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BIORDERED SET LANGUAGES

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A language $L \subseteq A^*$ is said to be a biordered set language if it is recognized by a biordered set. This is a class of languages which have several biordered set theoretical properties. Here we give some characterizations of biordered set languages and examine some of its relations with band languages. A procedure for constructing biordered set languages from band languages is also given.

1. INTRODUCTION

Kleene's theorem[3] is the fundamental result connecting algebraic properties of semigroups with combinatorial properties of languages. Study of languages determined by idempotents has interest both variety theoretic and otherwise. In variety theory certain varieties can be specified purely in terms of idempotents such as *Inv* the variety generated by inverse semigroups (Ash[1]), block groups *BG* (Margolis and Pin[4]) etc. Thus one can determine whether a language L belongs to such a variety by translating L into an idempotent language. Another area of interest for this type of language is in programming languages, where one needs the specific combinatorial properties involved between various words. This is a class of languages where such relations are easily recognized, in terms of the algebraic properties of the biordered set. Also if E is a biordered set and S_E is an idempotent generated regular semigroup on E, then the productions in any language recognized by S_E is determined by the biordered structure on E. This can lead to a study of those languages in which productions are precisely those determined by biordered sets. This paper aims at initiating a study of languages determined by idempotents in a semigroup.

2. PRELIMINARIES

In this section we recall the basic definitions, results and notations that will be used in the sequel. All undefined terms and notations are as in Howie[2] and

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Lallement[3]. Let A be a non empty set called alphabet and elements of A are called letters. A word over A is a finite sequence $a_1 a_2 \dots a_n$ of elements of A. The length of the word $w = a_1 a_2 \dots a_n$ is n and is denoted by $| w |$. The set of all words over A is denoted by A^+. The empty word is defined as the word of length zero and is denoted by 1. Then $A^* = A^+ \bigcup \{1\}$ is a monoid under the concatenation of words called the free monoid over A. A language L is a subset of A^*. L is called recognizable if there exist a monoid M and an onto morphism $\varphi : A^* \to M$ such that $L = P\varphi^{-1}$ for some $P \subseteq M$. In this case we say that L is recognized by (M, φ). Often we suppress the φ and say L is recognized by M. L is called an idempotent language or an elementary language if $P = \{e\} \subseteq E(M)$ and L is called a biorder language if $P \subseteq E(M)$ where $E(M) = \{e \in M : e^2 = e\}$ is the set of all idempotents of M. Also L is said to be an absolutely elementary biorder language if every monoid recognizing L recognizes it by an idempotent. That is L is an elementary language with respect to every monoid recognizing L (cf. Bingjun[8]). Clearly biorder language is a union of elementary languages. To each language we associate a congruence P_L called syntactic congruence defined as follows.

For $u, v \in A^*$

$$uP_Lv \text{ if for all } x, y \in A^*,\ xuy \in L \Leftrightarrow xvy \in L.$$

The quotient morphism $\eta_L : A^* \to A^*/P_L$ is called the syntactic morphism and the monoid $M(L) = A^*/P_L$ is called the syntactic monoid. The following result gives a characterization for $M(L)$.

Theorem 2.1 (cf. Lallement[3]) — Let $L \subseteq A^*$. Then we have the following :

(i) $M(L)$ is the smallest monoid recognizing L.

(ii) If M is any monoid recognizing L, then $M(L)$ is a homomorphic image of a submonoid of M. □

If ρ is a congruence on A^* then the quotient morphism $A^* \to A^*/\rho$ is also denoted often by ρ itself. The following theorem given in Bingjun[8] characterizes the congruence δ_L and the class of biorder languages.

Theorem 2.2 (cf. Bingjun[8], Theorem 5) — Let $L \subseteq A^*$ and let L be a biorder language. Let δ_L be the smallest congruence on A^* containing $\{(w, w^2) : w \in L\}$. Then A^*/δ_L is the maximal monoid recognizing L by its idempotents. That is if M is any monoid and $\varphi : A^* \to M$ is a surjective homomorphism such that $L = E\varphi^{-1}$ for some $E \subseteq E(M)$ then there exists a unique morphism $\psi : A^*/\delta_L \to M$ such that the following diagram is commutative

(A)

$$\begin{array}{ccc} A^* & \xrightarrow{\ \varphi\ } & M \\ {\scriptstyle \delta_L}\downarrow & & \| \\ A^*/\delta_L & \xrightarrow[\ \psi\ ]{} & M \end{array}$$

□

The following result given in Bingjun[8] characterizes the biorder languages.

Theorem 2.3 (cf. Bingjun[8], Corollary 6) — For any language $L \subseteq A^*$, the following are equivalent.

(i) L is a biorder language.

(ii) $M(L)$ recognizes L by its idempotents.

