + \frac{(\mathbf{c}_1\cdot\mathbf{c}_2)^2}{\Delta^2}\right)\left(1 + \frac{(|\mathbf{c}_1|^2 - \mathbf{c}_1\cdot\mathbf{c}_2)^2}{\Delta^2}\right),$$
where $\Delta = \det(\mathbf{S}^{\perp} \cap \mathbb{Z}^n)$.

Corollary 1 — If in Theorem 2, we have $\mathbf{c}_1\cdot\mathbf{c}_2 = 0$ then
$$\bar{\nu}^2(\mathbf{B}, \mathbf{S}) = \frac{1}{|\mathbf{c}_1|^2} + \frac{|\mathbf{c}_1|^2}{\Delta^2} = \frac{1}{|\mathbf{c}_1|^2} + \frac{1}{|\mathbf{c}_2|^2}.$$

Corollary 2 — If in Theorem 2, $\mathbf{S}^{\perp}$ contains $\mathbf{c} \in \mathbb{Z}^n$ with $|\mathbf{c}|^2 = 2$ then
$$\bar{\nu}^2(\mathbf{B}, \mathbf{S}) = \begin{cases} \dfrac{1}{2} + \dfrac{2}{\Delta^2} & \text{if } \Delta^2 \equiv 0 \pmod 2 \\[2ex] \dfrac{1}{2}\left(1 + \dfrac{1}{\Delta^2}\right)^2 & \text{if } \Delta^2 \equiv 1 \pmod 2. \end{cases}$$

Corollary 3 — Suppose that in Theorem 2 the subspace $\mathbf{S}^\perp$ does not contain $\mathbf{c} \in \mathbb{Z}^n$ with $|\mathbf{c}|^2 = 2$.

(a) If $\mathbf{S}^\perp \cap \mathbb{Z}^n$ has a basis $\mathbf{c}_1, \mathbf{c}_2$ such that

(i) $\mathbf{c}_1 \cdot \mathbf{c}_2 = 0$ and $|\mathbf{c}_1|^2 = |\mathbf{c}_2|^2 = 4$ or

(ii) $\mathbf{c}_1 \cdot \mathbf{c}_2 = 0$ and $|\mathbf{c}_1|^2 = 3$, $3 \le |\mathbf{c}_2|^2 \le 6$, or

(iii) $\mathbf{c}_1 \cdot \mathbf{c}_2 = 1$ and $|\mathbf{c}_1|^2 = |\mathbf{c}_2|^2 = 3$

then $\bar{\nu}^2(\mathbf{B}, \mathbf{S})$ is $\frac{1}{2}$ or $\frac{1}{3} + \frac{1}{|\mathbf{c}_2|^2}$ or $\frac{9}{16}$, respectively, and in each case $\bar{\nu}^2(\mathbf{B}, \mathbf{S}) \ge \frac{1}{2}$.

(b) If $\mathbf{S}^\perp \cap \mathbb{Z}^n$ does not have a basis satisfying one of the conditions mentioned in (a) then $\bar{\nu}^2(\mathbf{B}, \mathbf{S}) < \frac{1}{2}$.

For future use, we introduce the following subspaces :

$$\mathbf{S}_1 : x_1 - x_2 = 0,\ x_3 - x_4 = 0$$

$$\mathbf{S}_2 : x_1 - x_2 = 0,\ x_2 - x_3 = 0$$

$$\mathbf{S}_3 : x_1 - x_2 = 0,\ x_1 + x_2 + x_3 = 0.$$

In the spectrum $\{\bar{\nu}(\mathbf{B}, \mathbf{S}) : \dim \mathbf{S} = n - 2\}$ these subspaces (when available) give the highest values, namely 1, $\sqrt{8/9}$, $\sqrt{5/6}$ respectively.

Theorem 3 — Let $\mathbf{S}$ be a rational subspace of dimension $n - 2$ in $\mathbb{R}^n$, which does not lie in a co-ordinate hyperplane.

(a) For $n \ge 4$, $\bar{\nu}(\mathbf{B}, \mathbf{S}) \le 1$ and equality holds iff $\mathbf{S} \sim \mathbf{S}_1$. Further, $\nu(\mathbf{B}, \mathbf{S}_1 + \mathbf{p}) < 1$ except when $\mathbf{S}_1 + \mathbf{p} : x_1 - x_2 = m/2$, $x_3 - x_4 = m'/2$, for odd integers m, m'.

(b) For $n = 3$, $\bar{\nu}(\mathbf{B}, \mathbf{S}) \le \sqrt{8/9}$ and equality holds iff $\mathbf{S} \sim \mathbf{S}_2$. Further, $\nu(\mathbf{B}, \mathbf{S}_2 + \mathbf{p}) < \sqrt{8/9}$ except when $\mathbf{S}_2 + \mathbf{p} : x_1 - x_2 = m + \frac{1}{3}$, $x_2 - x_3 = m' + \frac{1}{3}$, for integers m, m' (or its reflection in the origin).

Corollary 4 — (a) $\bar{\nu}(\mathbf{B}, n - 2) = 1$ for $n \ge 4$.

(b) $\bar{\nu}(\mathbf{B}, n - 2) = \sqrt{8/9}$ for $n = 3$.

Theorem 3(b) and Corollary 4(b) give Schoenberg's conjecture for $n = 3$.

Now let $\mathbf{L}$ be a line in $\mathbb{R}^4$ passing through the origin but not lying in a co-ordinate hyperplane. We shall prove

Theorem 4 — If $\mathbf{L}$ is an irrational line in $\mathbb{R}^4$ which passes through the origin and does not lie in a co-ordinate hyperplane then $\bar{\nu}(\mathbf{B}, \mathbf{L}) \le 1$ and equality holds iff $\mathbf{L}$ lies in a 2-dimensional subspace equivalent to $\mathbf{S}_1$.

For rational lines, the situation is different. There are rational lines in $\mathbb{R}^4$ for which $\bar{\nu}(\mathbf{B}, \mathbf{L}) > 1$. In fact we shall show that the maximum value that can be taken by $\bar{v}(\mathbf{B}, \mathbf{L})$ is for rational lines $\mathbf{L}$ not lying in a co-ordinate hyperplane $\sqrt{5/4}$ and determine precisely such lines for which $\bar{v}(\mathbf{B}, \mathbf{L}) > 1$. Keeping this in view, we first determine $\bar{v}(\mathbf{B}, \mathbf{L})$ for rational lines lying in the subspaces $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$.

When $\mathbf{L}$ is a rational line, we write $\mathbf{L} = \langle \mathbf{a} \rangle$, where $\mathbf{a}$ is a primitive point in $\mathbb{Z}^4$. Clearly, $\mathbf{L} \sim \mathbf{L}'$ iff $\mathbf{a} \sim \mathbf{a}'$, where $\mathbf{L}' = \langle \mathbf{a}' \rangle$ with $\mathbf{a}' \in \mathbb{Z}^4$ primitive. Since $\mathbf{L}$ does not lie in a co-ordinate hyperplane each co-ordinate of $\mathbf{a}$ is non-zero.

We shall prove :

Theorem 5 — If $L = \langle \mathbf{a} \rangle$, where $\mathbf{a} \in \mathbb{Z}^4$ is primitive and $\mathbf{a}$ lies in a subspace equivalent to S_1 then

$$\bar{v}^2(\mathbf{B}, \mathbf{L}) = \begin{cases} 1 + \dfrac{1}{|\mathbf{a}|^2} & \text{if } |\mathbf{a}|^2 \equiv 0 \pmod 4 \\[2ex] 1 + \dfrac{2}{|\mathbf{a}|^2}\left(1 + \dfrac{1}{|\mathbf{a}|^2}\right) & \text{if } |\mathbf{a}|^2 \equiv 2 \pmod 4. \end{cases}$$

Theorem 6 — If $\mathbf{L} = \langle \mathbf{a} \rangle$, where $\mathbf{a} \in \mathbb{Z}^4$ is primitive and $\mathbf{a}$ lies in a subspace equivalent to $\mathbf{S}_2$ then

$$\bar{v}^2(\mathbf{B}, \mathbf{L}) = \begin{cases} \dfrac{8}{9} + \dfrac{3}{|\mathbf{a}|^2} & \text{if } |\mathbf{a}|^2 \equiv 0 \pmod 3 \\[2ex] \dfrac{8}{9} + \dfrac{11}{9|\mathbf{a}|^2} + \dfrac{8}{9|\mathbf{a}|^4} & \text{if } |\mathbf{a}|^2 \equiv 1 \pmod 3. \end{cases}$$

