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APPROXIMATION OF ABSTRACT FUNCTIONAL DIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAY

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In this work we study the approximation of solutions of abstract retarded functional differential equations (ARFDE) with unbounded delay by means of solutions of ARFDE with bounded delay. As consequence we establish some results of stability and existence of periodic solutions for the first one.

1. INTRODUCTION

Let X be a Banach space with norm $\| \cdot \|$ and suppose that $A : D(A) \to X$ is the infinitesimal generator of a strongly continuous operator semigroup of linear operators $T(t)$ defined on X. The objective of this work is to study approximation properties of the solutions of a class of partial functional differential equations with unbounded delay.

Let $\mathcal{B}$ be an abstract phase space. We will consider initial value problems which can be modelled as the abstract Cauchy problem :

$$\dot{x}(t) = Ax(t) + F(t, x_t), \quad t \geq 0, \qquad \text{... (1.1)}$$

with initial condition

$$x_0 = \varphi \in \mathcal{B}, \qquad \text{... (1.2)}$$

where $F : [0, a] \times \mathcal{B} \to X$, $a > 0$, is a continuous function and x_t represents the function defined from $(-\infty, 0]$ into X by $x_t(\theta) = x(t + \theta)$, $-\infty < \theta \leq 0$.

As a concrete model for this class of equations we consider the partial integro-differential Volterra equation with infinite delay

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$$\frac{\partial}{\partial t} w(\xi, t) = \frac{1}{\pi^2} \frac{\partial^2}{\partial \xi^2} w(\xi, t) + \int_{-\infty}^{t} q(t, s, \xi, w(\xi, s))\, ds + h(t), \quad t \geq 0, \quad 0 \leq \xi \leq 1, \qquad \text{... (1.3)}$$

with boundary condition

$$w(0, t) = w(1, t) = 0, \quad t \geq 0, \qquad \text{... (1.4)}$$

and initial condition

$$w(\xi, \theta) = \varphi(\xi, \theta), \quad 0 \leq \xi \leq 1, \ \theta \leq 0, \qquad \text{... (1.5)}$$

where q, h and φ are appropriated functions. As the space X for this equation we choose $L^2([0, 1])$. The operator A is given by

$$A\varphi := \frac{1}{\pi^2} \frac{d^2\varphi}{d\xi^2}$$

with domain

$$D(A) = \left\{ \varphi \in X : \frac{d^2\varphi}{d\xi^2} \in X, \ \frac{d\varphi}{d\xi}(0) = \frac{d\varphi}{d\xi}(1) = 0 \right\}$$

and it is well known that this operator generates a semigroup. The study of retarded functional differential equations (RFDE) with unbounded delay has attracted the attention of several authors in recent years. We only mention the works of Hale and Kato[4], Corduneanu and Lakshmikantham[2], Hino *et al.*[7] and Shin[11], for the general theory, and Henríquez[6] for the problem of existence of solutions and existence of periodic solutions of ARFDE with unbounded delay of type (1.1).

Throughout this paper we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato[4]. To establish the axioms of space $\mathcal{B}$ we follow the terminologies used by Hino *et al.*[7]. Thus, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$. We will assume that $\mathcal{B}$ satisfies the following axioms :

(A) If $x : (-\infty, \sigma + a) \to X$, $a > 0$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$ then for every t in $[\sigma, \sigma + a)$ the following conditions hold :

(i) x_t is in $\mathcal{B}$;

(ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$;

(iii) $\| x_t \|_{\mathcal{B}} \leq K(t - \sigma) \sup \{\| x(s) \| : \sigma \leq s \leq t\} + M(t - \sigma) \| x_\sigma \|_{\mathcal{B}}$,

where $H \geq 0$ is a constant; $K, M : [0, \infty) \to [0, \infty)$, K is continuous and M is locally bounded and H, K and M are independent of $x(\cdot)$.

(A-1) For the function $x(\cdot)$ in (A), x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + a)$.

(B) The space $\mathcal{B}$ is complete.

To obtain some of our results we will need additional properties of the space $\mathcal{B}$. Next we denote by C_{00} the space of continuous functions from $(-\infty, 0]$ to X with compact support. We consider the following axiom for the phase spaces :

(C-2) If a uniformly bounded sequence $(\varphi^n)_n$ in C_{00} converges to a function φ in the compact-open topology then φ belongs to $\mathcal{B}$ and $\| \varphi^n - \varphi \|_{\mathcal{B}} \to 0$, as $n \to \infty$.

We will denote by $\hat{\mathcal{B}}$ the quotient Banach space $\mathcal{B}/\| \cdot \|_{\mathcal{B}}$ and, if $\varphi \in \mathcal{B}$ we write $\hat{\varphi}$ for the coset determined by φ.

For the theory of strongly continuous semigroups of linear operators (abbreviated, C_0-semigroup) we refer to Nagel[9]. In particular, it is well known that there exist constants $\tilde{M} \geq 1$ and $\mu \in \mathbb{R}$ such that

$$\| T(t) \| \leq \tilde{M} e^{\mu t}, \quad t \geq 0. \qquad \text{... (1.6)}$$

Our first purpose in this paper will be to investigate the approximation of solutions of the abstract Cauchy problem (1.1)-(1.2) by solutions of ARFDE with finite delay. Lastly, we will show that our theory can be effectively applied to obtain new results on the asymptotic stability of solutions and the existence of periodic solutions and we apply these results to partial integro-differential equations with unbounded delay.

The paper is organized as follows. In section 2 we will establish a result of existence of solutions while in section 3 we will study the approximation problem. Finally, in section 4 we present some examples and applications. Throughout this paper we always assume that $\mathcal{B}$ is a phase space. The terminology and notations are those generally used in operator theory. In particular, if X and Y denote Banach spaces, we indicate by $\mathcal{L}(X, Y)$ the Banach space of bounded linear operators from X into Y and we abbreviate this notation to $\mathcal{L}(X)$ whenever $X = Y$. Moreover, we will denote by $B_r[x]$ the closed ball with centre at x and radius r.

2. Existence of Solutions

In a recent paper Zima[13] has established a comparison criteria to obtain the existence of a fixed point of an operator $\mathcal{A}$. In this section we present a slight extension of Zima's result which provides a unified approach to allow us to obtain existence of solutions for an ARFDE with bounded or unbounded delay.

Initially we consider two Banach spaces Y and Z with norm $\| \cdot \|_Y$ and $\| \cdot \|_Z$, respectively. We assume the existence of an order relation $\preceq$ in Z and the existence of a map $m : Y \to Z$ such that the conditions hold :

(i) For all $y \in Y$, $0 \preceq m(y)$;

(ii) the norm in Z is monotonic with respect to the order $\leq$, that is, if $0 \preceq z_1 \preceq z_2$ then $\| z_1 \|_Z \leq \| z_2 \|_Z$;

(iii) there exists constants $C_1, C_2 > 0$ such that

$$\| y \|_Y \leq C_1 \| m(y) \|_Z ,$$

$$\| m(y) \|_Z \leq C_2 \| y \|_Y,$$

for all $y \in Y$.

Theorem 1 — Let S be a non empty and closed subset of Y and let $\mathcal{A} : S \to S$ be a continuous map. If there exists a bounded linear operator $B : Z \to Z$ such that :

(a) The spectral radius $r(B) < 1$;

(b) the operator B is increasing with respect to $\preceq$, that is, if $0 \preceq z_1 \preceq z_2$ then $Bz_1 \preceq Bz_2$;

(c) For all $y_1, y_2 \in S$, $m(\mathcal{A}y_1 - \mathcal{A}y_2) \preceq B\, m(y_1 - y_2)$,

then $\mathcal{A}$ has an unique fixed point.