(iii) $L\delta_L \delta_L^{-1} = L$ where δ_L is the smallest congruence on A^* containing $\{(w, w^2) : w \in L\}$. □

Next we give some preliminaries about biordered set. By a partial algebra E we mean a set with a partial binary operation. That is a mapping from $D_E \subseteq E \times E$ into E. We call D_E the domain of the partial binary operation. Let E be a partial algebra. On E define

$$\omega^r = \{(e, f) : fe = e\}, \ \omega^l = \{(e, f) : ef = e\}$$

$$R = \omega^r \cap (\omega^r)^{-1}, \ L = \omega^l \cap (\omega^l)^{-1} \text{ and } \omega = \omega^r \cap \omega^l.$$

Definition 2.1 (cf. Nambooripad[6], Definition 1.1) — Let E be a partial algebra. Then E is a biordered set if the following axioms and their duals hold. For $e, f \in E$

(1) ω^r and ω^l are quasiorders on E and

$$D_E = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}.$$

(2a) $f \in \omega^r(e) \Rightarrow fR\, fe\omega e$

(2b) $g\omega^l f, f, g \in \omega^r(e) \Rightarrow ge\omega^l fe.$

(3a) $g\omega^r f\omega^r e \Rightarrow gf = (ge)f.$

(3b) $g\omega^l f, f, g \in \omega^r(e) \Rightarrow (fg)e = (fe)(ge).$

Let $M(e, f)$ denote the quasiordered set $(\omega^l(e) \cap \omega^r(f), \prec)$ where $\prec$ is defined by

$$g \prec h \Leftrightarrow eg\omega^r eh, gf\omega^l hf.$$

Then the set

$$S(e, f) = \{h \in M(e, f) : g \prec h \text{ for all } g \in M(e, f)\}$$

is called sandwich set of e and f in that order.

(4a) $f, g \in \omega^r(e) \Rightarrow S(f, g)e = S(fe, ge).$

Axiom (4a) of the above definition is equivalent to the following condition (cf. Nambooripad[6], Proposition 2.4)

(4a′) If $g, h \in \omega^r(e)$ and $ge\omega^l he$ then there exists $g_1 \in \omega^r(e)$ such that $g_1\omega^l h$ and $g_1 e = ge$ and this g_1 is unique whenever it exists.

If M (resp. S) is a monoid (resp. semigroup), then $E(M)$ (resp. $E(S)$) is a biordered set (cf. Nambooripad[6], Theorem 1.1). The following theorem given in Nambooripad[6] gives an algorithm to check whether a given subset of $E(M)$ is a biordered set or not.

Theorem 2.4 (cf. Nambooripad[6], Proposition 2.6) — Let E be a biordered set and $E' \subseteq E$. Then E' is a biordered set if and only if the following conditions and their duals hold.

(i) For all $e', f' \in E'$, $(e', f') \in D_E$ implies $e'f' \in E'$.

(ii) If $e' \in E'$, $f', g' \in \omega^r(e') \cap E'$ and $g'e' \, \omega^l f'e'$ then there exists $g_1' \in E'$ such that $g_1' \in \omega^l(f') \cap \omega^r(e')$ and $g_1'e' = g'e'$. □

The following lemma gives a sufficient condition for the inverse image of a biordered set to be a biordered set.

Lemma 2.5 — Let S and S' be semigroups and let $\varphi : S \to S'$ be an onto morphism. Let E be a biordered subset of $E(S')$ and let $\overline{E} = E\varphi^{-1} \subseteq E(S)$. Then $\overline{E}$ is a biordered set.

PROOF : Let $e, f \in \overline{E}$ and $f\omega^r e$. Then $e\varphi, f\varphi \in E$ and $f\varphi\omega^r e\varphi$. Since E is a biordered set, $f\varphi e\varphi \in E$. That is $f\varphi e\varphi = (fe)\varphi \in E$. Therefore $fe \in E\varphi^{-1} = \overline{E}$. Similarly if $f\omega^l e$, then $ef \in \overline{E}$. Let $e, f, g \in \overline{E}$, $f, g \in \omega^r(e)$ and $ge\omega^l fe$. Since $E(S)$ is a biordered set, there exist $\overline{g} \in E(S)$ such that $\overline{g}\omega^r e$, $\overline{g}\omega^l f$ and $\overline{g}e = ge$. Thus $\overline{g}\varphi\omega^r e\varphi, \overline{g}\varphi\omega^l f\varphi$ and $\overline{g}\varphi e\varphi = g\varphi e\varphi$. But $f, g \in \omega^r(e)$ and $ge\omega^l fe$ implies $f\varphi, g\varphi \in \omega^r(e\varphi)$ and $g\varphi e\varphi\omega^l f\varphi e\varphi$. By Theorem 2.4 there exist a $g_1 \in E$ such that $g_1\omega^l f\varphi, g_1\omega^r e\varphi$ and $g_1 e\varphi = g\varphi e\varphi$. By the uniqueness of g_1 it follows that $g_1 = \overline{g}\varphi$. Thus $\overline{g} \in E\varphi^{-1}$ and $E\varphi^{-1}$ is a biordered set. □

The next lemma shows that a one to one morphism between sets of idempotents preserves biorder structure.

Lemma 2.6 — Let S, S' be semigroups and $\varphi : S \to S'$ be a homomorphism. Let E and E' be subsets of $E(S)$ and $E(S')$ respectively. Let the restriction of φ to E be denoted by φ itself. Let $\varphi : E \to E'$ be one to one and onto. Then we have the following

(i) if E is a biordered set then E' is a biordered set.

(ii) if E' is a biordered set then E is a biordered set. □

3. Biordered Set Languages

Definition 3.1 — Let $L \subseteq A^*$. L is said to be a biordered set language if there exists a monoid M (not necessarily finite) and a surjective morphism $\varphi : A^* \to M$ such that $L = E\varphi^{-1}$ where E is a biordered subset of $E(M)$.

If L is a biordered set language, then L is a disjoint union of idempotent languages (elementary languages) (Kumar[7]). Since every biordered set language is a biorder language we can say that (by Theorem 2.3) every biordered set language is recognized by the idempotents of A^*/δ_L. All biorder languages need not be biordered set languages. If L is an idempotent language, then L is a biordered set language. The effect of the relations ω^r, ω^l, R, L and ω on a biordered set language was studied in Kumar[7].