Theorem 7 — If $\mathbf{L} = \langle \mathbf{a} \rangle$, where $\mathbf{a} \in \mathbb{Z}^4$ is primitive and $\mathbf{a}$ lies in a subspace equivalent to $\mathbf{S}_3$ then

$$\bar{v}^2(\mathbf{B}, \mathbf{L}) = \begin{cases} \dfrac{5}{6} + \dfrac{6}{|\mathbf{a}|^2} & \text{if } |\mathbf{a}|^2 \equiv 0 \pmod 6 \\[2ex] \dfrac{5}{6} + \dfrac{5}{3|\mathbf{a}|^2} + \dfrac{1}{2|\mathbf{a}|^4} & \text{if } |\mathbf{a}|^2 \equiv 1 \pmod 6 \\[2ex] \dfrac{5}{6} + \dfrac{3}{|\mathbf{a}|^2} + \dfrac{9}{2|\mathbf{a}|^4} & \text{if } |\mathbf{a}|^2 \equiv 3 \pmod 6 \\[2ex] \dfrac{5}{6} + \dfrac{10}{3|\mathbf{a}|^2} + \dfrac{16}{3|\mathbf{a}|^4} & \text{if } |\mathbf{a}|^2 \equiv 4 \pmod 6. \end{cases}$$

Theorem 8 — If $\mathbf{L}$ is a rational line in $\mathbb{R}^4$ which passes through the origin and does not lie in a co-ordinate hyperplane or in a subspace equivalent to $\mathbf{S}_1$ and is not equivalent to $\langle (1, 1, 1, 3) \rangle$, $\langle (1, 1, 1, 2) \rangle$, $\langle (1, 1, 2, 3) \rangle$ or $\langle (2, 2, 2, 3) \rangle$ then $\bar{v}(\mathbf{B}, \mathbf{L}) < 1$. The values of $\bar{v}^2(\mathbf{B}, \mathbf{L})$ for these four lines are $\dfrac{41}{36} > \dfrac{53}{49} > \dfrac{158}{150} > \dfrac{65}{63}$, respectively.

Theorem 9 — If $\mathbf{L}$ is a line in $\mathbb{R}^4$ which passes through the origin and does not lie in a co-ordinate hyperplane then $\bar{v}(\mathbf{B}, \mathbf{L}) \le \sqrt{5/4}$ and equality holds iff $\mathbf{L} \sim \langle (1, 1, 1, 1) \rangle$. Further, for $\mathbf{L} = \langle (1, 1, 1, 1) \rangle$, $v(\mathbf{B}, \mathbf{L} + \mathbf{p}) < \sqrt{5/4}$ except when $\mathbf{p} \in \mathbf{L} + \mathbf{q}_i$ or their translates through $\mathbb{Z}^4$, where points $\mathbf{q}_i$ are obtained from $\mathbf{q} = \left(0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}\right)$ by permuting the co-ordinates. In particular, $\bar{v}(\mathbf{B}, 1) = \sqrt{5/4}$ for $n = 4$.

This proves Schoenberg's conjecture for $n = 4$.

3. Irrational Subspaces — Proofs of Theorems 1 and 4

An irrational subspace $\mathbf{S}$ of dimension $n - 2$ is contained in a unique rational subspace $\mathbf{M}(\mathbf{S})$ and $\bar{v}(\mathbf{B}, \mathbf{S}) = \bar{v}(\mathbf{B}, \mathbf{M}(\mathbf{S}))$ (see Corollary 4 of Dumir *et al.*[5]). If dim $\mathbf{M}(\mathbf{S}) = n$ then $\bar{v}(\mathbf{B}, \mathbf{M}(\mathbf{S})) = 0$ and so $\bar{v}(\mathbf{B}, \mathbf{S}) = 0$. If dim $\mathbf{M}(\mathbf{S}) = n - 1$ and if $\mathbf{M}(\mathbf{S}) : \mathbf{c} \cdot \mathbf{x} = 0$, $\mathbf{c} \in \mathbb{Z}^n$, $\mathbf{c}$ primitive then $\bar{v}(\mathbf{B}, \mathbf{S}) = \bar{v}(\mathbf{B}, \mathbf{M}(\mathbf{S})) = \frac{1}{|\mathbf{c}|} \le \frac{1}{\sqrt{2}}$ and equality holds if and only if $|\mathbf{c}|^2 = 2$ (see Theorem 3(ii) of Dumir *et al.*[5]). This proves Theorem 1.

To prove Theorem 4 we observe that for an irrational line $\mathbf{L}$ in $\mathbb{R}^4$, $\mathbf{M}(\mathbf{L})$ is either $\mathbb{R}^4$ or a hyperplane or a two dimensional subspace. In the first case, $\bar{v}(\mathbf{B}, \mathbf{L}) = \bar{v}(\mathbf{B}, \mathbf{M}(\mathbf{L})) = 0$; in the second, $\bar{v}(\mathbf{B}, \mathbf{L}) \le \frac{1}{\sqrt{2}}$ as argued above; and in the third, we appeal to Theorem 3(a).

4. Rational Spaces : Reduction and Some Known Results

Here all spaces that we consider will be rational. The flats will also be rational in the sense that these will be translates of rational subspaces. If $\mathbf{U}$ is a rational subspace of dimension d in $\mathbb{R}^n$ then $\mathbf{U}^{\perp}$ is a rational subspace of dimension $n - d$. Let $\varphi_{\mathbf{U}}$ be the orthogonal projection of $\mathbb{R}^n$ onto $\mathbf{U}^{\perp}$. Then $\varphi_{\mathbf{U}}(\mathbf{B})$ is a ball of diameter 1 in the $(n - d)$-dimensional space $\mathbf{U}^{\perp}$. Also $\varphi_{\mathbf{U}}(\mathbb{Z}^n)$ is a lattice and $\varphi_{\mathbf{U}}(\Lambda) = \varphi_{\mathbf{U}}(\mathbb{Z}^n) + \varphi_{\mathbf{U}}(1/2)$. It is easy to see that $\bar{v}(\mathbf{B}, \mathbf{U})$ is the covering radius of the lattice $\varphi_{\mathbf{U}}(\mathbb{Z}^n)$ with respect to the ball with centre $\mathbf{o}$ and diameter 1. In particular, when $\mathbf{S}$ is a rational subspace of dimension $n - 2$ in $\mathbb{R}^n$, the determination of $\bar{v}(\mathbf{B}, \mathbf{S})$ is equivalent to the determination of the covering radius of a 2-dimensional lattice. The covering radius of such a lattice is easy to determine and we do this in Section 5. For rational lines $\mathbf{L}$ in $\mathbb{R}^4$, the determination of $\bar{v}(\mathbf{B}, \mathbf{L})$ is equivalent to the determination of the covering radius of a 3-dimensional lattice. Using a reduction of Voronoi, Barnes[2] obtained an expression for this which we now describe.

A positive definite quadratic form f is called reduced (in the sense of Voronoi) if it can be written as

$$f(x_1, x_2, x_3) = \rho_{01} x_1^2 + \rho_{02} x_2^2 + \rho_{03} x_3^2 + \rho_{12} (x_1 - x_2)^2 + \rho_{23} (x_2 - x_3)^2 + \rho_{31} (x_3 - x_1)^2,$$

where $\rho_{ij} \ge 0$ for $0 \le i, j \le 3$. From the results in Section 2 of Barnes[2] it follows that if f is a reduced form associated with the lattice Γ then the covering radius $r(\Gamma)$ of Γ with respect to $\mathbf{B}$ is given by

$$r^2(\Gamma) = \Sigma\rho_{ij} - \frac{\kappa + 4\lambda}{d(f)}, \qquad \dots (1)$$

where

$$d(f) = \det f = (\det(\Gamma))^2, \ \Sigma\rho_{ij} = \rho_{01} + \rho_{02} + \rho_{03} + \rho_{12} + \rho_{23} + \rho_{31},$$

$$\lambda = \min(\lambda_1 \lambda_2, \lambda_2 \lambda_3, \lambda_3 \lambda_1),$$

where $\lambda_1 = \rho_{01}\rho_{23}, \ \lambda_2 = \rho_{02}\rho_{31}, \ \lambda_3 = \rho_{03}\rho_{12}$

and $\kappa = \Sigma \rho_{01}\rho_{02}\rho_{03}(\rho_{12} + \rho_{23} + \rho_{31}),$

where the sum for κ contains four terms obtained by permuting 0, 1, 2, 3 cyclically and putting $\rho_{ij} = \rho_{ji}$.

Let f_0 be the form with all ρ_{ij} equal to 1 and $\mathbf{E}_0$, the ellipsoid $f_0(x) \le \frac{5}{4}$. Let $\mathbf{v}_i$, $1 \le i \le 6$, be the points obtained from $\left(\frac{3}{4}, \frac{2}{4}, \frac{1}{4}\right)$ by permuting the co-ordinates. It follows from Section 2 of Barnes[2] that the just covered points in the configuration $\mathbf{E}_0 + \mathbb{Z}^3$ are precisely the points $\mathbf{v}_i + \mathbb{Z}^3$, $1 \le i \le 6$.