PROOF : First we begin by observing that

$$0 \preceq m(\mathcal{A}y_1 - \mathcal{A}y_2) \preceq B\, m(y_1 - y_2)$$

implies that

$$\begin{aligned} m(\mathcal{A}^2 y_1 - \mathcal{A}^2 y_2) &\preceq B\, m(\mathcal{A}y_1 - \mathcal{A}y_2) \\ &\preceq B^2\, m\,(y_1 - y_2) \end{aligned}$$

and, proceeding by induction, we can assert that

$$m(\mathcal{A}^k y_1 - \mathcal{A}^k y_2) \preceq B^k\, m(y_1 - y_2),$$

for all $k \in \mathbb{N}$. Consequently, condition (iii) yields

$$\begin{aligned} \| \mathcal{A}^k y_1 - \mathcal{A}^k y_2 \|_Y &\leq C_1 \| m(\mathcal{A}^k y_1 - \mathcal{A}^k y_2) \|_Z \\ &\leq C_1 \| B^k m(y_1 - y_2) \|_Z \\ &\leq C_1 \| B^k \| \| m(y_1 - y_2) \|_Z \\ &\leq C_1 C_2 \| B^k \| \| y_1 - y_2 \|_Y \end{aligned}$$

for all $y_1, y_2 \in S$ and all $k \in \mathbb{N}$. On the other hand, since $r(B) < 1$ then $\| B^k \| \to 0$, as $k \to \infty$, and the above estimate implies that for some $k \in \mathbb{N}$ the map $\mathcal{A}^k$ is a contraction. Therefore, $\mathcal{A}$ has an unique fixed point in S. ■

Next, we will present an application of Theorem 1 to the problem of existence of solutions of ARFDE of type (1.1).

Henceforth, we will assume that $F : [0, a] \times \mathcal{B} \to X$ is a continuous function.

We will say that a function $x : (-\infty, a] \to X$ is a mild solution of the Cauchy problem (1.1)-(1.2) if $x_0 = \varphi$ and the restriction $x : [0, a] \to X$ is continuous and satisfies the integral equation :

$$x(t) = T(t)\,\varphi(0) + \int_0^t T(t-s)\,F(s, x_s)\,ds, \quad 0 \le t \le a. \qquad \text{... (2.1)}$$

In the rest of this work we will abbreviate our terminology calling solutions to the mild solutions.

Theorem 2 — Assume that F satisfies the Lipschitz condition

$$\| F(t, \varphi_1) - F(t, \varphi_2) \|_X \le L \, \| \varphi_1 - \varphi_2 \|_{\mathcal{B}} \text{ for all } 0 \le t \le a, \ \varphi_1, \varphi_2 \in \mathcal{B}, \qquad \text{... (2.2)}$$

where L is a positive constant. Then there exists a unique solution of (1.1)-(1.2) for every $\varphi \in \mathcal{B}$.

PROOF : We introduce the spaces $Y := C([0, a]; X)$ and $Z := C([0, a]; \mathbb{R})$ endowed with the norm of uniform convergence. We consider Z provided with the pointwise order relation and define the map $m : Y \to Z$ by the expression

$$m(u)\,(t) := \sup_{0 \le s \le t} \| u(s) \|_X, \quad 0 \le t \le a.$$

It is easy to see from this construction that condition (i), (ii) and (iii), mentioned above, are satisfied with constants $C_1 = C_2 = 1$.

Now, let us consider $\varphi \in \mathcal{B}$ fixed and let us introduce the set $S(\varphi) := \{u \in Y : u(0) = \varphi(0)\}$. We define on S an operator $\mathcal{A}$ by the formula

$$(\mathcal{A}u)\ (t) := T(t)\ \varphi(0) + \int_0^t T(t-s)\,F(s, \tilde{u}_s)\,ds, \quad 0 \le t \le a \qquad \text{... (2.3)}$$

where we have denoted by $\tilde{u} : (-\infty, a] \to X$ the function defined by $\tilde{u}(\theta) := \varphi(\theta)$, $-\infty \le \theta \le 0$, and $\tilde{u}(s) := u(s)$, $0 \le s \le a$. It is clear that $\mathcal{A} : S \to S$. Let N be a constant such that $\| T(t) \| \le N$ for all $0 \le t \le a$. We define the map $B : Z \to Z$ by the expression

$$(Bf)\,(t) := NL \int_0^t K(s)\,f(s)\,ds. \qquad \text{... (2.4)}$$

It is easy to show that B is a bounded linear operator on Z which satisfies condition (b) of Theorem 1. Furthermore, in order to prove condition (c), it is sufficient to observe that

$$\| (\mathcal{A}u - \mathcal{A}v)(t) \| = \| \int_0^t T(t-s)\,[F(s, \tilde{u}_s) - F(s, \tilde{v}_s)]\,ds \|$$

$$\le NL \int_0^t \| \tilde{u}_s - \tilde{v}_s \|_{\mathcal{B}}\,ds$$

$$\leq NL \int_0^t K(s) \sup_{0 \leq \xi \leq s} \| u(\xi) - v(\xi) \| \, ds$$

so that

$$m(\mathcal{A}u - \mathcal{A}v)\,(t) = \sup_{0 \leq \eta \leq t} \| (\mathcal{A}(u) - \mathcal{A}(v))\,(\eta) \|$$

$$\leq NL \sup_{0 \leq \eta \leq t} \int_0^\eta K(s) \sup_{0 \leq \xi \leq s} \| u(\xi) - v(\xi) \| \, ds$$

$$\leq NL \int_0^t K(s)\, m(u - v)\,(s)\, ds$$

$$= Bm(u - v)\,(t)$$

for each $0 \leq t \leq a$. Finally, it is well known that B is a compact operator with spectrum $\sigma(B) = \{0\}$. Thus, $r(B) < 1$ and all conditions of Theorem 1 hold. Consequently, $\mathcal{A}$ has a fixed point which is the unique solution of (1.1)-(1.2). ■

In the next section we will study the approximation of solutions of (1.1) by a sequence of solutions of ARFDE with finite delay. The ARFDE with finite delay $r > 0$ (in short, ARFDEr) have been studied by several authors. We refer to the works of Travis and Webb[12], Grabosch and Moustakas[3] and Hale[5] and also to their references. Usually these equations are considered on the spaces of continuous or integrable functions from $[-r, 0]$ into X. In order to present a unified theory we prefer here an axiomatic approach.

Let $\mathcal{P}$ be a linear space of functions mapping $[-r, 0]$ into X endowed with a seminorm $\| \cdot \|_{\mathcal{P}}$. We will assume that $\mathcal{P}$ satisfies the following axioms :

(A′) If $x : [\sigma - r, \sigma + a] \to X$, $a > 0$, is continuous on $[\sigma, \sigma + a]$ and $x_\sigma \in \mathcal{P}$ then for every t in $[\sigma, \sigma + a]$ the following conditions hold :

(i) x_t is in $\mathcal{P}$;

(ii) $\| x(t) \|_X \leq H' \| x_t \|_{\mathcal{P}}$;

(iii) $\| x_t \|_{\mathcal{P}} \leq K' \sup \{ \| x(s) \| : t' \leq s \leq t \} + M'(t - \sigma) \| x_\sigma \|_{\mathcal{P}}$

where $t' := \max \{t - r, \sigma\}$ and both H' as K' are positive constants; $M' : [0, \infty) \to [0, \infty)$ is a bounded function that vanishes on $[r, +\infty)$, and H', K' and M' are independent of $x(\cdot)$.

(A-1′) For the function $x(\cdot)$ considered in (A′), x_t is a $\mathcal{P}$-valued continuous function on $[\sigma, \sigma + a]$.

(B′) The space $\mathcal{P}$ is complete.

We will denote by $\hat{\mathcal{P}}$ the quotient Banach space $\mathcal{P}/\| \cdot \|_{\mathcal{P}}$ and, if $\varphi \in \mathcal{P}$ we write $\hat{\varphi}$ for the coset determined by φ.

In the sequel we call phase space for ARFDEr those spaces $\mathcal{P}$ which satisfy axioms (A′), (A-1′) and (B′). Clearly the spaces $\mathcal{P} := C([-r, 0]; X)$, endowed with the norm of uniform convergence, and $\mathcal{P} := L^p([-r, 0]; X)$, $1 \le p < \infty$, provided with the seminorm

$$|||\varphi|||_p := \left[\|\varphi(0)\|^p + \int_{-r}^{0} \|\varphi(\theta)\|^p \, d\theta \right]^{1/p}$$

satisfy the preceeding conditions.