Example 3.1 — All elementary languages are biordered set languages.

Example 3.2 — Let $A = \{a, b\}$. Consider the monoid $C(a, b)$ with presentation $\langle\, a, b : ab = 1 \,\rangle$. $C(a, b)$ is the quotient monoid A^*/P_L where L is the correct parentheses language[3]. Let $\eta_L : A^* \to C(a, b)$ be the syntactic morphism of L. Then the collection of idempotents $\{b^n a^n : n = 1, 2, 3, ...\}$ is a semilattice. Then $L_1 = \{b^n a^n : n = 1, 2, 3, ...\}\ \eta_L^{-1}$ is a biordered set language and

$$L_1 = \{(a^n b^{2n} a^n)^+ \bigcup (b^n a^{2n} b^n)^+ \bigcup (b^n a^n)^+ : n = 1, 2, 3, \ ...\}.$$

Now we show that every biordered set language L is recognized by a biordered subset of $E(A^*/\delta_L)$ (cf. Theorem 2.2).

Theorem 3.1 — Let L be a biordered set language. Then $L = E\delta_L^{-1}$ where E is a biordered subset of $E(A^*/\delta_L)$.

PROOF : Since L is a biordered set language, there exists an onto morphism $\varphi : A^* \to M$ such that $L = E_1\,\varphi^{-1}$ where $E_1 \subseteq E(M)$ is a biordered set. Then from the commutative diagram (A) (in Theorem 2.2), we have $(L\delta_L)\psi = L\varphi$ for a homomorphism $\psi : A^*/\delta_L \to M$. So

$$\begin{aligned} E_1\,\psi^{-1}\,\delta_L^{-1} &= E_1\,(\delta_L\,\psi)^{-1} \\ &= E_1\,\varphi^{-1} = L \\ &= (L\delta_L)\,\delta_L^{-1}. \end{aligned}$$

Since δ_L is onto we have $E_1\,\psi^{-1} = L\delta_L$. Thus $E_1\,\psi^{-1} = L\delta_L$. Since $L\delta_L \subseteq E(A^*/\delta_L)$ (Theorem 2.2), $E_1\,\psi^{-1} \subseteq E(A^*/\delta_L)$. Therefore by Lemma 2.5, $E = E_1\,\psi^{-1}$ is a biordered set. □

Let L be a biordered set language and M be a monoid and $\varphi : A^* \to M$ a surjective morphism such that $L = L\varphi\varphi^{-1}$ and $L\varphi \subseteq E(M)$. Since A^*/δ_L is the maximal monoid recognizing L by idempotents, there exists a morphism $\psi : A^*/\delta_L \to M$ (cf. Lallement[3]) such that the following diagram commutes.

$$\text{(B)} \qquad \begin{array}{ccc} A^* & \xrightarrow{\ \varphi\ } & M \\ {\scriptstyle \delta_L}\downarrow & & \| \\ A^*/\delta_L & \xrightarrow[\ \psi\ ]{} & M \end{array}$$

For the following results (Lemma 2.2, Lemma 2.3, Lemma 2.4 and Theorem 2.5) we need the morphism ψ and the above commutative diagram.

Lemma 3.2 — Let L be a biordered set language and M be a monoid and $\varphi : A^* \to M$ a surjective morphism such that $L = L\varphi\varphi^{-1}$ and $L\varphi \subseteq E(M)$. Let $e, f \in L\delta_L, \bar{e} = e\psi$, $\bar{f} = f\psi$ and let $\bar{e}\omega^r \bar{f}$. Then there exists an $e' \in L\delta_L$ such that $e'\omega^r f$ and $e'\psi = e\psi$.

PROOF : Let $u \in \bar{e}\,\varphi^{-1}$ and $v \in f\,\delta_L^{-1}$ (and this implies $v \in \bar{f}\varphi^{-1}$). Then

$$(vu) \in f\,\delta_L^{-1}\,\bar{e}\varphi^{-1}$$
$$= \bar{f}\varphi^{-1}\,\bar{e}\varphi^{-1}$$
$$= (\bar{f}\bar{e})\,\varphi^{-1} = \bar{e}\,\varphi^{-1} \qquad (\text{since } \bar{e}\omega^r\bar{f}).$$

Let $(vu)\delta_L = e'$. Then $e'\psi = (vu)\,\delta_L\,\psi = (vu)\varphi = \bar{e} = e\psi$ and

$$(v \cdot vu)\,\delta_L = (v^2u)\,\delta_L = (v^2)\,\delta_L\,(u)\,\delta_L \quad (\text{since } \delta_L \text{ is a morphism})$$
$$\subseteq (v)\,\delta_L\,(u)\,\delta_L \quad (\text{since } v^2\,\delta_L = v\delta_L)$$
$$= (vu)\,\delta_L.$$

Thus $(v)\,\delta_L\,(vu)\,\delta_L = (vu)\,\delta_L$ and so $f \cdot e' = e'$ and this implies $e'\omega^r f$. □

The following lemma is dual to the above lemma and the proof follows from the dual arguments.