If $\mathbf{S}$ is a rational $(n-2)$-dimensional subspace of $\mathbb{R}^n$ and $\mathbf{S}^\perp \cap \mathbb{Z}^n$ is a lattice of determinant $\Delta(\mathbf{S})$ (say) containing a primitive point $\mathbf{c}$ then it follows from Corollary 10(ii) of Dumir *et al.*[5] that

$$\bar{v}(\mathbf{B}, \mathbf{S}) \le \frac{1}{|\mathbf{c}|^2} + \frac{|\mathbf{c}|^2}{\Delta^2(\mathbf{S})}. \qquad \dots (2)$$

In the sequel $\mathbf{L}$ will stand for a rational line through $\mathbf{o}$ in $\mathbb{R}^4$, not lying in a co-ordinate hyperplane. We shall always write $\mathbf{L} = \langle \mathbf{a} \rangle$, $\mathbf{a}$ primitive in $\mathbb{Z}^4$. If $\mathbf{S}$ is a 2-dimensional subspace containing $\mathbf{L}$, then by Corollary 9(ii) and Remark 1 of Dumir *et al.*[5] we obtain

$$\bar{v}^2(\mathbf{B}, \mathbf{L}) \le \bar{v}^2(\mathbf{B}, \mathbf{S}) + \Delta^2(\mathbf{S})/|\mathbf{a}|^2. \qquad \dots (3)$$

For the 3-dimensional lattice $\Gamma\mathbf{a} = \mathbf{L}^\perp \cap \mathbb{Z}^n$ we choose a reduced basis $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ by Hermite's reduction process (see Section 10.3 of Gruber and Lekkerkerker[7]). These points are chosen successively to satisfy the following conditions :

(i) $|\mathbf{c}_1| = \min\{|\mathbf{p}| : \mathbf{p} \in \Gamma\mathbf{a}, \ \mathbf{p} \ne \mathbf{o}\}$.

(ii) $|\mathbf{c}_2| = \min\{|\mathbf{p}| : \mathbf{c}_1, \mathbf{p}$ can be extended to a basis of $\Gamma\mathbf{a}\}$.

(iii) $|\mathbf{c}_3| = \min\{|\mathbf{p}| : \mathbf{c}_1, \mathbf{c}_2, \mathbf{p}$ is a basis of $\Gamma\mathbf{a}\}$.

Then the following inequalities are satisfied :

$$|\mathbf{c}_1| \le |\mathbf{c}_2| \le |\mathbf{c}_3| \qquad \dots (4)$$

$$|\mathbf{c}_i \cdot \mathbf{c}_j| \leq \frac{1}{2}|\mathbf{c}_i|^2 \text{ for } i < j \qquad \ldots (5)$$

$$|\mathbf{c}_1||\mathbf{c}_2| \leq \sqrt{4/3}\,\Delta \qquad \ldots (6)$$

where Δ is the determinant of the lattice generated by $\mathbf{c}_1, \mathbf{c}_2$, and

$$|\mathbf{c}_1||\mathbf{c}_2||\mathbf{c}_3| \leq \sqrt{2}\,|\mathbf{a}|. \qquad \ldots (7)$$

Inequality (6) goes back to Lagrange and (7) to Gauss.

On replacing $\mathbf{c}_2$ by $-\mathbf{c}_2$ and $\mathbf{c}_3$ by $-\mathbf{c}_3$, if necessary, we can suppose that

$$\mathbf{c}_1 \cdot \mathbf{c}_2 \geq 0, \ \mathbf{c}_2 \cdot \mathbf{c}_3 \geq 0. \qquad \ldots (8)$$

The inequalities (6) and (7) together with Hadamard's inequality give

$$\sqrt{3/4}\,|\mathbf{c}_1||\mathbf{c}_2| \leq \Delta \leq |\mathbf{c}_1||\mathbf{c}_2| \leq \frac{\sqrt{2}\,|\mathbf{a}|}{|\mathbf{c}_3|}. \qquad \ldots (9)$$

Using (2) and (3) we get

$$\bar{\nu}^2(\mathbf{B}, \mathbf{L}) \leq \frac{1}{|\mathbf{c}_1|^2} + \frac{|\mathbf{c}_1|^2}{\Delta^2} + \frac{\Delta^2}{|\mathbf{a}|^2}. \qquad \ldots (10)$$

We notice that for $0 < \alpha \leq x \leq \beta$,

$$f(x) = \frac{|\mathbf{c}_1|^2}{x} + \frac{x}{|\mathbf{a}|^2} \leq \max(f(\alpha), f(\beta)).$$

Thus (10) together with (9) gives

$$\bar{\nu}^2(\mathbf{B}, \mathbf{L}) \leq \max\left(\frac{1}{|\mathbf{c}_1|^2} + \frac{4}{3|\mathbf{c}_2|^2} + \frac{3}{2|\mathbf{c}_3|^2}, \ \frac{1}{|\mathbf{c}_1|^2} + \frac{1}{|\mathbf{c}_2|^2} + \frac{2}{|\mathbf{c}_3|^2}\right). \qquad \ldots (11)$$

Using (4) we obtain

$$\bar{\nu}^2(\mathbf{B}, \mathbf{L}) \leq \frac{1}{|\mathbf{c}_1|^2} + \frac{3}{|\mathbf{c}_2|^2} \leq \frac{4}{|\mathbf{c}_1|^2}. \qquad \ldots (12)$$

For later use in section 6.2, we observe that if $\mathbf{p} = \alpha_1\mathbf{c}_1 + \alpha_2\mathbf{c}_2 + \alpha_3\mathbf{c}_3 \in \mathbf{Z}^4$, for some real $\alpha_1, \alpha_2, \alpha_3$ then $\mathbf{p} \in \Gamma\mathbf{a}$ and hence $\alpha_1, \alpha_2, \alpha_3$ are integers. In particular

$$\frac{1}{2}\mathbf{c}_i \pm \frac{1}{2}\mathbf{c}_j \notin \mathbf{Z}^4 \quad \text{for} \quad i \neq j. \qquad \ldots (13)$$

5. Rational Subspaces of Dimension $n-2$

When $\mathbf{S}$ is a rational subspace of dimension $n-2$, $\mathbf{S}^{\perp} \cap \mathbb{Z}^n$ is a 2-dimensional lattice. Let $\mathbf{c}_1, \mathbf{c}_2$ be a basis of $\mathbf{S}^{\perp} \cap \mathbb{Z}^n$ with $|c_1| = \min \{ |\mathbf{p}|; \mathbf{p} \in \mathbf{S}^{\perp} \cap \mathbb{Z}^n, \mathbf{p} \neq \mathbf{o}\}$ and $0 \le \mathbf{c}_1 \cdot \mathbf{c}_2 \le \frac{1}{2} |\mathbf{c}_1|^2$. We obtain an orthogonal basis $\mathbf{c}_1, \mathbf{c}_2'$ of $\mathbf{S}^{\perp}$ by defining $\mathbf{c}_2' = \mathbf{c}_2 - \frac{\mathbf{c}_1 \cdot \mathbf{c}_2}{|\mathbf{c}_1|^2} \mathbf{c}_1$. Then

$$\Delta = \det(\langle \mathbf{c}_1, \mathbf{c}_2 \rangle \cap \mathbb{Z}^n) = |\mathbf{c}_1| \, |\mathbf{c}_2'|.$$

Write $g = g.c.d.(\mathbf{c}_1 \cdot \mathbf{c}_2, |\mathbf{c}_1|^2)$, $h = \mathbf{c}_1 \cdot \mathbf{c}_2 / g$ and $k = |\mathbf{c}_1|^2/g$. Then $0 \le h \le k/2$ and $g.c.d.(h,k) = 1$. Let $\mathbf{d} = k\mathbf{c}_2 - k\mathbf{c}_1 = k\mathbf{c}_2'$. So $|\mathbf{d}| = k\,|\mathbf{c}_2'| = k\Delta/|\mathbf{c}_1|$.

The projection of $\mathbb{Z}^n$ on $\mathbf{S}^{\perp}$ is essentially the 2-dimensional lattice

$$\varphi_S(\mathbb{Z}^n) = \left\{ \left(\frac{\mathbf{c}_1 \cdot \mathbf{x}}{|\mathbf{c}_1|}, \frac{\mathbf{d} \cdot \mathbf{x}}{|\mathbf{d}|} \right) : \mathbf{x} \in \mathbb{Z}^n \right\}$$

$$= \left\{ \left(\frac{u}{|\mathbf{c}_1|}, \frac{v}{|\mathbf{d}|} \right) : u, v \in \mathbb{Z} \text{ and } uh + v \equiv 0 \pmod{k} \right\}.$$

Let $r = r(\varphi_S(\mathbb{Z}^n))$ be the covering radius of the lattice $\varphi_S(\mathbb{Z}^n)$ with respect to the ball $\mathbf{B}$.