We close this section with a result of existence of solutions for ARFDEr with finite delay, which includes the existence results presented in the already mentioned references. We consider the equation

$$\dot{x}(t) = Ax(t) + f(t, x_t), \quad t \ge 0 \qquad \text{... (2.5)}$$

with initial condition

$$x_0 = \varphi \in \mathcal{P} \qquad \text{... (2.6)}$$

where $f : [0, a] \times \mathcal{P} \to X$, $a > 0$, is a continuous function and x_t represents the function defined from $[-r, 0]$ into X by $x_t(\theta) = x(t + \theta)$, $-r \le \theta \le 0$.

We will say that a function $x : [-r, a] \to X$ is a (mild) solution of the Cauchy problem (2.5)-(2.6) if $x_0 = \varphi$ and the restriction $x : [0, a] \to X$ is continuous and satisfies the integral equation :

$$x(t) = T(t)\,\varphi(0) + \int_0^t T(t-s)\, f(s, x_s)\, ds, \qquad 0 \le t \le a. \qquad \text{... (2.7)}$$

Proceeding as in the demonstration carried out in Theorem 2, we can establish the following result.

Theorem 3 — Assume that f satisfies the Lipschitz condition

$$\| f(t, \varphi_1) - f(t, \varphi_2) \|_X \le L' \| \varphi_1 - \varphi_2 \|_{\mathcal{P}} \text{ for all } 0 \le t \le a,\ \varphi_1, \varphi_2 \in \mathcal{P}, \cdots \text{ (2.8)}$$

where L' is a positive constant. Then there exists a unique solution of (2.5)-(2.6) for every $\varphi \in \mathcal{P}$.

The existence of solutions of eqn. (1.1) has been established by Henríquez[6] under compactness conditions on the composition $T(\cdot) \circ F(\cdot)$. In particular, we have obtained the following result :

Proposition 1 — Let $F : [0, \infty) \times \mathcal{B} \to X$ be a locally Lipschitz continuous function which satisfies the condition

$$\| F(t, \varphi) \|_X \le N_1 \| \varphi \|_{\mathcal{B}} + N_2, \quad \varphi \in \mathcal{B}, \qquad \text{... (2.9)}$$

for some constants $N_1, N_2 \ge 0$. If $T(\cdot)$ is a compact semigroup then for each $\varphi \in \mathcal{B}$ there exists a unique solution of (1.1)-(1.2) defined on $(-\infty, \infty)$.

Furthermore, the same result, but substituting $\mathcal{B}$ by $\mathcal{P}$ and F by f in the above statement, it remains true for problem (2.5)-(2.6).

3. The Approximation Problem

In this section we study the problem of approximating the solutions of the abstract Cauchy problem (1.1)-(1.2) by solutions of ARFDE with finite delay of type (2.5)-(2.6).

Let $(r_n)_n$ be a monotonically increasing sequence of positive real numbers such that $r_n \to +\infty$ as $n \to +\infty$. For each $n \in \mathbb{N}$, we will denote by $\mathcal{P}_n$ a phase space for the ARFDEr_n. In this case the parameters H', K' and M' introduced in the definition of phase space for equations with finite delay will be denoted by H^n, K^n and M^n, respectively.

We consider the Cauchy problems

$$\dot{x}(t) = Ax(t) + F_n(t, x_t), \quad t \geq 0 \qquad \text{... (3.1)}$$

$$x_0 = \varphi^n \in \mathcal{P}_n \qquad \text{... (3.2)}$$

where $F_n : [0, a] \times \mathcal{P}_n \to X$ are continuous functions that satisfy appropriated hypotheses to guarantee existence and uniqueness of solutions. Next we set $S_n(\varphi^n) := \{u \in C([0, a]; X) : u(0) = \varphi^n(0)\}$ and, for $u \in S_n(\varphi^n)$ we denote by $\tilde{u} : [-r_n, a] \to X$ the function defined by $\tilde{u}(\theta) := \varphi^n(\theta)$, $-r_n \leq \theta \leq 0$ and $\tilde{u}(s) : u(s)$, $0 \leq s \leq a$.

To obtain our results we need to relate the phase space $\mathcal{B}$ with the spaces $\mathcal{P}_n$. For this reason we introduce the following terminology :

Definition 1 — We will say that a sequence $(\mathcal{P}_n, P_n)_n$ is an approximation scheme of $\mathcal{B}$ if there exist bounded linear operators $P_n : \mathcal{B} \to \mathcal{P}_n$ that satisfy the conditions

(AS-1) For every $\varphi \in \mathcal{B}$, $(P_n\varphi)(0) \to \varphi(0)$ as $n \to \infty$.

(AS-2) For each integer n there exist positive functions $\alpha_n, \beta_n : [0, \infty) \to \mathbb{R}$ such that $\alpha_n(\cdot)$ is continuous, $\beta_n(\cdot)$ is locally bounded and the following condition holds :

$$\| \tilde{v}_t - P_n \tilde{u}_t \|_{\mathcal{P}_n} \leq \alpha_n(t) \sup_{0 \leq s \leq t} \| v(s) - u(s) \|_X + \beta_n(t) \| \varphi^n - P_n \varphi \|_{\mathcal{P}_n}$$

for every $\varphi \in \mathcal{B}$, $\varphi^n \in \mathcal{P}_n$, $u \in S(\varphi)$, $v \in S_n(\varphi^n)$.

Now, we can state our first result on the approximation of the solution of the abstract Cauchy problem (1.1)-(1.2) by solutions of equations (3.1). To establish this result we need to relate the approximation scheme with the functions F and F_n. For this reason, we introduce the following condition :

Assumption (F) — For every $\varphi \in \mathcal{B}$ the following conditions hold :

(F-1) For each $u \in S(\varphi)$, there exists a positive and integrable function η such that

$$\| F_n (t, P_n \tilde{u}_t) \| \leq \eta(t), \quad 0 \leq t \leq a,$$

for every $n \in \mathbb{N}$, and

(F-2) $F_n (t, P_n \varphi)$ converges pointwise to $F(t, \varphi)$ on $[0, a]$.

In the results that follows, for a fixed $\varphi \in \mathcal{B}$, we represent by u the solution of (1.1)-(1.2) and by u^n the solution of (3.1)-(3.2), with $\varphi^n := P_n \varphi$.

Theorem 4 — Assume that $(\mathcal{P}_n, P_n)_n$ is an approximation scheme of $\mathcal{B}$. If each function F_n satisfies a Lipschitz condition with constant L_n, Assumption (F) holds and $(L_n \alpha_n)_n$ is a uniformly bounded sequence on $[0, a]$ then for every $\varphi \in \mathcal{B}$ the sequence $(u^n)_n$ converges uniformly to u on $[0, a]$.

PROOF : It is clear that for $0 \leq t \leq a$

$$u^n (t) - u(t) = T(t) (\varphi^n (0) - \varphi(0)) + \int_0^t T(t-s) [F_n (s, u_s^n) - F(s, u_s)]\, ds$$

$$= T(t) (\varphi^n (0) - \varphi(0)) + \int_0^t T(t-s) [F_n (s, u_s^n) - F_n (s, P_n u_s)]\, ds$$

$$+ \int_0^t T(t-s) [F_n (s, P_n u_s) - F(s, u_n)]\, ds.$$

Thus, from condition (AS-2) of Definition 1, we obtain that

$$\| u^n (t) - u(t) \|_X \leq N \| \varphi^n (0) - \varphi(0) \|_X + N \int_0^t \| F_n (s, P_n u_s) - F(s, u_s) \|\, ds$$

$$+ NL_n \int_0^t \alpha_n (s) \sup_{0 \leq \xi \leq s} \| u^n (\xi) - u(\xi) \|_X\, ds$$

where $N := \sup_{0 \leq t \leq a} \| T(t) \|$. Introducing the functions

$$w_n (t) := \sup_{0 \leq s \leq t} \| u^n (s) - u(s) \|_X$$

and

$$g_n (t) := N \| \varphi^n (0) - \varphi(0) \|_X + N \int_0^t \| F_n (s, P_n u_s) - F(s, u_s) \|\, ds$$

the Gronwall-Bellman's lemma, applied to the above inequality, yields that

$$w_n(t) \le g_n(t) + NL_n \int_0^t g_n(s)\, \alpha_n(s)\, e^{NL_n \int_s^t \alpha_n(\xi)\, d\xi}\, ds.$$

On the other hand, the Lebesgue's dominated convergence theorem and Assumption (F) imply that $g_n(t) \to 0$, as $n \to \infty$, uniformly on $[0, a]$. Using, now, the boundedness of $L_n \alpha_n(\cdot)$, we conclude that $w_n(t) \to 0$, as $n \to \infty$, uniformly on $[0, a]$, which completes the proof. ■

Henceforth, we always assume that $(\mathcal{P}_n, P_n)_n$ is an approximation scheme of $\mathcal{B}$. In our next result we consider a special situation of Theorem 4. To present this result we suppose the existence of a sequence of bounded linear operator $E_n : \mathcal{P}_n \to \mathcal{B}$ which satisfies the following conditions :

(E-1) There exists a constant $C \ge 0$ such that $\| E_n \| \le C$, for every $n \in \mathbb{N}$.