Lemma 3.3 — Let L be a biordered set language and M be a monoid and $\varphi : A^* \to M$ a surjective morphism such that $L = L\varphi\varphi^{-1}$ and $L\varphi \subseteq E(M)$. Let e, $f \in L\delta_L$, $\bar{e} = e\psi$, $\bar{f} = f\psi$ and let $\bar{e}\omega^l\bar{f}$. Then there exists an $e' \in L\delta_L$ such that $e'\omega^l f$ and $e'\psi = e\psi$. □

Lemma 3.2 and Lemma 3.3 show that the bimorphism $\psi \,|\, E : E \to E(M)$ weakly reflects (cf. Nambooripad[6], pp. 24-25) the quasiorders ω^r and ω^l, where $E = L\delta_L$.

Lemma 3.4 — Let L be a biordered set language and M be a monoid and $\varphi : A^* \to M$ a surjective morphism such that $L = L\varphi\varphi^{-1}$ and $L\varphi \subseteq E(M)$. Let e, g, $f \in L\delta_L$, $\bar{g} = g\psi$, $\bar{f} = f\psi$, $\bar{e} = e\psi$, and $\bar{g}$, $\bar{f} \in \omega^r(\bar{e})$ and $\bar{g}\;\bar{e}\omega^l\;\bar{f}\;\bar{e}$. Then there exists $g' \in L\delta_L$ such that $(g')\,\psi\omega^r\;\bar{e}$, $(g')\,\psi\omega^l\;\bar{f}$ and $(g')\psi\bar{e} = \bar{g}\;\bar{e}$.

PROOF : By Lemma 3.2, there exists g_1, f_1 such that $f_1\omega^r e$, $g_1\;\omega^r\;e$ and $f_1\psi = f\psi$, $g_1\psi = g\psi$. Let $w_1 \in (g_1)\,\delta_L^{-1}$, $w_2 \in (f_1)\,\delta_L^{-1}$ and $u \in (e)\,\delta_L^{-1}$ (and this implies $w_1 \in (g_1\,\psi)\varphi^{-1}$, $w_2 \in (f_1\,\psi)\,\varphi^{-1}$ and $u \in (e\psi)\varphi^{-1}$). Then

$$w_1\,uw_2\,u \in (g_1\,\psi)\varphi^{-1}\,(e\psi)\varphi^{-1}\,(g_2\,\psi)\,\varphi^{-1}\,(e\psi)\varphi^{-1}$$
$$\subseteq ((g_1)\psi\,(e)\psi\,(f_1)\psi\,(e)\,\psi)\varphi^{-1}$$
$$= (\bar{g}_1\;\bar{e}\;\bar{f}_1\;\bar{e})\varphi^{-1} \qquad (\text{where } \bar{g}_1 = g_1\psi\,, \bar{f}_1 = f_1\psi)$$
$$= (\bar{g}_1\;\bar{e})\varphi^{-1} \qquad (\text{since } \bar{g}_1\;\bar{e}\;\omega^l\;\bar{f}_1\;\bar{e})$$
$$= ((g_1e)\,\psi)\varphi^{-1}.$$

Let $(w_1uw_2u)\,\delta_L = g''$. Then $(g'')\,\psi = (g_1e)\psi = (g_1)\psi\,(e)\psi = (g)\psi\,(e)\psi = \bar{g}\;\bar{e}$, and

$$(uw_1uw_2u)\,\delta_L = (uw_1)\,\delta_L\,(uw_2u)\,\delta_L$$
$$= (w_1)\,\delta_L\,(uw_2u)\,\delta_L \quad (\text{since } (uw_1)\,\delta_L = w_1\delta_L)$$
$$= (w_1uw_2u)\,\delta_L$$

i.e., $(u)\,\delta_L\,(w_1uw_2u)\,\delta_L = (w_1uw_2u)\,\delta_L$

implies $eg'' = g''$ so $g''\,\omega^r e$. But

$$\begin{aligned}(w_1uw_2uuw_2u)\,\delta_L &= (w_1uw_2u)\,\delta_L\,(uw_2)\,\delta_L\,(u)\,\delta_L\\ &= (w_1uw_2u)\,\delta_L\,(w_2)\,\delta_L\,(u)\,\delta_L \quad (\text{since } (uw_2)\,\delta_L = (w_2)\,\delta_L)\\ &= (w_1uw_2)\,\delta_L\,(uw_2)\,\delta_L\,(u)\,\delta_L \quad (\text{since } \delta_L \text{ is a morphism})\\ &= (w_1uw_2)\,\delta_L\,(uw_2)\,\delta_L\,(u)\,\delta_L \quad (\text{since } (uw_2)\,\delta_L = (w_2)\,\delta_L)\\ &= (w_1uw_2^2)\,\delta_L\,(u)\,\delta_L\\ &= (w_1uw_2)\,\delta_L\,(u)\,\delta_L \quad (\text{since } (w_2^2)\,\delta_L = (w_2)\,\delta_L)\\ &= (w_1uw_2u)\,\delta_L\,(u)\,\delta_L \quad (\text{since } (w_2u)\,\delta_L = (w_2)\,\delta_L)\end{aligned}$$

i.e., $(w_1uw_2u)\,\delta_L\,(u)\,\delta_L\,(w_2u)\,\delta_L = (w_1uw_2u)\,\delta_L\,(u)\,\delta_L$

and this implies $g''\,ef_1e = g''e$, so $g''\,e\omega^l f_1e$. By (4a') there exists $g' \in L\varphi$ such that $g' \in \omega^l(f_1) \bigcap \omega^r(e)$ and $g''e = g'e$. Then $(g')\,\psi \in \omega^l(f_1\psi) \bigcap \omega^r(e\psi)$ and $(g')\psi\,(e)\psi = (g'')\psi\,(e)\psi = \bar{g}\;\bar{e}\;(e)\psi = \bar{g}\;\bar{e}\;\bar{e} = \bar{g}\;\bar{e} = (g)\psi\,(e)\psi$. Thus $(g')\psi\bar{e} = \bar{g}\;\bar{e}$. □

The next theorem shows that if L is a biordered set language then L is recognized by a biordered subset of $E(M)$ for any monoid which recognizes L by its idempotents.