Lemma 1 — $r^2 = |\mathbf{c}_1|^2 \left(\frac{1}{|\mathbf{c}_1|^2} + \frac{h^2}{|\mathbf{d}|^2} \right) \left(\frac{1}{|\mathbf{c}_1|^2} + \frac{(k-h)^2}{|\mathbf{d}|^2} \right)$

$$= \frac{1}{|\mathbf{c}_1|^2} \left(1 + \frac{(\mathbf{c}_1 \cdot \mathbf{c}_2)^2}{\Delta^2} \right) \left(1 + \frac{(|\mathbf{c}_1|^2 - \mathbf{c}_1 \cdot \mathbf{c}_2)^2}{\Delta^2} \right).$$

Proof : The lattice $\varphi_S(\mathbb{Z}^n)$ is generated by $\mathbf{p} = \left(0, \frac{k}{|\mathbf{d}|}\right)$ and $\mathbf{q} = \left(-\frac{1}{|\mathbf{c}_1|}, \frac{h}{|\mathbf{d}|} \right)$. The circumradius of the triangle $\mathbf{opq}$ is

$$\frac{\mathbf{pq}}{2 \sin \mathbf{poq}} = \frac{\mathbf{pq}}{2} |\mathbf{c}_1| \, |\mathbf{q}| = \frac{1}{2} |\mathbf{c}_1| \sqrt{\left(\frac{1}{|\mathbf{c}_1|^2} + \frac{h^2}{|\mathbf{d}|^2} \right) \left(\frac{1}{|\mathbf{c}_1|^2} + \frac{(k-h)^2}{|\mathbf{d}|^2} \right)}.$$

Since $0 \le \mathbf{c}_1 \cdot \mathbf{c}_2 < |\mathbf{c}_1|^2$, the angle $\mathbf{oqp}$ is acute and so the triangle $\mathbf{opq}$ is acute angled or right angled. It is well known that the covering radius of the lattice with respect to $\mathbf{B}$ is twice the circumradius of such a triangle and so the Lemma follows.

Remark 1 : For later use in Theorem 3(b), we notice that in the case when $|\mathbf{c}_1|^2 = |\mathbf{c}_2|^2 = 2$ and $\mathbf{c}_1 \cdot \mathbf{c}_2 = 1$, $\mathbf{r} = \left(-\frac{1}{3\sqrt{2}}, \frac{1}{\sqrt{6}} \right)$ is the circumcentre of the isosceles triangle $\mathbf{opq}$ and $r' = \left(\frac{1}{3\sqrt{2}}, \frac{1}{\sqrt{6}} \right)$ is the circumcentre of the triangle

$\mathbf{o}$, $\mathbf{p} - \mathbf{q}$, $\mathbf{p}$. The points of the plane which are furthest from $\varphi_S(\mathbf{Z}^n)$ are precisely the translates of $\mathbf{r}$ or $\mathbf{r}'$ through points of $\varphi_S(\mathbf{Z}^n)$.

PROOF OF THEOREM 2, COROLLARIES 1 AND 2 : Theorem 2 follows at once from Lemma 1 and the result stated in the beginning of section 4 which gives $\bar{v}(\mathbf{B}, \mathbf{S}) = r(\varphi_S(\mathbf{Z}^n))$.

Corollary 1 follows from Theorem 2 on substituting $\mathbf{c}_1 \cdot \mathbf{c}_2 = 0$ and $\Delta^2 = |\mathbf{c}_1|^2 |\mathbf{c}_2|^2$. For the proof of Corollary 2 we observe that since by hypothesis $\mathbf{S}$ does not lie in a co-ordinate hyperplane, any point $\mathbf{c} \in \mathbf{S}^\perp \cap \mathbf{Z}^n$ with $|\mathbf{c}|^2 = 2$ is a minimal point of this lattice. So we can take $\mathbf{c}_1 = \mathbf{c}$ in Theorem 2. Then $\mathbf{c}_1 \cdot \mathbf{c}_2 = 0$ or 1 and so $\Delta^2 = |\mathbf{c}_1|^2 |\mathbf{c}_2|^2 - (\mathbf{c}_1 \cdot \mathbf{c}_2)^2 = 2|\mathbf{c}_2|^2$ or $2|\mathbf{c}_2|^2 - 1$. Therefore $\Delta^2 \equiv 0$ or 1 (mod 2) according as $\mathbf{c}_1 \cdot \mathbf{c}_2 = 0$ or 1. Hence Corollary 2 follows.

PROOF OF COROLLARY 3 : In each case mentioned in part (a), $\mathbf{c}_1$ is a minimal point of $\mathbf{L}^\perp \cap \mathbf{Z}^n$. So the value of $\bar{v}^2(\mathbf{B}, \mathbf{S})$ is given by the formula in Theorem 2 and is seen to be at least $\frac{1}{2}$.

For part (b), we observe that $\Delta^2 = |\mathbf{c}_1|^2 |\mathbf{c}_2|^2 - (\mathbf{c}_1 \cdot \mathbf{c}_2)^2 \geq \frac{3}{4} |\mathbf{c}_1|^2 |\mathbf{c}_2|^2$. Therefore inequality (2) of Section 4 gives

$$\bar{v}^2(\mathbf{B}, \mathbf{S}) \leq \frac{1}{|\mathbf{c}_1|^2} + \frac{|\mathbf{c}_1|^2}{\Delta^2} \leq \frac{1}{|\mathbf{c}_1|^2} + \frac{4}{3|\mathbf{c}_2|^2} \leq \frac{7}{3|\mathbf{c}_1|^2} < \frac{1}{2}$$

if $|\mathbf{c}_1|^2 \geq 5$. So let us now consider $|\mathbf{c}_1|^2 = 3$ or 4.

If $\mathbf{c}_1 \cdot \mathbf{c}_2 = 0$ then by Corollary 1, $\bar{v}^2(\mathbf{B}, \mathbf{S}) = \frac{1}{|\mathbf{c}_1|^2} + \frac{1}{|\mathbf{c}_2|^2} < \frac{1}{2}$ except in the cases mentioned in part (a).

If $\mathbf{c}_1 \cdot \mathbf{c}_2 = 1$, then by Theorem 2

$$\bar{v}^2(\mathbf{B}, \mathbf{S}) = \frac{1}{|\mathbf{c}_1|^2} \left(1 + \frac{1}{\Delta^2} \right) \left(1 + \frac{(|\mathbf{c}_1|^2 - 1)^2}{\Delta^2} \right).$$

Also $\Delta^2 = |\mathbf{c}_1|^2 \, |\mathbf{c}_2|^2 - 1$. If $|\mathbf{c}_1|^2 = 4$, then $\Delta^2 \geq 15$ and so $\bar{v}^2(\mathbf{B}, \mathbf{S}) \leq \frac{1}{4} \cdot \frac{16}{15} \cdot \frac{8}{5} < \frac{1}{2}$. If $|\mathbf{c}_1|^2 = 3$, then in this case $|\mathbf{c}_2|^2 \geq 4$ and $\Delta^2 \geq 11$. Therefore $\bar{v}^2(\mathbf{B}, \mathbf{S}) \leq \frac{1}{3} \cdot \frac{12}{11} \cdot \frac{15}{11} < \frac{1}{2}$.

In case $\mathbf{c}_1 \cdot \mathbf{c}_2 = 2$, we have $|\mathbf{c}_1|^2 = 4$ because $|\mathbf{c}_1|^2 \geq 2\,\mathbf{c}_1 \cdot \mathbf{c}_2$. Then $\Delta^2 \geq 12$ and by Theorem 2 it follows that $\bar{v}^2(\mathbf{B}, \mathbf{L}) = \frac{1}{4}\left(1 + \frac{4}{\Delta^2} \right)^2 \leq \frac{4}{9} < \frac{1}{2}$.

PROOF OF THEOREM 3 : It follows from Corollary 3 that $\overline{\nu}^2(\mathbf{B}, \mathbf{S}) \le \frac{2}{3} < \frac{8}{9}$ if $|\mathbf{c}_1|^2 \ne 2$. When $|\mathbf{c}_1|^2 = 2$, $\mathbf{c}_1 \cdot \mathbf{c}_2 = 0$ or 1. If $\mathbf{c}_1 \cdot \mathbf{c}_2 = 1$, then $\Delta^2 \ge 3$ and $\overline{\nu}^2(\mathbf{B}, \mathbf{S}) = \frac{1}{2}\left(1 + \frac{1}{\Delta^2}\right)^2 \le \frac{8}{9}$. If $\mathbf{c}_1 \cdot \mathbf{c}_2 = 0$, then Δ^2 is even and at least 4.

Hence $\overline{\nu}^2(\mathbf{B}, \mathbf{S}) = \frac{1}{2} + \frac{2}{\Delta^2} < \frac{8}{9}$ unless $\Delta^2 = 4$. If $\Delta^2 = 4$ then $\overline{\nu}(\mathbf{B}, \mathbf{S}) = 1$ and $|\mathbf{c}_1|^2 = |\mathbf{c}_2|^2 = 2$. This case does not arise when $n = 3$, but for $n \ge 4$ it gives $\mathbf{S} \sim \mathbf{S}_1$.

For $n \ge 4$ we determine the translates $\mathbf{S}_1 + \mathbf{p}$ of $\mathbf{S}_1$ for which $\nu(\mathbf{B}, \mathbf{S}_1 + \mathbf{p}) = 1$. Let $\mathbf{S}_1 + \mathbf{p}$ be defined by the equations $x_1 - x_2 - \alpha = 0$, $x_3 - x_4 - \beta = 0$. For any $\mathbf{x} \in \Lambda$ the Euclidean distance of $\mathbf{x}$ from $\mathbf{S}_1 + \mathbf{p}$ is given by

$$d^2(\mathbf{x}, \mathbf{S}_1 + \mathbf{p}) = \frac{(x_1 - x_2 - \alpha)^2}{2} + \frac{(x_3 - x_4 - \beta)^2}{2}.$$

It is clear that we can choose $x \in \Lambda$ such that $|x_1 - x_2 - \alpha| \le \frac{1}{2}$, $|x_3 - x_4 - \beta| \le \frac{1}{2}$ with strict inequality at one place except when 2α and 2β are both odd integers. So $\nu(\mathbf{B}, \mathbf{S}_1 + \mathbf{p}) < 1$ except in the case stated in part (a) of the Theorem.