(E-2) For every $\varphi \in \mathcal{B}$, $\| \varphi - E_n P_n \varphi \|_{\mathcal{B}} \to 0$ as $n \to \infty$.

Now, we consider eqn. (3.1), with the functions F_n defined by

$$F_n(t, \varphi) := F(t, E_n \varphi), \; n \in \mathbb{N}. \qquad \text{... (3.3)}$$

Corollary 1 — Assume that conditions (E-1) and (E-2) hold. If F satisfies a Lipschitz condition with constant $L \ge 0$ and $(\alpha_n)_n$ is a uniformly bounded on $[0, a]$ sequence then for every $\varphi \in \mathcal{B}$ the sequence $(u^n)_n$ converges uniformly to u on $[0, a]$.

PROOF : Since for each $n \in \mathbb{N}$ the function F_n defined in (3.3) verifies a Lipschitz condition with constant $L_n := LC$, it is sufficient to show that condition (F) holds. Using (E-2) and the Uniform Boundedness Principle, we conclude that the set $\{E_n P_n : n \in \mathbb{N}\}$ is bounded in $\mathcal{L}(\mathcal{B})$. Thus, condition (F-1) follows from the estimate

$$\| F(t, E_n P_n u_t) \|_X \le L \, \| E_n P_n \| \, \| u_t \|_{\mathcal{B}} + \| F(t, 0) \|_X$$

and condition (F-2) is an immediate consequence of (E-2) and the continuity of F. ■

Next, we consider the linear nonhomogeneous ARFDE with unbounded delay

$$\dot{x}(t) = Ax(t) + \Lambda(x_t) + f(t), \quad t \ge 0 \qquad \text{... (3.4)}$$

$$x_0 = \varphi \in \mathcal{B} \qquad \text{... (3.5)}$$

and the linear nonhomogeneous ARFDEr_n

$$\dot{x}(t) = Ax(t) + \Lambda_n(x_t) + f_n(t), \quad t \ge 0 \qquad \text{... (3.6)}$$

$$x_0 = \varphi^n \in \mathcal{P}_n \qquad \text{... (3.7)}$$

where both $\Lambda : \mathcal{B} \to X$ as $\Lambda_n : \mathcal{P}_n \to X$, $n \in \mathbb{N}$ are bounded linear operators and f, $f_n : [0, \infty) \to X$ are continuous functions. The solutions of (3.4)-(3.5) are defined on

$\mathbb{R}$ while the solutions of (3.6)-(3.7) are defined on $[-r_n, +\infty)$. Furthermore, when we take $f := 0$ and $f_n := 0$, the operators $V(t)$ and $V_n(t)$, defined by the expressions

$$V(t)\varphi := x_t(\cdot, \varphi), \quad \varphi \in \mathcal{B},$$

$$V_n(t)\varphi^n := x_t(\cdot, \varphi^n), \quad \varphi^n \in \mathcal{P}_n,$$

are strongly continuous semigroups of linear operators on $\mathcal{B}$ and $\mathcal{P}_n$, respectively. The following results are immediate consequences of Theorem 4 :

Corollary 2 — Assume that $(\mathcal{P}_n, P_n)_n$ is an approximation scheme of $\mathcal{B}$ and that $(\|\Lambda_n\| \alpha_n)_n$ is a uniformly bounded sequence on each bounded interval. If both $f_n \to f$ and $\Lambda_n P_n \to \Lambda$, when $n \to \infty$, in the pointwise sense then for every $\varphi \in \mathcal{B}$ and $a > 0$ the sequence $(u^n)_n$ converges uniformly to u on $[0, a]$.

PROOF : From the Uniform Boundedness Principle we can conclude that $\{\Lambda_n P_n : n \in \mathbb{N}\}$ is a uniformly bounded set of linear operators, which implies that conditions of Theorem 4 hold. ■

If we take $f := 0$ and $f_n := 0$ for every $n \in \mathbb{N}$, from the previous corollary we infer the following result of convergence of semigroups :

Corollary 3 — Assume that $(\mathcal{P}_n, P_n)_n$ is an approximation scheme of $\mathcal{B}$, the sequence $(\|\Lambda_n\|)_n$ is bounded and that $(\alpha_n)_n$ is a uniformly bounded sequence on each bounded interval. If $\Lambda_n P_n \to \Lambda$, as $n \to \infty$, in the pointwise sense then $\| P_n V(t)\varphi - V_n(t) P_n \varphi \|_{\mathcal{P}_n} \to 0$ as $n \to \infty$, uniformly for t in bounded intervals and for each $\varphi \in \mathcal{B}$.

PROOF : From the condition (AS-2) of Definition 1, we know that

$$\| P_n V(t)\varphi - V_n(t) P_n \varphi \|_{\mathcal{P}_n} \le \alpha_n(t) \sup_{0 \le s \le t} \| u(s) - u^n(s) \|,$$

where u denotes the solution of (3.4)-(3.5) and u^n represents the solution of (3.6)-(3.7), with $\varphi^n := P_n \varphi$. The assertion follows from Corollary 2. ■

Let us now consider equations (3.6) with the operators $\Lambda_n := \Lambda \circ E_n$, for each $n \in \mathbb{N}$, where $E_n : \mathcal{P}_n \to \mathcal{B}$ are bounded linear operators that satisfy conditions (E-1) and (E-2).

Corollary 4 — Assume that $(\mathcal{P}_n, P_n)_n$ is an approximation scheme of $\mathcal{B}$ and that conditions (E-1) and (E-2) hold. If $(\alpha_n)_n$ is a uniformly bounded sequence on bounded intervals then $(E_n V_n(t) P_n)_n$ converges to $V(t)$, in the strong topology of operators and uniformly on bounded intervals.

PROOF : It is clear that

$$\| V(t)\varphi - E_n V_n(t) P_n \varphi \|_{\mathcal{B}} \le \| V(t)\varphi - E_n P_n V(t)\varphi \|_{\mathcal{B}} + \| E_n \| \, \| P_n V(t)\varphi - V_n(t) P_n \varphi \|_{\mathcal{P}_n}.$$

Since $(E_n)_n$ and $(\Lambda_n)_n$ are bounded sequences of linear operators, from Corollary 3, we deduce that the second term of the right hand side converges to zero, uniformly on $[0, a]$, for each $a > 0$. Furthermore, in view of $\{V(t)\,\varphi : 0 \le t \le a\}$ is a compact subset of $\mathcal{B}$, condition (E-2) implies that the first term of the right hand side also converges to zero uniformly on $[0, a]$. ■

4. Applications

In this section we will present some applications and examples of the preceeding results. Initially, we will study the asymptotic stability problem for linear equations. Hereafter, we assume that $(\mathcal{P}_n, P_n)_n$ is an approximation scheme for the phase space $\mathcal{B}$ and that $E_n : \mathcal{P}_n \to \mathcal{B}$ are bounded linear maps. Let us consider the linear ARFDE with infinite delay

$$\dot{x}(t) = Ax(t) + \Lambda(x_t), \quad t \ge 0 \qquad \ldots (4.1)$$

and the ARFDE with finite delay

$$\dot{x}(t) = Ax(t) + \Lambda_n(x_t), \quad t \ge 0 \qquad \ldots (4.2)$$

where $\Lambda_n := \Lambda \circ E_n$.

The stability properties of eqns. (4.2), when the phase space $\mathcal{P}_n$ is the space of continuous functions $C([-r_n, 0]; X)$, have been studied by Travis and Webb[12] and by Grabosch and Moustakas[3].