Theorem 3.5 — Let $L \subseteq A^*$ be a biordered set language and M be a monoid and $\varphi : A^* \to M$ be a surjective morphism such that $L = L\varphi\varphi^{-1}$ and $L\varphi \subseteq E(M)$. Then $L\varphi$ is a biordered subset of $E(M)$.

PROOF : By Theorem 3.1, $L = (L\delta_L)\,\delta_L^{-1}$ and $L\delta_L = E$ is a biordered subset of $E(A^*/\delta_L)$. From the commutative diagram (B), it follows that $(L)\varphi = ((L)\delta_L)\psi = (E)\psi \subseteq M$. Let $(e)\psi, (f)\psi \in L\varphi$ and let $(e)\psi\,\omega^r\,(f)\psi$. Then by Lemma 3.2, there exists an $e' \in (L)\delta_L$ such that $(e')\psi = (e)\psi$ and $e'\omega^r f$. Since E is a biordered set, $e'f \in E$, so $(e')\psi\,(f)\psi \in L\varphi$. That is $e\psi f\psi \in L\varphi$. Similarly if $(e)\psi\,\omega^l.(f)\psi$, then $f\psi e\psi \in L\varphi$. Let $\bar{g}$, $\bar{f}$, $\bar{e} \in L\varphi$, $\bar{g}$, $\bar{f} \in \omega^r(\bar{e})$ and $\bar{g}\;\bar{e}\omega^l\bar{f}\;\bar{e}$. Then by Lemma 3.3, there exist $g'\psi \in L\varphi$ such that $g'\psi\,\omega^r\bar{e}$, $(g')\psi\,\omega^l\bar{f}$ and $g'\psi\bar{e} = \bar{g}\;\bar{e}$. Therefore by Theorem 2.4, $L\varphi$ is a biordered set. □

Theorem 3.6 — Let L be a biordered language. Then the following are equivalent.

(i) L is a biordered set language

(ii) $L\delta_L$ is a biordered subset of $E(A^*/\delta_L)$

(iii) $L\eta_L$ is a biordered subset of $E(M(L))$ where $\eta_L : A^* \to M(L)$ is the canonical morphism and $M(L)$ is the syntactic monoid of L.

(iv) If M is any monoid recognizing L by its idempotents then $L = E\varphi^{-1}$ for a biordered subset $E \subseteq E(M)$ and an onto morphism $\varphi : A^* \to M$.

PROOF : (i) ⇒ (ii) Follows from Theorem 3.1.

(ii) $\Rightarrow$ (iii) By Theorem 2.2 there exists a morphism $\psi : A^*/\delta_L \rightarrow M(L)$ such that the following diagram is commutative

$$\text{(C)} \qquad \begin{array}{ccc} A^* & \xrightarrow{\eta_L} & M(L) \\ \downarrow{\scriptstyle \delta_L} & & \| \\ A^*/\delta_L & \xrightarrow[\psi]{} & M(L) \end{array}$$

By Theorem 3.5, $(L\delta_L)\psi$ is a biordered set in $M(L)$. Thus $(L)\eta_L$ is a biordered subset in $E(M(L))$.

(iii) $\Rightarrow$ (iv) Let $\varphi : A^* \rightarrow M$ be the morphism recognizing L by its idempotents of M. Then there exists a morphism $\psi : M \rightarrow M(L)$ such that the following diagram is commutative (cf. Lallement[3], Lemma VI.5.1)

$$\text{(D)} \qquad \begin{array}{ccc} A^* & \xrightarrow{\eta_L} & M(L) \\ {\scriptstyle \varphi}\downarrow & & \| \\ M & \xrightarrow[\psi]{} & M(L) \end{array}$$

So $L = (L\eta_L)\,\eta_L^{-1} = ((L\eta_L)\psi^{-1})\varphi^{-1}$. Since M recognizes L by its idempotents, $(L\eta_L)\psi^{-1} \subseteq E(M)$. Therefore by Lemma 2.5, $(L\delta_L)\psi^{-1}$ is a biordered set in $E(M)$, since $(L)\eta_L$ is a biordered set in $M(L)$. Take $E = (L\eta_L)\psi^{-1}$. Then we get $L = E\varphi^{-1}$.

(iv) $\Rightarrow$ (i) Follows from the definition. □

The following theorem distinguishes biordered set languages within the class of biorder languages.

Theorem 3.7 — Let $L \subseteq A^*$ and δ_L be the congruence on A^* generated by the relation $\{(w, w^2) : w \in L\}$. Let $L = L\delta_L\,\delta_L^{-1}$ and $\sim$ denote the restriction of δ_L to L. Then L is a biordered set language if and only if the following hold.

(C1) For $u, v \in L$, if $uv \sim u$ or $uv \sim v$ then $vu \in L$.

(C2) Let $u, w_1, w_2 \in L$ with $uw_1 \sim w_1, uw_2 \sim w_2$ and $w_1u \sim w_1w_2u$. Then there exists $t \in L$ with $t \sim utw_2$ and $w_1u \sim tw_2u$.