For $n = 3$, the analysis above shows that $\overline{\nu}^2(\mathbf{B}, \mathbf{S}) \le 8/9$ and equality holds if and only if $\mathbf{c}_1 \cdot \mathbf{c}_2 = 1$ and $\Delta^2 = 3$ i.e. $\mathbf{S} \sim \mathbf{S}_2$.

To determine the points $\mathbf{p}$ such that $\nu^2(\mathbf{B}, \mathbf{S}_2 + \mathbf{p}) = 8/9$ we appeal to Remark 1. When we choose $\mathbf{c}_1 = (1, -1, 0)$ and $\mathbf{c}_2 = (0, -1, 1)$ then $\varphi_{\mathbf{S}_2}\left(0, \frac{1}{3}, \frac{2}{3}\right) = \left(\frac{-1}{3\sqrt{2}}, \frac{1}{\sqrt{6}}\right) = \mathbf{r}$ and $\varphi_{\mathbf{S}_2}\left(\frac{1}{3}, 0, \frac{2}{3}\right) = \left(\frac{1}{3\sqrt{2}}, \frac{1}{\sqrt{6}}\right) = \mathbf{r}'$. It is clear that $\nu(\mathbf{B}, \mathbf{S}_2 + \mathbf{p}) = \sqrt{8/9}$ if and only if $\varphi_{\mathbf{S}_2}(\mathbf{S}_2 + \mathbf{p}) = \varphi_{\mathbf{S}_2}(\mathbf{p})$ is a just covered point and so the equality cases are as stated in the Theorem.

This completes the proof of Theorem 3. Corollary 4 is an immediate consequence of Theorems 1 and 3.

6. RATIONAL LINES IN $\mathbb{R}^4$

6.1. Proofs of Theorems 5, 6, and 7

We consider rational lines lying in special subspaces and prove Theorems 5, 6 and 7 making use of the results on covering radius stated in section 4. In particular, we use expression (1) repeatedly. Here $\mathbf{L} = \langle \mathbf{a} \rangle$, where $\mathbf{a} \in \mathbb{Z}^4$ is primitive and $\mathbf{L}^\perp \cap \mathbb{Z}^4$ is a 3-dimensional lattice of determinant $|\mathbf{a}|$. For each theorem we choose a suitable orthogonal basis of the subspace $\mathbf{L}^\perp$ and write $\varphi_{\mathbf{L}}(\mathbb{Z}^4)$ explicitly.

Then we choose a suitable basis of $\varphi_L(\mathbb{Z}^4)$ so that the associated quadratic form f is reduced in the sense of Voronoi and use (1) to determine $\bar{\nu}(\mathbf{B}, \mathbf{L}) = r(\varphi_L(\mathbb{Z}^4))$.

PROOF OF THEOREM 5 : Without loss of generality we can suppose that $\mathbf{L}$ lies in $\mathbf{S}_1$. We can suppose that $\mathbf{a} = (a, a, b, b)$, where a, b are relatively prime integers and $a \neq 0$, $b \neq 0$. Let $\mathbf{d}_1 = (1, -1, 0, 0)$, $\mathbf{d}_2 = (0, 0, 1, -1)$, and $\mathbf{d}_3 = (b, b, -a, -a)$. Then $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$ is an orthogonal basis of the subspace $\mathbf{L}^{\perp}$ (though not of the lattice $\mathbb{Z}^4 \cap \mathbf{L}^{\perp}$).

The projection of $\mathbb{Z}^4$ on $\mathbf{L}$ can be described as

$$\varphi_L(\mathbb{Z}^4) = \left\{ \left(\frac{x_1 - x_2}{\sqrt{2}}, \frac{x_3 - x_4}{\sqrt{2}}, \frac{b(x_1 + x_2) - a(x_3 + x_4)}{\sqrt{2(a^2 + b^2)}} \right) : x_i \in \mathbb{Z}, 1 \le i \le 4 \right\}$$

$$= \left\{ \left(\frac{u}{\sqrt{2}}, \frac{v}{\sqrt{2}}, \frac{bu - av + 2(bx_2 - ax_4)}{|\mathbf{a}|} \right) : u, v, x_2, x_4 \in \mathbb{Z} \right\}$$

$$= \left\{ \left(\frac{u}{\sqrt{2}}, \frac{v}{\sqrt{2}}, \frac{bu - av + 2w}{|\mathbf{a}|} \right) : u, v, w \in \mathbb{Z} \right\}.$$

Case (i) : $|\mathbf{a}|^2 \equiv 0 \pmod 4$

Here a, b are both odd. The lattice $\varphi_L(\mathbb{Z}^4)$ is generated by $\left(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{|\mathbf{a}|} \right)$, $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{|\mathbf{a}|} \right)$ and $\left(0, \frac{-1}{\sqrt{2}}, \frac{1}{|\mathbf{a}|} \right)$. The associated quadratic form is

$$f(y_1, y_2, y_3) = \frac{1}{2} y_1^2 + \frac{1}{2} (y_2 - y_3)^2 + \frac{1}{|\mathbf{a}|^2} (-y_1 + y_2 + y_3)^2$$

$$= \left(\frac{1}{2} - \frac{1}{|\mathbf{a}|^2} \right) y_1^2 + \frac{1}{|\mathbf{a}|^2} y_2^2 + \frac{1}{|\mathbf{a}|^2} y_3^2 + \frac{1}{|\mathbf{a}|^2} (y_1 - y_2)^2$$

$$+ \left(\frac{1}{2} - \frac{1}{|\mathbf{a}|^2} \right) (y_2 - y_3)^2 + \frac{1}{|\mathbf{a}|^2} (y_3 - y_1)^2.$$

This is reduced in the sense of Voronoi since $|\mathbf{a}|^2 \ge 4$. The related parameters are

$$d(f) = (\det \varphi_L(\mathbb{Z}^4))^2 = |\mathbf{a}|^{-2}, \quad \Sigma \rho_{ij} = 1 + 2|\mathbf{a}|^{-2},$$

$$\lambda_2 \lambda_3 = \frac{1}{|\mathbf{a}|^8}, \quad \lambda_1 \lambda_3 = \lambda_1 \lambda_2 = \left(\frac{1}{2} - \frac{1}{|\mathbf{a}|^2} \right)^2 \frac{1}{|\mathbf{a}|^4}.$$

Since $|\mathbf{a}|^2 \ge 4$, it follows that $\lambda_2 \lambda_3 \le \lambda_1 \lambda_3 = \lambda_1 \lambda_2$ and so $\lambda = |\mathbf{a}|^{-8}$ and

$$\kappa = |\mathbf{a}|^{-4} (1 - 4|\mathbf{a}|^{-4}).$$

Thus $$\bar{\nu}^2(\mathbf{B}, \mathbf{L}) = \Sigma \rho_{ij} - |\mathbf{a}|^2 (\kappa + 4\lambda) = 1 + |\mathbf{a}|^{-2}.$$

Case (ii) : $|\mathbf{a}|^2 \equiv 2 \pmod 4$

Here a and b are of opposite parity. Without loss of generality we can suppose that a is odd and b is even. Then $\varphi_L(\mathbf{Z}^4)$ is generated by $\left(\frac{1}{\sqrt{2}}, 0, 0\right)$, $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{|\mathbf{a}|}\right)$, $\left(0, \frac{-1}{\sqrt{2}}, \frac{1}{|\mathbf{a}|}\right)$. The associated quadratic form can be written as

$$f(y_1, y_2, y_3) = \frac{1}{2} y_1^2 + \frac{2}{|\mathbf{a}|^2} y_2^2 + \frac{2}{|\mathbf{a}|^2} y_3^2 + \left(\frac{1}{2} - \frac{1}{|\mathbf{a}|^2}\right)(y_2 - y_3)^2.$$

The related parameters are

$$d(f) = \frac{1}{|\mathbf{a}|^2},\ \Sigma \rho_{ij} = 1 + \frac{3}{|\mathbf{a}|^2},\ \kappa = \frac{1}{|\mathbf{a}|^4} - \frac{2}{|\mathbf{a}|^6},\ \text{and } \lambda = 0.$$

Therefore $\quad \bar{\nu}^2(\mathbf{B}, \mathbf{L}) = \Sigma\rho_{ij} - |\mathbf{a}|^2(\kappa + 4\lambda) = 1 + \frac{2}{|\mathbf{a}|^2} + \frac{2}{|\mathbf{a}|^4}.$