We remind here the following concepts of stability for operator functions :

Definition 2 — A strongly continuous operator function $U : \mathbb{R}^+ \to \mathcal{L}(X, Y)$ is said to be :

(a) Uniformly stable if $\| U(t) \| \to 0$, as $t \to \infty$.

(b) Strongly stable if $U(t)x \to 0$, as $t \to \infty$ for every $x \in X$.

Proposition 2 — Assume that conditions (E-1) and (E-2) hold and that the sequence $(\alpha_n)_n$ is uniformly bounded on bounded intervals. If the operator functions $V_n(\cdot)P_n$ are strongly stable, uniformly on $n \in \mathbb{N}$, then $V(\cdot)$ is also strongly stable.

Proof : The proof is an immediate consequence of the inequality

$$\| V(t)\,\varphi \|_{\mathcal{B}} \le \| E_n \| \, \| V_n(t) P_n \varphi \|_{\mathcal{P}_n} + \| V(t)\,\varphi - E_n P_n V(t)\,\varphi \|_{\mathcal{B}}$$
$$+ \| E_n \| \, \| P_n V(t)\,\varphi - V_n(t) P_n \varphi \|_{\mathcal{B}}$$

and of Corollary 3. ■

The property of strong stability, uniformly on $n \in \mathbb{N}$, for $V_n(\cdot)P_n$ is difficult to verify in concrete situations. For this reason we will study other conditions for the stability of solutions of eqn. (4.1).

Proposition 3 — Assume that the following conditions hold :

(a) There exists $\mu > 0$ such that $\| T(t) \| \le \tilde{M} e^{-\mu t}$ for all $t \ge 0$;

(b) the functions K and M are bounded and $\dfrac{\tilde{M}\,\|\Lambda\|\,\|K\|_\infty}{\mu} < 1.$

Then the solutions of eqn. (4.1) are bounded on $[0, \infty)$.

PROOF : Let $x(\cdot)$ be the solution of (4.1) with initial condition $x_0 := \varphi \in \mathcal{B}$. From (2.1), we can write that

$$x(t) = T(t)\,\varphi(0) + \int_0^t T(t-s)\,\Lambda(x_s)\,ds.$$

Therefore,

$$\|x(t)\| \le \tilde{M}\,\|\varphi(0)\| + \tilde{M}\,\|\Lambda\|$$

$$\times \int_0^t e^{-\mu(t-s)}\left[\|K\|_\infty \max_{0\le\xi\le s}\|x(\xi)\| + \|M\|_\infty\,\|\varphi\|_{\mathcal{B}}\right] ds$$

$$\le \tilde{M}\left(H + \frac{\|\Lambda\|\,\|M\|_\infty}{\mu}\right)\|\varphi\|_{\mathcal{B}} + \frac{\tilde{M}\,\|\Lambda\|\,\|K\|_\infty}{\mu}\max_{0\le s\le t}\|x(s)\|,$$

which implies that

$$\|x(t)\| \le \left(1 - \frac{\tilde{M}\,\|\Lambda\|\,\|K\|_\infty}{\omega}\right)^{-1}\tilde{M}\left(H + \frac{\|\Lambda\|\,\|M\|_\infty}{\mu}\right)\|\varphi\|$$

for every $t \ge 0$. ■

It is clear that conditions (a) and (b), mentioned above, are not sufficient to guarantee strong stability of the semigroup V. In fact, assume that the phase space $\mathcal{B}$ contains the constant functions and that there exists an element $u \in X$, $u \ne 0$, such that $Au + \Lambda(\bar{u}) = 0$, where $\bar{u}$ denotes the constant function everywhere equal to u. Then the function $x(t) := u$, $t \in \mathbb{R}$, is the solution of (4.1) with initial condition $x_0 = \bar{u}$.

In order to obtain strong stability of semigroup V we need a strengthen version of condition (E-2). Specifically, we consider the following property :

(E-3) If $x : \mathbb{R} \to X$ is a function such that $x_0 \in \mathcal{B}$ and $x|_{[0,\infty)}$ is continuous and bounded, then $\|x_s - E_n P_n x_s\|_{\mathcal{B}} \to 0$ as $n \to \infty$, uniformly on $s \ge 0$.

Proposition 4 — Assume that conditions (E-1) and (E-3) hold and that $\varphi \in \mathcal{B}$ is an element for which the solution $x(\cdot, \varphi)$ of (4.1) with initial condition $x(\cdot, \varphi) = \varphi$ is bounded on $[0, \infty)$. If the following properties are verified :

(a) There exists $\mu > 0$ such that $\|T(t)\| \le \tilde{M} e^{-\mu t}$ for all $t \ge 0$;

(b) the functions α_n are bounded and $\sup_{n\in\mathbb{N}} \dfrac{\tilde{M}\,\|\Lambda\|\,\|E_n\|\,\|\alpha_n\|_\infty}{\mu} < 1;$

(c) each semigroup V_n is strongly stable, then $x(t, \varphi) \to 0$, as $t \to \infty$.

PROOF : Let us abbreviate $x(t) := x(t, \varphi)$ and $x^n(t) := x^n(t, \varphi^n)$, where $\varphi^n := P_n \varphi$, for the solution of equation (4.2) with initial condition φ^n. Consequently, we can write

$$x(t) - x^n(t) = T(t)[\varphi(0) - \varphi^n(0)] + \int_0^t T(t-s)\, \Lambda(x_s - E_n x_s^n)\, ds$$

which, together with condition (AS-2), implies

$$\begin{aligned}\| x(t) - x^n(t) \| &\le \tilde{M} \| \varphi(0) - \varphi^n(0) \| \\ &\quad + \tilde{M} \| \Lambda \| \int_0^t e^{-\mu(t-s)} \| x_s - E_n P_n x_s \|_{\mathcal{B}}\, ds \\ &\quad + \tilde{M} \| \Lambda \| \| E_n \| \int_0^t e^{-\mu(t-s)} \| P_n x_s - x_s^n \|_{\mathcal{P}_n}\, ds \\ &\le \tilde{M} \| \varphi(0) - \varphi^n(0) \| + \tilde{M} \| \Lambda \| \int_0^t e^{-\mu(t-s)} \| x_s - E_n P_n x_s^n \|_{\mathcal{B}}\, ds \\ &\quad + \frac{\tilde{M} \| \Lambda \| \| E_n \| \| \alpha_n \|_\infty}{\mu} \max_{0 \le s \le t} \| x(s) - x^n(s) \|.\end{aligned}$$

The first term of the above right-hand side converges to zero by (AS-1) and, since $x(\cdot)$ is bounded on $[0, \infty)$, from condition (E-3), we obtain that the second term also converges to zero. Thus, we complete the proof using (b) and (c). ■

Corollary 5 — If the hypotheses of Proposition 3 and Proposition 4 hold then V is a strongly stable semigroup.

We consider now two special cases whenever the phase spaces $\mathcal{P}_n = C([-r_n, 0]; X)$.

Corollary 6 — Assume that the hypotheses of Proposition 3 and (a) and (b) in Proposition 4 hold. If $T(\cdot)$ is a compact semigroup and for each $n \in \mathbb{N}$ the solutions of the equation

$$\lambda x - Ax - \Lambda_n (e^{\lambda\theta} x) = 0, \quad x \in D(A), \quad x \ne 0,$$

satisfy the condition $\mathrm{Re}(\lambda) < 0$ then $V(\cdot)$ is strongly stable.

PROOF : From Proposition 4.1 in Travis and Webb[12], we know that each semigroup $V_n(\cdot)$ is uniformly stable. The assertion is now consequence of Corollary 5. ■

Next, we assume that X is a Banach lattice. We will represent by $\tilde{\Lambda}_n$ the operator defined in the form

$$\tilde{\Lambda}_n(x) := \Lambda \circ E_n(\bar{x}),$$

where $\bar{x}$ denotes the constant function $\bar{x}(\theta) := x$ for $-r_n \le \theta \le 0$. Moreover, if the operator B generates a semigroup, we indicate by $s(B)$ the spectral bound of B.

Corollary 7 — Assume that the hypotheses of Proposition 3 and Proposition 4 hold. Let $T(\cdot)$ be a positive semigroup on X and let Λ_n be a positive operator, for each $n \in \mathbb{N}$. If the semigroups $V_n(\cdot)$ are uniformly bounded and $s(A + \tilde{\Lambda}_n) < 0$ then $V(\cdot)$ is a strongly stable semigroup.