(C2*) Let $u, w_1, w_2 \in L$ with $w_1u \sim w_1, w_2u \sim w_2$ and $uw_1 \sim uw_2w_1$. Then there exists $t \in L$ with $t \sim w_2tu$ and $uw_1 \sim uw_2t$.

PROOF : Let $E = L\delta_L$. Then by hypothesis $E \subseteq E(A^*/\delta_L)$ and $L = E\delta_L^{-1}$. Assume that L satisfies (C1), (C2) and (C2*). We prove that E is a biordered subset of $E(A^*/\delta_L)$. It is sufficient to prove that E satisfies conditions of Theorem 2.4. Let $u, v \in L$ and $(u)\,\delta_L = e, (v)\delta_L = f$. Let $e\omega^l f$. Then $ef = e$. That is

$$(u)\delta_L\,(v)\delta_L = (u)\delta_L$$

$$(uv)\delta_L = (u)\delta_L \qquad \text{(since } \delta_L \text{ is a homomorphism).}$$

$$uv \sim u \qquad \text{(by the definition of } \sim)$$

implies $vu \in L$ (by (C1)).
So $(vu)\delta_L \in E$ and hence $(v)\delta_L (u)\delta_L \in E$. That is $ef \in E$. Similarly if $f\omega^r e$, then also $fe \in E$. Let $u, w_1, w_2 \in L$ and let $(u)\delta_L = e, (w_1)\delta_L = g, (w_2)\delta_L = f$. Let $f, g \in \omega^r(e)$ and $ge\omega^l fe$. Then $ef = f$, $eg = g$ and $ge\ fe = ge$. That is

$$w_1\delta_L\, u\delta_L\, w_2\delta_L\, u\delta_L = w_1\delta_L\, u\delta_L$$

$$(w_1uw_2u)\,\delta_L = (w_1u)\delta_L \qquad \text{(since } \delta_L \text{ is a homomorphism)}$$

$$w_1\delta_L\,(uw_2)\,\delta_L\, u\delta_L = (w_1u)\,\delta_L$$

$$w_1\,\delta_L\, w_2\,\delta_L\, u\delta_L = (w_1u)\,\delta_L \qquad \text{(since } (uw_2)\delta_L = w_2\delta_L)$$

$$w_1w_2u \sim uw_1. \qquad \text{(by the definition of } \sim).$$

By (C2) there exists a $t \in L$ with $t \sim utw_2$ and $w_1u \sim tw_2u$. Let $t\delta_L = g_1$. Then

$$g_1 = t\delta_L\,(utw_2)\,\delta_L$$

$$= u\delta_L\, t\delta_L\, w_2\delta_L$$

$$= eg_1f$$

$g_1 \in E, eg_1 = g_1$ and $g_1f = g_1$. That is $g_1 \in \omega^l(f) \cap \omega^r(e)$ (by the definition of ω^r and ω^l). Since $w_1u \sim tw_2u$, we have

$$gew_1\,\delta_L\, u\delta_L = t\delta_L\, w_2\delta_L\, u\delta_L$$

$$= g_1fe$$

$$= g_1e \qquad \text{(since } g_1f = g).$$

The uniqueness of g_1 follows from the fact that $E \subseteq E(A^*/\delta_L)$. That is E satisfies (4a′) of Definition 2.1. (C2*) gives the condition dual to (4a′). Thus E is a biordered set.

Conversely assume that E is a biordered set. Let $u, v \in L$ such that $vu \sim u$. Let $u\delta_L = e$ and $v\delta_L = f$. Then

$$u\delta_L\, v\delta_L = u\delta_L \qquad \text{(since } uv \sim u)$$

$$ef = e$$

and this implies $e\omega^l f$. Since E is a biordered set, $fe \in E$. So $v\delta_L\, u\delta_L \in E$. That is $(vu)\delta_L \in E$. That is $vu \in L$. Similarly if $vu \sim v$, then also $vu \in E$. Let $u, w_1, w_2 \in L$ with $u\delta_L = e, w_1\delta_L = g, w_2\delta_L = f$. Let $uw_1 \sim w_1$, $uw_2 \sim w_2$ and $w_1u \sim w_1w_2u$. Then $uw_1 \sim w_1, uw_2 \sim w_2$ gives $f, g \in \omega^r(e)$ and $w_1u \sim w_1w_2u$ gives

$$w_1\delta_L\, w_2\delta_L\, u\delta_L = w_1\delta_L\, u\delta_L$$

$$w_1\delta_L\, u\delta_L\, w_2\delta_L\, u\delta_L = w_1\delta_L\, u\delta_L \qquad (\text{since } w_2\delta_L = u\delta_L\, w_2\delta_L)$$

$$gefe = ge$$

implies $ge\omega^l fe$. Since E is a biordered set there exists a $g_1 \in E$ such that $g_1 \in \omega^r(e) \bigcap \omega^l(f)$ and $ge = g_1 e$. Choose $t \in L$ such that $t\delta_L = g_1$ (and such words exists in L because $g_1 \in E$). Then

$$\begin{aligned} t\delta_L &= g_1 = g_1 f \\ &= t\delta_L\, w_2\delta_L \\ &= u\delta_L\, t\delta_L\, w_2\delta_L \qquad (\text{since } u\delta_L\, t\delta_L = eg_1 = g_1) \\ &= (utw_2)\delta_L \end{aligned}$$

$$t \sim utw_2 \qquad (\text{by the definition of } \sim)$$

and

$$\begin{aligned} (tw_2u)\,\delta_L &= t\delta_L\,(w_2u)\,\delta_L \\ &= u\delta_L\, t\delta_L\,(w_2u)\,\delta_L \\ &= (utw_2)\delta_L\, u\delta_L \qquad (\text{since } t \sim utw_2) \\ &= g_1 e = ge \\ &= w_1\delta_L\, u\delta_L = (w_1u)\,\delta_L \end{aligned}$$

$$tw_2u \sim w_1u \qquad (\text{by the definition of } \sim).$$

(C2*) follows from the dual condition of (2) in Theorem 2.4. □

The set of all biordered set languages over A is denoted by $BSL(A)$ (or BSL).