PROOF OF THEOREM 6 : Here we can suppose that $\mathbf{L}$ lies in S_2. We can write $a = (a, a, a, b)$, where a and b are non-zero coprime integers. We take $\mathbf{d}_1 = (1, -1, 0, 0)$, $\mathbf{d}_2 = (1, 1, -2, 0)$ and $\mathbf{d}_3 = (b, b, b, -3a)$ as an orthogonal basis of $\mathbf{L}^\perp$. Therefore

$$\varphi_L(\mathbf{Z}^4)$$

$$= \left\{\left(\frac{x_1 - x_2}{\sqrt{2}}, \frac{x_1 + x_2 - 2x_3}{\sqrt{6}}, \frac{b(x_1 + x_2 + x_3) - 3ax_4}{\sqrt{3b^2 + 9a^2}}\right) : x_i \in \mathbf{Z}, 1 \le i \le 4\right\}$$

$$= \left\{\left(\frac{u}{\sqrt{2}}, \frac{v}{\sqrt{6}}, \frac{vb + 3w}{\sqrt{3}|\mathbf{a}|}\right) : u, v, w \in \mathbf{Z}, u \equiv v \pmod 2\right\}.$$

Case (i) : $|\mathbf{a}|^2 \equiv 0 \pmod 3$ i.e. $b \equiv 0 \pmod 3$

Here $\varphi_L(\mathbf{Z}^4)$ is generated by $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, 0\right)$, $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, 0\right)$, and $\left(0, 0, \frac{\sqrt{3}}{|\mathbf{a}|}\right)$. The associated quadratic form can be written as

$$f(y_1, y_2, y_3) = \frac{1}{3} y_1^2 + \frac{1}{3} y_2^2 + \frac{3}{|\mathbf{a}|^2} y_3^2 + \frac{1}{3}(y_1 - y_2)^2.$$

This is reduced in the sense of Voronoi.

Here

$$d(f) = \frac{1}{|\mathbf{a}|^2},\ \Sigma\rho_{ij} = 1 + \frac{3}{|\mathbf{a}|^2},\ \kappa = \frac{1}{9|\mathbf{a}|^2} \text{ and } \lambda = 0.$$

Therefore $\quad \bar{\nu}^2(\mathbf{B}, \mathbf{L}) = \Sigma\rho_{ij} - |\mathbf{a}|^2(\kappa + 4\lambda) = \frac{8}{9} + \frac{3}{|\mathbf{a}|^2}.$

Case (ii) : $|\mathbf{a}|^2 \equiv 1 \pmod 3$ i.e. $b \equiv \pm 1 \pmod 3$

On replacing a by $-a$, if necessary, we can suppose that $b \equiv 1 \pmod 3$. Then $\varphi_L(\mathbf{Z}^4)$ is generated by $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}\,|\mathbf{a}|}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}\,|\mathbf{a}|}\right)$ and $\left(0, 0, \frac{-\sqrt{3}}{|\mathbf{a}|}\right)$. The associated quadratic form can be written as

$$f(y_1, y_2, y_3) = \frac{1}{3}\left(1 - \frac{1}{|\mathbf{a}|^2}\right)(y_1^2 + y_2^2) + \frac{1}{|\mathbf{a}|^2} y_3^2 + \frac{1}{3}\left(1 - \frac{1}{|\mathbf{a}|^2}\right)(y_1 - y_2)^2$$
$$+ \frac{1}{|\mathbf{a}|^2}(y_2 - y_3)^2 + \frac{1}{|\mathbf{a}|^2}(y_3 - y_1)^2.$$

Here

$$d(f) = \frac{1}{|\mathbf{a}|^2}, \ \Sigma\rho_{ij} = 1 + \frac{2}{|\mathbf{a}|^2}, \ \lambda = \frac{1}{9|\mathbf{a}|^4}\left(1 - \frac{1}{|\mathbf{a}|^2}\right)^2,$$

and $$\kappa = \frac{1}{3|\mathbf{a}|^2}\left(1 - \frac{1}{|\mathbf{a}|^2}\right)^2\left(\frac{1}{3} + \frac{5}{3|\mathbf{a}|^2}\right) + \frac{1}{|\mathbf{a}|^6}\left(1 - \frac{1}{|\mathbf{a}|^2}\right).$$

Then $$\bar{\nu}^2(\mathbf{B}, \mathbf{L}) = \Sigma\rho_{ij} - |\mathbf{a}|^2(\kappa + 4\lambda) = \frac{8}{9} + \frac{11}{9|\mathbf{a}|^2} + \frac{8}{9|\mathbf{a}|^4}.$$

PROOF OF THEOREM 7 : Here we suppose that $\mathbf{L}$ lies in $\mathbf{S}_3$. We can write $\mathbf{a} = (a, a, -2a, b)$, where a and b are non-zero coprime integers. Then the vectors $\mathbf{d}_1 = (1, -1, 0, 0)$, $\mathbf{d}_2 = (1, 1, 1, 0)$, and $\mathbf{d}_3 = (b, b, -2b, -6a)$ give an orthogonal basis of $\mathbf{L}$. The projected lattice is

$$\varphi_L(\mathbf{Z}^4) = \left\{\left(\frac{x_1 - x_2}{\sqrt{2}}, \frac{x_1 + x_2 + x_3}{\sqrt{3}}, \frac{b(x_1 + x_2 - 2x_3) - 6ax_4}{\sqrt{6(b^2 + 6a^2)}}\right) : x_i \in \mathbf{Z}^4, \ 1 \le i \le 4\right\}$$
$$= \left\{\left(\frac{u}{\sqrt{2}}, \frac{v}{\sqrt{3}}, \frac{b(3u - 2v) + 6w}{\sqrt{6}\,|\mathbf{a}|}\right) : u, v, w \in \mathbf{Z}\right\}.$$

On replacing $\mathbf{a}$ by $-\mathbf{a}$, if necessary, we can suppose that $b \equiv t \pmod 6$ with $0 \le t \le 3$.

Case (i) : $|\mathbf{a}|^2 \equiv 0 \pmod 6$ i.e. $b \equiv 0 \pmod 6$

Here $\varphi_L(\mathbf{Z}^4)$ is a rectangular lattice generated by $\left(\frac{1}{\sqrt{2}}, 0, 0\right), \left(0, \frac{1}{\sqrt{3}}, 0\right)$ and $\left(0, 0, \frac{\sqrt{6}}{|\mathbf{a}|}\right)$. Therefore

$$\bar{\nu}^2(\mathbf{B}, \mathbf{L}) = \frac{1}{2} + \frac{1}{3} + \frac{6}{|\mathbf{a}|^2} = \frac{5}{6} + \frac{6}{|\mathbf{a}|^2}.$$

Case (ii) : $|\mathbf{a}|^2 \equiv 1 \pmod 6$ i.e. $b \equiv 1 \pmod 6$

A basis of $\varphi_L(\mathbb{Z}^4)$ is given by $\left(\frac{1}{\sqrt{2}}, 0, \frac{3}{\sqrt{6}\,|\mathbf{a}|}\right), \left(0, \frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{6}\,|\mathbf{a}|}\right)$ and $\left(0, \frac{-1}{\sqrt{3}}, \frac{-4}{\sqrt{6}\,|\mathbf{a}|}\right)$.

The associated quadratic form can be written as

$$f(y_1, y_2, y_3) = \left(\frac{1}{2} - \frac{3}{2|\mathbf{a}|^2}\right) y_1^2 + \frac{1}{|\mathbf{a}|^2} y_2^2 + \frac{2}{|\mathbf{a}|^2} y_3^2 + \frac{1}{|\mathbf{a}|^2}(y_1 - y_2)^2$$
$$+ \left(\frac{1}{3} - \frac{4}{3|\mathbf{a}|^2}\right)(y_2 - y_3)^2 + \frac{2}{|\mathbf{a}|^2}(y_3 - y_1)^2.$$

It is reduced in the sense of Voronoi. Here

$$d(f) = \frac{1}{|\mathbf{a}|^2}, \quad \Sigma \rho_{ij} = \frac{5}{6} + \frac{19}{6|\mathbf{a}|^2},$$

$$\lambda_1 = \left(\frac{1}{2} - \frac{3}{2|\mathbf{a}|^2}\right)\left(\frac{1}{3} - \frac{4}{3|\mathbf{a}|^2}\right), \quad \lambda_2 = \lambda_3 = \frac{2}{|\mathbf{a}|^4}.$$

Since $|\mathbf{a}|^2 = 6a^2 + b^2 \geq 7$; $\frac{1}{2} - \frac{3}{2|\mathbf{a}|^2} \geq \frac{2}{|\mathbf{a}|^2}$ and $\frac{1}{3} - \frac{4}{3|\mathbf{a}|^2} \geq \frac{1}{|\mathbf{a}|^2}$ and so $\lambda_1 \geq \lambda_2 = \lambda_3$. Therefore $\lambda = \lambda_2 \lambda_3 = \frac{4}{|\mathbf{a}|^8}$. Also

$$\kappa = \frac{2}{|\mathbf{a}|^4}\left(1 - \frac{3}{|\mathbf{a}|^2}\right)\left(\frac{1}{3} + \frac{5}{3|\mathbf{a}|^2}\right) + \frac{1}{|\mathbf{a}|^4}\left(\frac{1}{3} - \frac{4}{3|\mathbf{a}|^2}\right)\left(\frac{1}{2} + \frac{5}{2|\mathbf{a}|^2}\right)$$
$$+ \frac{2}{|\mathbf{a}|^4}\left(\frac{1}{3} - \frac{4}{3|\mathbf{a}|^2}\right)\left(1 + \frac{1}{|\mathbf{a}|^2}\right).$$