PROOF : Since $V_n(\cdot)$ is uniformly bounded, from Corollary B-IV, 3.8 and Theorem C-IV, in Nagel[9] we can conclude that $V_n(\cdot)$ is strongly stable. The assertion is, now, consequence of Corollary 5. ■

Next, we consider the problem of existence of periodic solutions of an ARFDE with infinite delay

$$\dot{x}(t) = Ax(t) + F(t, x_t), \quad t \ge 0 \qquad \text{... (4.3)}$$

where the function $F : [0, \infty) \times \mathcal{B} \to X$ is continuous and $F(t, \varphi)$ is ω-periodic in t. To study this problem we introduce an approximation scheme $(\mathcal{P}_n, P_n)_n$ for $\mathcal{B}$, where $\mathcal{P}_n$ is a space of functions from $[-n\omega, 0]$ into X and a family of ARFDE with bounded delay $r_n := n\omega$

$$\dot{x}(t) = Ax(t) + F_n(t, x_t), \quad t \ge 0 \qquad \text{... (4.4)}$$

where each function $F_n : [0, \infty) \times \mathcal{P}_n \to X$, $n \in \mathbb{N}$, is continuous and $F_n(t, \varphi^n)$ is ω-periodic in t. We will assume that for every $\varphi \in \mathcal{B}$ there exists a unique solution $x(\cdot, \varphi)$ of (4.3), with initial condition $x_0(\cdot, \varphi) = \varphi$ and defined on $\mathbb{R}$, and that a similar condition holds for each equation (4.4). Our result of existence of periodic solutions of (4.3) depends on the existence of periodic solutions of equations (4.4). For this reason initially we study the problem of existence of periodic solutions for an ARFDE of type (2.5) with delay $r = l\omega$, $l \in \mathbb{N}$, where the function $f(t, \varphi)$ is continuous and ω-periodic in t. Let $\mathcal{P}$ be a phase space of functions from $[-r, 0]$ into X. Next, we will assume that the conditions of existence and uniqueness of solutions hold and, for each $\varphi \in \mathcal{P}$ we will represent by $x(\cdot, \varphi)$ the solution of (2.5) with initial condition (2.6). We define the Poincaré map $Q_\tau : \mathcal{P} \to \mathcal{P}$, $\tau > 0$, by the expression

$$Q_\tau \varphi := x_\tau(\cdot, \varphi).$$

It is easy to see that if $Q_\omega \varphi = \varphi$ then $x(\cdot, \varphi)$ is an ω-periodic function from $[-r, \infty]$ into X. In fact, if $-r \le \theta \le 0$ then $x(\theta) = \varphi(\theta) = x_\omega(\theta) = x(\omega + \theta)$. Furthermore, if we define $y(t) := x(t + \omega)$ for $t \ge -r$ then, for each $t \ge 0$,

$$y(t) = T(t + \omega)\varphi(0) + \int_0^{t+\omega} T(t + \omega - s) f(s, x_s)\, ds$$

$$= T(t)\left[T(\omega)\,\varphi(0) + \int_0^{\omega} T(\omega - s)\, f(s, x_s)\, ds \right] + \int_0^t T(t-s)\, f(s+\omega, x_{s+\omega})\, ds$$

$$= T(t)\, x(\omega) + \int_0^t T(t-s)\, f(s, y_s)\, ds,$$

which yields that $y(\cdot)$ is a solution of eqn. (2.5) and since $y_0 = x_\omega = \varphi$, by the uniqueness of solution, we obtain that $y(t) = x(t + \omega) = x(t)$ for all $t \geq 0$. Similarly, it is easy to check that if $Q_{m\omega}\,\varphi = \varphi$ for some $m \in \mathbb{N}$, then $x(\cdot, \varphi)$ is $m\omega$-periodic from $[0, \infty)$ into X. Therefore, in order to obtain a $m\omega$-periodic solution, it is sufficient to prove the existence of a fixed point of the map $Q_{m\omega}$. Since for this purpose the continuity of $Q_{m\omega}$ is essential, our next result establishes a couple of conditions to assure the continuity of map Q_τ.

Proposition 5 — Each of the following conditions implies the continuity of the application Q_τ :

(a) The map f satisfies the Lipschitz condition (2.8).

(b) The semigroup $T(\cdot)$ is compact, the function f takes bounded sets into bounded sets and for each $\varphi \in \mathcal{P}$ there exists $\delta > 0$ such that $\{x_t(\cdot, \psi) : 0 \leq t \leq \tau,\ \psi \in B_\delta[\varphi]\}$ is included in a bounded and closed subset of $\mathcal{P}$.

PROOF : Let us fix $\varphi \in \mathcal{P}$ and let $(\varphi^k)_k$ be a sequence convergent to φ in $\mathcal{P}$. We represent by $x(\cdot)$, $x^k(\cdot)$ the solution of (2.5) with initial condition φ and φ^k, respectively. If we assume condition (a) then we can write

$$x(t) - x^k(t) = T(t)\,[\varphi(0) - \varphi^k(0)] + \int_0^t T(t-s)\,[f(s, x_s) - f(s, x_s^k)]\, ds$$

for all $0 \leq t \leq \tau$, which, using the axiom (A'-iii), yields that

$$\| x(t) - x^k(t) \| \leq N \| \varphi(0) - \varphi^k(0) \| + NL' \int_0^t \| x_s - x_s^k \|_{\mathcal{P}}\, ds$$

$$\leq N \| \varphi(0) - \varphi^k(0) \| + NL'K' \int_0^t \max_{0 \leq \xi \leq s} \| x(\xi) - x^k(\xi) \|\, ds + \tau NL' M_\tau' \| \varphi - \varphi^k \|_{\mathcal{P}},$$

where $N := \sup_{0 \leq t \leq \tau} \| T(t) \|$ and $M_\tau' := \sup_{0 \leq t \leq \tau} M'(t)$. Thus, the Gronwall-Bellman's lemma implies that

$$\max_{0 \le s \le t} \| x(s) - x^k(s) \| \to 0, \quad k \to \infty$$

and, turning to use axiom (A'-iii), we obtain that $\| x_\tau - x_\tau^k \|_{\mathcal{P}} \to 0$, as $k \to \infty$.

On the other hand, if condition (b) holds we can prove that the set $\{x^k(\cdot) : k \in \mathbb{N}\}$ is relatively compact in $C([0, \tau); X)$. In fact, it is clear that $\{\varphi^k(0) : k \in \mathbb{N}\}$ is relatively compact in X and we can assume that $\| \varphi - \varphi^k \|_{\mathcal{P}} \le \delta$. Furthermore, let $0 < t \le \tau$ be fixed and choosing $0 < \varepsilon < t$ enough small, we can write

$$x^k(t) = T(t)\, \varphi^k(0) + \int_0^t T(t-s) f(s, x_s^k)\, ds$$

$$= T(t)\, \varphi^k(0) + T(\varepsilon) \int_0^{t-\varepsilon} T(t-s-\varepsilon) f(s, x_s^k)\, ds$$

$$+ \int_{t-\varepsilon}^{t} T(t-s) f(s, x_k^k)\, ds.$$

In view of the first and second term of the right hand side of the above expression are included in a compact set and the norm of the third term is of order ε, we infer that the set $\{x^k(t) : k \in \mathbb{N}\}$ is relatively compact. Proceeding in similar way we can prove that the functions $x^k(\cdot)$, $k \in \mathbb{N}$, are equicontinuous on $[0, \tau]$, which completes the proof of our assertion. Hence, if $(\psi^n)_n$ is a subsequence of $(\varphi^n)_n$ then there exists a subsequence $(x(\cdot, \psi^{nk}))_k$ of $(x(\cdot, \psi^n))_n$ which converges to some function $u(\cdot) \in C([0, \tau]; X)$. Next, we represent by $\tilde{u}(\cdot)$ the extension of u defined by $\tilde{u}(\theta) := \varphi(\theta)$ for $\theta < 0$. From the axioms of the space $\mathcal{P}$, we obtain that $x_s(\cdot, \psi^{nk}) \to \tilde{u}_s$ as $k \to \infty$, for all $0 \le s \le \tau$. Since $\{x_s(\cdot, \psi^{nk}) : 0 \le s \le \tau,\ k \in \mathbb{N}\}$ is a bounded subset of $\mathcal{P}$ and f takes bounded sets into bounded sets, using the Lebesgue's dominated converges theorem for the integration in the sense of Bochner (see Marle[8]), we obtain that $u(\cdot) = x(\cdot, \varphi)$, which shows that $Q_\tau \psi^{n_k} \to Q_\tau \varphi$ as $k \to \infty$. Since $(\psi^n)_n$ was an arbitrary subsequence of $(\varphi^n)_n$, this proves that Q_τ is a continuous map.