Theorem 3.8 — BSL is closed under intersection.

PROOF : Let $\{L_i : i \in I\}$ be a collection of languages with $L_i \in BSL$. Then there exist monoids M_i and surjective morphisms $\psi_i : A^* \to M_i$ such that $L_i = E_i\,\psi_i^{-1}$ where E_i is a biordered subset of $E(M_i)$. Define $\psi : A^* \to \prod_i M_i$ by $w \mapsto (w_1\psi_1, w_2\psi_2, \ldots)$. Then ψ is a well defined morphism and $\bigcap_{i\in I} L_i = \left(\prod_{i\in I} E_i\right)\psi^{-1}$. Since $\prod_i E_i$ is a biordered set, $\bigcap_{i\in I} L_i \in BSL$. □

4. BAND CLOSURE OF BIORDERED SET LANGUAGES

Here we relate biordered set languages with band languages and provide some procedure for extracting biordered set languages from band languages.

Definition 4.1 — Let $L \subseteq A^*$. L is called a band language if its syntactic monoid is a band.

Remark 4.1 : It is easy to see that if L is recognized by a band B then $M(L)$ being the homomorphic image of B is also a band or L is a band language.

Since the class of bands forms a variety, the class of languages whose syntactic monoids are bands forms a language variety (*-variety). Milito[5] gives some properties of this variety.

Let δ be the congruence on A^* generated by the relation $\{(w, w^2) : w \in A^*\}$. That is δ is the band congruence on A^* such that A^*/δ is a free band (cf. Howie[2]). Since $\delta_L \subseteq \delta$, there exists a homomorphism $\eta : A^*/\delta_L \to A^*/\delta$ such that the following diagram commutes (cf. Howie[2]).

(E)
$$\begin{array}{ccc} A^* & \xrightarrow{\ \delta\ } & A^*/\delta \\ {\scriptstyle \delta_L}\downarrow & & \| \\ A^*/\delta_L & \xrightarrow[\ \eta\ ]{} & A^*/\delta \end{array}$$

Theorem 4.1 — Let $L \subseteq A^*$. The following are equivalent.

(i) L is a band language.

(ii) L is recognized by the free band A^*/δ and by the morphism δ.

PROOF : (1) $\Rightarrow$ (2) Let L be a band language and $\eta_L : A^* \to M(L)$ be the syntactic morphism of L. Then $L = (L\eta_L)\eta_L^{-1}$. For any $x \in A^*$, $x^2\eta_L = (x\eta_L)^2 = x\eta_L$, so $(x^2, x) \in \eta_L \circ \eta_L^{-1}$. Since δ is the smallest congruence generated by the relation $\{(x, x^2) : x \in A^*\}$, $\delta \subseteq \eta_L \circ \eta_L^{-1}$. Therefore there exists a unique homomorphism $\varphi : A^*/\delta \to M(L)$ such that the diagram commutes

(F)
$$\begin{array}{ccc} A^* & \xrightarrow{\ \eta_L\ } & M(L) \\ {\scriptstyle \delta}\downarrow & & \| \\ A^*/\delta & \xrightarrow[\ \varphi\ ]{} & M(L) \end{array}$$

Thus $L = (L\eta_L)\eta_L^{-1} = ((L\eta_L)\varphi^{-1})\delta^{-1}$. Since $(L\eta_L)\varphi^{-1} \subseteq A^*/\delta$, L is recognized by the free band.

(2) $\Rightarrow$ (1) Let $L = E\delta^{-1}$ where $E \subseteq A^*/\delta$. Since L is recognized by A^*/δ, $M(L)$ is a homomorphic image of A^*/δ (cf. Theorem 2.1). Therefore $M(L)$ is a band. □

In view of Theorem 3.6 and the above we have the following corollary.

Corollary 4.2 — Let L be a band language. Then L is a biordered set language if and only if it is recognized by a biordered set in the free band. □

Now we associate a band language with every biorder language.

Definition 4.2 — Let L be a biordered set language. The band closure of L is the smallest band language containing L. We denote it by L^c.

Remark 4.2 : Since intersection of band languages is a band language there always exists a smallest band language containing L.

The following theorem gives an explicit description of band closure.

Theorem 4.3 — Let L be a biordered set language. Then the band closure is given by $L^c = (L\delta_L\eta)\delta^{-1}$ where δ_L, η, δ are as in the commutative diagram (E) with $\delta = \delta_L\eta$.