Then

$$\bar{v}^2(\mathbf{B}, \mathbf{L}) = \Sigma \rho_{ij} - |\mathbf{a}|^2(\kappa + 4\lambda) = \frac{5}{6} + \frac{5}{3|\mathbf{a}|^2} + \frac{1}{2|\mathbf{a}|^4}.$$

Case (iii) : $|\mathbf{a}|^2 \equiv 3 \pmod 6$ i.e. $b \equiv 3 \pmod 6$

Here $\varphi_L(\mathbb{Z}^4)$ is generated by $\left(\frac{1}{\sqrt{2}}, 0, \frac{3}{\sqrt{6}\,|\mathbf{a}|}\right), \left(0, \frac{1}{\sqrt{3}}, 0\right)$ and $\left(0, 0, \frac{-6}{\sqrt{6}\,|\mathbf{a}|}\right)$. The associated quadratic form can be written as

$$f(y_1, y_2, y_3) = \left(\frac{1}{2} - \frac{3}{2|\mathbf{a}|^2}\right) y_1^2 + \frac{1}{3} y_2^2 + \frac{3}{|\mathbf{a}|^2} y_3^2 + \frac{3}{|\mathbf{a}|^2}(y_3 - y_1)^2.$$

This is reduced in the sense of Voronoi, since each coefficient is non-negative. Here

$$d(f) = \frac{1}{|\mathbf{a}|^2}, \quad \Sigma \rho_{ij} = \frac{5}{6} + \frac{9}{2|\mathbf{a}|^2}, \quad \lambda = 0, \text{ and } \kappa = \frac{3}{|\mathbf{a}|^4}\left(\frac{1}{2} - \frac{3}{2|\mathbf{a}|^2}\right).$$

Thus

$$\bar{v}^2(\mathbf{B},\ \mathbf{L}) = \Sigma\, \rho_{ij} - |\mathbf{a}|^2(\kappa + 4\lambda) = \frac{5}{6} + \frac{3}{|\mathbf{a}|^2} + \frac{9}{2|\mathbf{a}|^4}.$$

Case (iv) : $|\mathbf{a}|^2 \equiv 4 \pmod 6$ i.e. $b \equiv 2 \pmod 6$

Here $\varphi_{\mathbf{L}}(\mathbf{Z}^4)$ has $\left(\frac{1}{\sqrt{2}}, 0, 0\right), \left(0, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{6}\,|\mathbf{a}|}\right)$ and $\left(0, 0, \frac{-6}{\sqrt{6}\,|\mathbf{a}|}\right)$ as a basis. The associated quadratic form can be written as

$$f(y_1, y_2, y_3) = \frac{1}{2} y_1^2 + \left(\frac{1}{3} - \frac{4}{3|\mathbf{a}|^2}\right) y_2^2 + \frac{4}{|\mathbf{a}|^2} y_3^2 + \frac{2}{|\mathbf{a}|^2}(y_2 - y_3)^2$$

and since $|\mathbf{a}|^2 > 4$, all coefficients are non-negative. The related parameters are

$$d(f) = \frac{1}{|\mathbf{a}|^2},\quad \Sigma\, \rho_{ij} = \frac{5}{6} + \frac{14}{3|\mathbf{a}|^2},\quad \lambda = 0, \text{ and } \kappa = \frac{4}{|\mathbf{a}|^4}\left(\frac{1}{3} - \frac{4}{3|\mathbf{a}|^2}\right).$$

Therefore

$$\bar{v}^2(\mathbf{B},\ \mathbf{L}) = \Sigma\, \rho_{ij} - |\mathbf{a}|^2(\kappa + 4\lambda) = \frac{5}{6} + \frac{10}{3|\mathbf{a}|^2} + \frac{16}{3|\mathbf{a}|^4}.$$

6.2. *Proof of Theorem* 8

Here $\mathbf{L} = \langle \mathbf{a} \rangle$ does not lie in a subspace equivalent to $\mathbf{S}_1$.

First let us suppose that $\mathbf{L}$ lies on a subspace equivalent to $\mathbf{S}_2$. Then by Theorem 6

$$\bar{v}^2(\mathbf{B},\ \mathbf{L}) \le \frac{8}{9} + \frac{3}{|\mathbf{a}|^2} < 1 \text{ if } |\mathbf{a}|^2 > 27.$$

We can suppose that $\mathbf{a}$ lies on $\mathbf{S}_2$ and $\mathbf{a} = (a, a, a, b)$, where a, b are positive co-prime integers. If $|\mathbf{a}|^2 \le 27$ then $\mathbf{a} = (1, 1, 1, b)$, $1 \le b \le 4$, or (2, 2, 2, 1) or (2, 2, 2, 3). Using Theorem 6, we can check that if $\mathbf{a} = (2, 2, 2, 1)$ or (1, 1, 1, 4) then $\bar{v}^2(\mathbf{B}, \mathbf{L}) < 1$. In all other cases $\bar{v}^2(\mathbf{B}, \mathbf{L}) > 1$. We notice that (1, 1, 1, 1) lies on $\mathbf{S}_1$; the values of all other $\bar{v}(\mathbf{B}, \mathbf{L})$ are as listed.

Now suppose that $\mathbf{L}$ lies on $\mathbf{S}_3$. Then by Theorem 7

$$\bar{v}^2(\mathbf{B},\ \mathbf{L}) \le \frac{5}{6} + \frac{6}{|\mathbf{a}|^2} < 1 \text{ if } |\mathbf{a}|^2 > 36.$$

Let $|\mathbf{a}|^2 \le 36$ and suppose $\mathbf{a} = (a, a, -2a, b)$, where a, b are positive co-prime integers. The only possibilities are (2, 2, − 4, 1), (2, 2, − 4, 3) and (1, 1, − 2, b) with $1 \le b \le 5$. $\mathbf{a} = (1, 1, -2, 2)$ lies on a subspace equivalent to $\mathbf{S}_1$. Using Theorem 7 it is easily checked that except for $\mathbf{a} = (1, 1, -2, 1)$ and (1, 1, − 2, 3), $\bar{v}(\mathbf{B}, \mathbf{L}) < 1$.

Now we can suppose that $\mathbf{L}$ does not lie in a subspace equivalent to $\mathbf{S}_1$, $\mathbf{S}_2$ or $\mathbf{S}_3$. Let $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ be a reduced basis of $\mathbf{L}^\perp \cap \mathbf{Z}^4$ as described in section 4. We break up the rest of the proof into a sequence of Lemmas.

Lemma 2 — If $|\mathbf{c}_1|^2 \geq 4$ then $\bar{v}(\mathbf{B}, \mathbf{L}) < 1$.

PROOF : By inequality (12), $\bar{v}(\mathbf{B}, \mathbf{L}) < 1$ if $|\mathbf{c}_1|^2 \geq 5$.

When $|\mathbf{c}_1|^2 = 4$, the same inequality gives $\bar{v}(\mathbf{B}, \mathbf{L}) < 1$ if $|\mathbf{c}_2|^2 \geq 5$. Now we notice that it is not possible to have $|\mathbf{c}_2|^2 = |\mathbf{c}_1|^2 = 4$ because then both $\mathbf{c}_1, \mathbf{c}_2$ are equivalent to (1, 1, 1, 1) and so $\frac{1}{2}\mathbf{c}_1 + \frac{1}{2}\mathbf{c}_2 \in \mathbf{Z}^4$, which is a contradiction (see (13)).

Lemma 3 — If $|\mathbf{c}_1|^2 = 3$ then $\bar{v}(\mathbf{B}, \mathbf{L}) < 1$.

PROOF : Since $|\mathbf{c}_1|^2 > 2$, all co-ordinates of $\mathbf{a}$ are distinct and therefore $|\mathbf{a}|^2 \geq 30$. By (5) and (8) $\mathbf{c}_1 \cdot \mathbf{c}_2 = 0$ or 1. By inequality (12), $\bar{v}(\mathbf{B}, \mathbf{L}) < 1$ if $|\mathbf{c}_2|^2 \geq 5$ and so it remains to consider $|\mathbf{c}_2|^2 = 3$ and 4.

Let $|\mathbf{c}_2|^2 = 4$. Here $\mathbf{c}_1 \sim (1, 1, 1, 0)$, $\mathbf{c}_2 \sim (1, 1, 1, 1)$ and so $\mathbf{c}_1 \cdot \mathbf{c}_2$ is not 0. It follows that $\Delta^2 = |\mathbf{c}_1|^2 |\mathbf{c}_2|^2 - (\mathbf{c}_1 \cdot \mathbf{c}_2)^2 = 11$ and (10) then gives $\bar{v}(\mathbf{B}, \mathbf{L}) < 1$.