■

Next we introduce the following condition :

Assumption (EBS) — There exists a closed, bounded and convex subset $E \subseteq \mathcal{P}$ such that for all $\varphi \in E$ there exists a unique solution $x(\cdot, \varphi)$ of problem (2.5)-(2.6) defined on $[-r, \infty)$ such that $x_\omega(\cdot, \varphi) \in E$ and the set $\{x_t(\cdot, \varphi) : 0 \le t \le \omega,\ \varphi \in E\}$ is bounded.

***Theorem** 5* — Let assume that T is a compact semigroup, the assumption (EBS) holds and f is a ω-periodic function which takes bounded subsets of $[0, \infty) \times E$ into bounded subsets of X. If

$$\inf_{0 < \sigma < \omega} M'(\omega - \sigma)\, [H' K' \sup_{0 \le t \le \sigma} \| T(t) \| + M'(\sigma)] < 1 \qquad \text{... (4.5)}$$

then the set of $\varphi \in E$ such that the solution $x(\cdot, \varphi)$ is ω-periodic is non-empty and compact.

We omit the proof of this result because it is very similar to that of Theorem 2 of Henríquez[6]. The essential idea is to prove that Q_ω is a condensing map which, by Sadovkii's Theorem[10], has a fixed point.

The preceeding result has the following consequences :

Corollary 8 — Assume that $T(\cdot)$ is a compact semigroup, the function f is locally Lipschitz continuous and ω-periodic and the following conditions are fulfilled :

(a) There exist constants $\tilde{M} \geq 1$ and $\mu > 0$ such that $\| T(t) \| \leq \tilde{M} e^{-\mu t}$ for $t \geq 0$;

(b) there exist positive constants N_1 and N_2 such that

$$\| f(t, \varphi) \| \leq N_1 \| \varphi \|_{\mathcal{P}} + N_2, \quad t \geq 0; \qquad \text{... (4.6)}$$

(c) $\tilde{M} K' N_1 e^{\mu r} < \mu$.

Then there exists $m \in \mathbb{N}$ such that the equation (2.5) has a $m\omega$-periodic solution on $[0, \infty)$.

PROOF : The existence and uniqueness of solutions of problem (2.5)-(2.6) follows from the remark at the end of section 2. It is clear that f takes bounded subsets of $[0, \infty) \times \mathcal{P}$ into bounded subsets of X. First we will prove that Assumption (EBS), with $m\omega$ in place of ω, is satisfied. Specifically, we will show that there exists $R > 0$ and $m \in \mathbb{N}$ such that $Q_{m\omega}(B_R[0]) \subseteq B_R[0]$ and the set $\{x_t(\cdot, \varphi) : 0 \leq t \leq m\omega, \varphi \in B_R[0]\}$ is bounded.

If $x(\cdot)$ denotes the solution of (2.5) with initial condition φ then from the relation

$$x(t) = T(t)\varphi(0) + \int_0^t T(t-s) f(s, x_s)\, ds$$

and using conditions (a) and (b), we can estimate the norm of x_t. In fact,

$$\| x(t) \| \leq \tilde{M} e^{-\mu t} \| \varphi(0) \| + \tilde{M} \int_0^t e^{-\mu(t-s)} (N_1 \| x_s \|_{\mathcal{P}} + N_2)\, ds.$$

Hence we obtain, for $t \geq r$

$$\| x_t \|_{\mathcal{P}} \leq K' \max_{t-r \leq s \leq t} \| x(s) \|$$

$$\leq K' \max_{t-r \leq s \leq t} [\tilde{M} e^{-\mu s} \| \varphi(0) \| + \tilde{M} \int_0^s e^{-\mu(s-\xi)} (N_1 \| x_\xi \|_{\mathcal{P}} + N_2)\, d\xi]$$

$$\leq K' \tilde{M} e^{\mu(r-t)} \| \varphi(0) \| + \frac{K' \tilde{M} N_2}{\mu} + K' \tilde{M} N_1 e^{\mu r} \int_0^t e^{-\mu(t-s)} \| x_s \|_{\mathcal{P}}\, ds$$

whereas, for $t \leq r$

$$\| x_t \|_{\mathcal{P}} \leq K' \max_{0 \leq s \leq t} \| x(s) \| + M'(t) \| \varphi \|_{\mathcal{P}}$$

$$\leq K' \tilde{M} \| \varphi(0) \| + \frac{K' \tilde{M} N_2}{\mu} + M'(t) \| \varphi \|_{\mathcal{P}}$$

$$+ K' \tilde{M} N_1 e^{\mu r} \int_0^t e^{-\mu(t-s)} \| x_s \|_{\mathcal{P}} \, ds.$$

Collecting these expressions, we infer that

$$\| x_t \|_{\mathcal{P}} \leq K' \tilde{M} \rho(t) \| \varphi(0) \| + \frac{K' \tilde{M} N_2}{\mu} + M'(t) \| \varphi \|_{\mathcal{P}}$$

$$+ K' \tilde{M} N_1 e^{\mu r} \int_0^t e^{-\mu(t-s)} \| x_s \|_{\mathcal{P}} \, ds$$

for every $t \geq 0$, where we have introduced $\rho(t) := e^{\mu(r-t)}$. Now, a simple calculation using the Gronwall-Bellman's lemma shows that

$$\| x_t \|_{\mathcal{P}} \leq K' \tilde{M} \rho(t) \| \varphi(0) \| + \frac{K' \tilde{M} N_2}{\mu} + M'(t) \| \varphi \|_{\mathcal{P}}$$

$$+ \nu \int_0^t \left[K' \tilde{M} \rho(s) \| \varphi(0) \| + \frac{K' \tilde{M} N_2}{\mu} + M'(s) \| \varphi \|_{\mathcal{P}} \right] e^{-(\mu - \nu)(t-s)} \, ds,$$

where $\nu := K' \tilde{M} N_1 e^{\mu r}$. Since $\rho(\cdot)$ is a function that is vanish at infinity, the preceeding inequality and condition (c) imply that we can write

$$\| x_t \|_{\mathcal{P}} \leq C_t \| \varphi \|_{\mathcal{P}} + C, \quad t \geq 0,$$

where C is a constant and $C_t \to 0$, as $t \to \infty$. This implies that the solution $x(\cdot)$ is defined on $[0, \infty)$. Furthermore, if we choose a constant $R \geq 2C$ and $\tau := m\omega$ such that $C_\tau \leq \frac{1}{2}$, we obtain the assertion.

On the other hand, we can assume that $\tau \geq 2r$ and, since $M'(t) = 0$, for $t \geq r$, then condition (4.5) holds with $m\omega$ instead of ω. This completes the proof. ■

Next, we consider the linear nonhomogeneous case. That is, we suppose that

$$f(t, \varphi) := L(t, \varphi) + h(t),$$

where $h(\cdot)$ is continuous and ω-periodic; $L(t, \cdot)$ is a bounded linear map for each $t \geq 0$ and $L(\cdot, \varphi)$ is a ω-periodic function in t. We put $N_1 := \sup_{0 \leq t \leq \omega} \| L(t, \cdot) \|$.

Corollary 9 — Assume that $T(\cdot)$ is a compact semigroup, the function f is locally Lipschitz continuous and that the following conditions hold :

(a) There exist constants $\tilde{M} \geq 1$ and $\mu > 0$ such that $\| T(t) \| \leq \tilde{M} e^{-\mu t}$ for $t \geq 0$;

(b) $\tilde{M} K' N_1 e^{\mu r} < \mu$.

Then there exists an ω-periodic solution of (2.5).