PROOF : From the commutative diagram (E) it follows that $(L)\delta_L\eta \subseteq A^*/\delta$. Let $L_c = ((L\delta_L)\eta)\delta^{-1}$. Then L_c is recognized by A^*/δ and so $M(L_c)$ is a homomorphic image of A^*/δ. Since A^*/δ is a band we see that $M(L_c)$ is also a band. Thus L_c is a band language. Now let L' be a band language such that $L \subseteq L' \subseteq L_c$. Then by Theorem 4.1, there exists $P \subseteq ((L)\delta_L)\eta \subseteq A^*/\delta$ such that $L' = P\delta^{-1}$ and $L\delta_L \subseteq L'\delta_L$ and $L'\delta_L \subseteq L_c\delta_L$. Since $L\delta = (L\delta_L)\eta = (L_c)\delta$, we have $L'\delta = (L_c)\delta = L\delta$. So $P = L'\delta = L\delta = (L\delta_L)\eta$. Therefore $L' = L_c$. Thus L_c is the smallest band language containing L and so $L_c = L^c$. Hence $L^c = ((L\delta_L)\eta)\delta^{-1}$. □

Now we go over to a construction of biordered set languages from band languages.

Lemma 4.4 — Let L be a biordered set language and let δ_L, δ and η as in the commutative diagram (E). Let $B = L\delta$ and let $L_b = L \cap (b)\delta^{-1}$ for $b \in B$. Then L_b is a biordered set language. If for every $b \in B, L_b$ is an absolutely elementary biorder language, then $\eta \,|\, L\delta_L : L\delta_L \to B$ is one to one.

PROOF : By Theorem 3.8, L_b is a biordered set language. Let $e', f' \in (L)\delta_L$ and let $e'\eta = f'\eta$. Let $e'\delta_L^{-1} = L_{e'}$, $f'\delta_L^{-1} = L_{f'}$. Then $(L_{e'})\delta = (L_{f'})\delta = g$ where $g \in A^*/\delta$. Since $L_{e'} \subseteq L$ and $L_{e'} \subseteq g\delta^{-1}, L_{e'} \subseteq L \cap g\delta^{-1}$. Similarly $L_{f'} \subseteq L \cap g\delta^{-1}$. Assume that $g\eta^{-1} = \{e', f'\}$. Then $L_{e'} \cup L_{f'} = L \cap g\delta^{-1}$. That is $\{e', f'\}$ recognizes $L \cap g\delta^{-1}$. This is a contradiction, since $L \cap g\delta^{-1}$ is an absolutely elementary biorder language. So $\varphi \,|\, L\delta_L : L\delta_L \to B$ is one to one. If $|g\varphi^{-1}| > 2$, then also by the same argument $\varphi \,|\, L\delta_L : L\delta_L \to B$ is one to one. □

Theorem 4.5 — Let $L \subseteq A^*$ be a biordered set language and let $B = L\delta$ where δ is the free band congruence on A^*. Let $L_b = L \cap (b)\delta^{-1}$ and assume that L_b is an absolutely elementary biorder language for all $b \in B$. Then B is a biordered set.

PROOF : Since L is a biordered set language, $L\delta_L$ is a biordered set. Since L_b is an absolutely elementary biorder language, by Lemma 2.6, $(L\delta_L)\varphi$ is a biordered set. Thus B is a biordered set. □

The next Theorem provides a construction of biordered set languages from absolutely elementary biorder languages. Here δ is the free band congruence on A^*. All languages are supposed to be in a fixed A^*.

Theorem 4.6 — Let $\{L_i : i \in I\}$ be a collection of absolutely elementary biorder languages satisfying the following.

(i) $L = \bigcup_i L_i$ is a biorder language.

(ii) $L\delta$ is a biordered subset of the free band A^*/δ.

(iii) For $x, y \in L$, $x\delta y$ if and only if $x, y \in L_i$ for some i.

Then L is a biordered set language.

PROOF : Let $(L_i)\delta_L = e$. Then from the commutative diagram (E), $e\delta_L^{-1} \subseteq (e\eta)\delta^{-1}$ where $e\eta = b$. Since $L_i = e\delta_L^{-1} \subseteq L$, $e\delta_L^{-1} \subseteq L \bigcap b\delta^{-1}$. But by (iii), $L \bigcap b\delta^{-1} \subseteq e\delta_L^{-1}$. So $e\delta_L^{-1} = L \bigcap b\delta^{-1}$. That is $(L \bigcap b\delta^{-1})\, \delta_L\, \delta_L^{-1} = L \bigcap b\delta^{-1}$.

Thus $L \bigcap b\delta^{-1}$ is recognized by δ_L. Next we prove that $\eta\,|\,L\eta_L$ is one to one. Let $e', f' \in (L)\delta_L$ and $e'\eta = f'\eta$. Then

$$e' = (e'\eta\delta^{-1})\,\delta_L \qquad \text{(from the diagram (E))}$$

$$= (L \bigcap (e'\eta\delta^{-1}))\,\delta_L \qquad (\text{since } e'\eta\delta^{-1} \subseteq e'\,\delta_L^{-1} \subseteq L)$$

$$= (L \bigcap b'\,\delta^{-1})\,\delta_L \qquad (\text{since } e'\eta = b')$$

and $$f' = (L \bigcap b'\delta^{-1})\delta_L \qquad (\text{since } f'\eta = e'\eta = b').$$

Thus $e'\,\delta_L^{-1} = L \bigcap b'\,\delta^{-1}$ and $f'\,\delta_L^{-1} = L \bigcap b'\,\delta^{-1}$, so $e'\,\delta_L^{-1} = f'\,\delta_L^{-1}$. Hence $e' = f'$. Let $L\delta_L = E$. Then $(E)\eta = (L)\delta$. By (ii), $L\delta$ is a biordered set. So by Lemma 2.6, $E = (L\delta)\eta^{-1}$ is a biordered set. Hence the result. □

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