When $|\mathbf{c}_2|^2 = 3$, $\mathbf{c}_1$ and $\mathbf{c}_2$ are both equivalent to (1, 1, 1, 0). We notice that $\mathbf{c}_1 \cdot \mathbf{c}_2 = 1$ would imply $\frac{1}{2}\mathbf{c}_1 + \frac{1}{2}\mathbf{c}_2 \in \mathbf{Z}^4$, which is a contradiction. Therefore $\mathbf{c}_1 \cdot \mathbf{c}_2 = 0$ and $\Delta^2 = 9$. Then (10) given $\bar{v}(\mathbf{B}, \mathbf{L}) < 1$.

Lemma 4 — If $|\mathbf{c}_1|^2 = 2$ and L does not lie in a subspace equivalent to $\mathbf{S}_1, \mathbf{S}_2$ or $\mathbf{S}_3$, then $\bar{v}(\mathbf{B}, \mathbf{L}) < 1$.

PROOF : Here $\mathbf{c}_1 \cdot \mathbf{c}_2 = 0$ or 1 follows from (5) and (8). Inequality (12) gives $\bar{v}(\mathbf{B}, \mathbf{L}) < 1$ when $|\mathbf{c}_2|^2 \geq 7$. We have to discuss the cases $2 \leq |\mathbf{c}_2|^2 \leq 6$ individually.

The case $|\mathbf{c}_2|^2 = 2$ does not arise because then the subspace generated by $\mathbf{c}_1, \mathbf{c}_2$ is equivalent to $\mathbf{S}_1$ or $\mathbf{S}_2$. When $|\mathbf{c}_2|^2 = 3$ and $\mathbf{c}_1 \cdot \mathbf{c}_2 = 0$, the space generated by $\mathbf{c}_1, \mathbf{c}_2$ is equivalent to $\mathbf{S}_3$ and so is not to be considered here. So let $|\mathbf{c}_2|^2 = 3$ and $\mathbf{c}_1 \cdot \mathbf{c}_2 = 1$. Then $\Delta^2 = 5$. Then Corollary 2 and inequality (3) give

$$\bar{v}^2(\mathbf{B}, \mathbf{L}) \leq \frac{18}{25} + \frac{5}{|\mathbf{a}|^2} < 1 \quad \text{if} \quad |\mathbf{a}|^2 \geq 18.$$

It can be easily seen that $|\mathbf{a}|^2 \geq 18$, since $\mathbf{a}$ does not lie in a subspace equivalent to $\mathbf{S}_1, \mathbf{S}_2$ or $\mathbf{S}_3$.

When $|\mathbf{c}_2|^2 = 4$, $\mathbf{c}_1 \cdot \mathbf{c}_2$ cannot be 1 and so $\mathbf{c}_1 \cdot \mathbf{c}_2 = 0$ and $\Delta^2 = 8$. We observe that $|\mathbf{c}_3|^2 = 4$ would give $\frac{1}{2}\mathbf{c}_2 + \frac{1}{2}\mathbf{c}_3 \in \mathbf{Z}^4$, which is not possible by (13). Thus $|\mathbf{c}_3|^2 \geq 5$. When $|\mathbf{c}_3|^2 = 5$ we cannot have $\mathbf{c}_2 \cdot \mathbf{c}_3$ equal to 0 or 2. Inequalities (5) and (8) then give $\mathbf{c}_2 \cdot \mathbf{c}_3 = 1$ and also $|\mathbf{c}_1 \cdot \mathbf{c}_3| \leq 1$. Since $|\mathbf{a}|$ equals the determinant of the lattice $\mathbf{L}^\perp \cap \mathbf{Z}^4$, we get $|\mathbf{a}|^2 = 8|\mathbf{c}_3|^2 - 4(\mathbf{c}_1 \cdot \mathbf{c}_3)^2 - 2(\mathbf{c}_2 \cdot \mathbf{c}_3)^2 \geq 34$. Therefore (10) gives $\bar{v}(\mathbf{B}, \mathbf{L}) < 1$.

Now let $|\mathbf{c}_2|^2 = 5$. If $\mathbf{c}_1 \cdot \mathbf{c}_2 = 0$ then $\Delta^2 = 10$ and $|\mathbf{a}|^2 = 10|\mathbf{c}_3|^2 - 5(\mathbf{c}_1 \cdot \mathbf{c}_3)^2 - 2(\mathbf{c}_2 \cdot \mathbf{c}_3)^2 \geq 37$, because $|\mathbf{c}_1 \cdot \mathbf{c}_3| \leq 1$ and $|\mathbf{c}_2 \cdot \mathbf{c}_3| \leq 2$. Then (10) gives $\bar{v}(\mathbf{B}, \mathbf{L}) < 1$. If $\mathbf{c}_1 \cdot \mathbf{c}_2 = 1$, then $\Delta^2 = 9$. Since $|\mathbf{a}|^2 \geq 25$ by (7), it follows from Corollary 2 and inequality (3) that

$$\bar{\nu}^2(\mathbf{B}, \mathbf{L}) \le \frac{50}{81} + \frac{9}{25} < 1.$$

When $|\mathbf{c}_2|^2 = 6$, $\mathbf{c}_2 \sim (1, -1, 2, 0)$. So $\mathbf{c}_1 \cdot \mathbf{c}_2 = 0$ would give $\frac{1}{2}\mathbf{c}_1 + \frac{1}{2}\mathbf{c}_2 \in \mathbf{Z}^4$ which is a contradiction. Therefore $\mathbf{c}_1 \cdot \mathbf{c}_2 = 1$ and $\Delta^2 = 11$. Since $|\mathbf{a}|^2 \ge 36$ by (7), inequality (10) gives $\bar{\nu}(\mathbf{B}, \mathbf{L}) < 1$. This completes the proof of Lemma 4 and hence of Theorem 8.

PROOF OF THEOREM 9 : It follows from Theorems 5 and 8 that $\bar{\nu}^2(\mathbf{B}, \mathbf{L}) < 5/4$ except when $\mathbf{L}$ lies in a subspace equivalent to $\mathbf{S}_1$ and $|\mathbf{a}|^2 \le 6$. Since $\mathbf{a}$ does not lie in a co-ordinate hyperplane, this gives $\mathbf{a} \sim (1, 1, 1, 1)$ in which case $\bar{\nu}^2(\mathbf{B}, \mathbf{L}) = 5/4$.

Now let us suppose $\mathbf{a} = (1, 1, 1, 1)$, $\mathbf{d}_1 = (1, -1, 0, 0)$, $\mathbf{d}_2 = (0, 0, 1, -1)$, $\mathbf{d}_3 = (1, 1, -1, -1)$. The proof of Theorem 5 Case (i) shows that relative to the orthonormal basis $\mathbf{d}_i/|\mathbf{d}_i|$, $i = 1, 2, 3$, the projected lattice is

$$\varphi_{\mathbf{L}}(\mathbf{Z}^4) = \left\{\left(\frac{x_1 - x_2}{\sqrt{2}}, \frac{x_3 - x_4}{\sqrt{2}}, \frac{x_1 + x_2 - x_3 - x_4}{2}\right) : x_i \in \mathbf{Z}\right\}$$

with lattice generators

$$\varphi_{\mathbf{L}}(-1, 0, 0, 0) = \left(\frac{-1}{\sqrt{2}}, 0, -\frac{1}{2}\right) = \mathbf{g}_1$$

$$\varphi_{\mathbf{L}}(0, 0, 0, -1) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{2}\right) = \mathbf{g}_2$$

$$\varphi_{\mathbf{L}}(0, 0, -1, 0) = \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{2}\right) = \mathbf{g}_3$$

and associated quadratic form $\frac{1}{4}f_0$. Since $\mathbf{B}$ has diameter 1, the remarks in Section 4 show that $\sqrt{5/4}\,\mathbf{B} + \varphi_{\mathbf{L}}(\mathbf{Z}^4)$ covers $\mathbf{L}^\perp$ with just-covered points

$$u_1\mathbf{g}_1 + u_2\mathbf{g}_2 + u_3\mathbf{g}_3 + \varphi_{\mathbf{L}}(\mathbf{Z}^4),$$

where $\{u_1, u_2, u_3\} = \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}$. It follows that the lines $\mathbf{L} + \mathbf{p}$ satisfying $\nu^2(\mathbf{B}, \mathbf{L} + \mathbf{p}) = 5/4$ must have $\mathbf{p}$ of the form

$$\mathbf{p} = u_1(-1, 0, 0, 0) + u_2(0, 0, 0, -1) + u_3(0, 0, -1, 0) + k(1, 1, 1, 1) + \mathbf{Z}^4,$$

with $\{u_1, u_2, u_3\} = \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}$ and $k \in \mathbb{R}$. Suitable choices of the free parameters give precisely the points described in the statement of Theorem 9.

The cases of equality in the $\bar{v}$-problem are identical with those in the Schoenberg problem because of the happy accident that for $\mathbf{L} = \langle (1,1,1,1) \rangle$, $\varphi_{\mathbf{L}}(\mathbf{Z}^4) = \varphi_{\mathbf{L}}(\Lambda)$.

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