PROOF : It is clear that f satisfies condition (b) of Corollary 8, with the constant N_2 defined by $N_2 := \sup_{0 \leq t \leq \omega} \| h(t) \|$. Therefore, there exists a solution $x(\cdot)$ which is $m\omega$-periodic on $[0, \infty)$ for some $m \in \mathbb{N}$. Let us introduce the set

$$S := \overline{c(\{ \hat{x}_t (\cdot) : 0 \leq t \leq m\omega \})},$$

where c is used to denote the convex hull. Since $\{x_t (\cdot) : 0 \leq t \leq m\omega\}$ is compact in $\mathcal{P}$, then S is also a compact and convex subset of the Banach space $\hat{\mathcal{P}}$. On the other hand, if Q denotes the map $Q\varphi := x_\omega (\cdot, \varphi)$, then Q is a continuous and affine map. The last property is a consequence of the uniqueness of solutions of equation (2.5) with initial condition (2.6). Consequently, the map $\hat{Q}$ induced by Q on $\hat{\mathcal{P}}$ also verifies these properties, which implies that $\hat{Q}(S) \subseteq S$. Now, the Schauder's fixed point theorem assert that $\hat{Q}$ has a fixed point, so that Q also has a fixed point, which in turn implies that equation (2.5) has an ω-periodic solution. ■

It is clear that the constants C and C_t of Corollary 8 depends on the delay r. In concrete phase spaces $\mathcal{P}$ it is possible to establish this dependence in a explicit form more appropriated for our objectives. Next, we present an example of this situation.

Example 1 — Let $g : [-r, 0] \to \mathbb{R}$ be a positive continuous function and let $\mathcal{P}$ be the space of continuous functions from $[-r, 0]$ into X. We define the functional $\| \cdot \|_g$ on $\mathcal{P}$ by

$$\| \varphi \|_g := \sup_{-r \leq \theta \leq 0} \frac{\| \varphi(\theta) \|}{g(\theta)}.$$

It is easy to see that $\| \cdot \|_g$ is a norm and $\mathcal{P}$ is a phase space for the equation (2.5). Assume that $T(\cdot)$ is a compact semigroup, f is a continuous function and that the following conditions are fulfilled :

(a) There exist constants $\tilde{M} \geq 1$ and $\mu > 0$ such that $\| T(t) \| \leq \tilde{M} e^{-\mu t}$, for $t \geq 0$;

(b) there exist constants N_1 and N_2 such that

$$\| f(t, \varphi) \| \leq N_1 \| \varphi \|_g + N_2, \quad t \geq 0; \qquad \ldots (4.7)$$

(c) $\tilde{M} N_1 \| e^{-\mu\theta} \|_g < \mu$.

Then the initial value problem (2.5)-(2.6) has solution $x(\cdot) = x(\cdot, \varphi)$ defined on $[-r, \infty)$ and

$$\| x_t \|_g \le G_r(t) \| \varphi \|_g + \tilde{M} \| \varphi(0) \| e_r e^{-(\mu - \nu)t} + \frac{\tilde{M} N_2}{\mu - \nu} \| g^{-1} \|_\infty$$

$$+ \nu \| \varphi \|_g \int_0^t G_r(s) e^{-(\mu-\nu)(t-s)} ds, \qquad \dots (4.8)$$

where we have introduced the notations $e_r := \| e^{-\mu\theta} \|_g$; $\nu := \tilde{M} N_1 e_r$ and $G_r(t) := \sup_{-r \le \theta \le -t} \frac{g(t+\theta)}{g(\theta)}$ for $0 \le t \le r$, and $G_r(t) := 0$ for $t > r$.

Consequently, there exist $R > 0$ and $\tau > 0$ such that $Q_\tau(B_R[0]) \subseteq B_R[0]$ and the set $\{x_t(\cdot, \varphi) : 0 \le t \le \tau,\ \varphi \in B_R[0]\}$ is bounded in $\mathcal{P}$.

PROOF : First, we will prove that the solution $x(\cdot) := x(\cdot, \varphi)$ is defined in $[0, \infty)$ for all $\varphi \in \mathcal{P}$. In fact, from (4.7) we obtain

$$\| x(t) \| \le \tilde{M} e^{-\mu t} \| \varphi(0) \| + \tilde{M} \int_0^t e^{-\mu(t-s)} (N_1 \| x_s \|_g + N_2)\, ds$$

$$\le \tilde{M} e^{-\mu t} \| \varphi(0) \| + \frac{\tilde{M} N_2}{\mu} + \tilde{M} N_1 \int_0^1 e^{-\mu(t-s)} \| x_s \|_g\, ds.$$

On the other hand, the definition of the norm in $\mathcal{P}$ yields

$$\| x_t \|_g = \sup_{-r \le \theta \le 0} \frac{\| x(t+\theta) \|}{g(\theta)}.$$

Thus, if $t \ge r$ then

$$\| x_t \|_g \le \tilde{M} e^{-\mu t} \| \varphi(0) \| e_r + \frac{\tilde{M} N_2}{\mu} \left\| \frac{1}{g} \right\|_\infty + \tilde{M} N_1 e_r \int_0^t e^{-\mu(t-s)} \| x_s \|_g\, ds.$$

Moreover, when $0 \le t \le r$,

$$\| x_t \|_g = \max \left\{ \sup_{-r \le \theta \le -t} \frac{\| \varphi(t+\theta) \|}{g(\theta)},\ \sup_{-t \le \theta \le 0} \frac{\| x(t+\theta) \|}{g(\theta)} \right\}$$

and since $\displaystyle \sup_{-r \le \theta \le -t} \frac{\| \varphi(t+\theta) \|}{g(\theta)} \le \| \varphi \|_g G_r(t),$

then

$$\| x_t \|_g \le G_r(t) \| \varphi \|_g + \tilde{M} e^{-\mu t} \| \varphi(0) \| e_r + \frac{\tilde{M} N_2}{\mu} \| g^{-1} \|_\infty$$

$$+ \tilde{M} N_1 e_r \int_0^t e^{-\mu(t-s)} \| x_s \|_g\, ds,$$

for each $t \ge 0$. Now, a simple calculation using the Gronwall-Bellman's lemma shows (4.8). Consequently, in this case we can also write

$$\| x_t \|_g \leq C_t \| \varphi \|_g + C,$$

where C is a constant and $C_t \to 0$, as $t \to \infty$, and we conclude the demonstration as in the proof of Corollary 8. ■

Now, in Example 4, we consider a situation, where the constants C_t and C that arise in the previous example can be chosen independent of the delay r.

Next, we return to eqn. (4.4). Additionally to the already mentioned conditions about existence and uniqueness of solutions, we will assume that for each $n \in \mathbb{N}$ eqn. (4.4) has a ω-periodic solution $u^n(\cdot)$ with initial condition $u_0^n(\cdot) = \varphi^n \in \mathcal{P}_n$ and that the sequence $(u^n(\cdot))_n$ is uniformly bounded on $[0, \omega]$.

We introduce another conditions for the approximation scheme $(\mathcal{P}_n, P_n)_n$ and for the approximation of F by the sequence $(F_k)_k$:

(AS-3) For each $k \in \mathbb{N}$ there exists a positive constant C_k such that

$$\| \varphi^k - P_k \varphi \|_{\mathcal{P}_k} \leq C_k \max_{-k\omega \leq \theta \leq 0} \| \varphi^k(\theta) - \varphi(\theta) \|$$

for every continuous functions $\varphi^k \in \mathcal{P}_k$ and $\varphi \in \mathcal{B}$.

(F-3) The sequence $F_k(t, \varphi^k) \to F(t, \varphi)$ as $k \to \infty$ for each $\varphi \in \mathcal{B}$ and each sequence $(\varphi^k)_k$, with $\varphi^k \in \mathcal{P}_k$, such that $\| \varphi^k - P_k \varphi \|_{\mathcal{P}_k} \to 0$ as $k \to \infty$.

It is clear that condition (F-3) implies (F-2), while (F-2) implies (F-3) when the family $\{F_k(t, \cdot) : k \in \mathbb{N}\}$ is uniformly equicontinuous.

Now we can establish the main result of this part.

Theorem 6 — Assume that the following conditions are fulfilled :

(i) The phase space $\mathcal{B}$ satisfies axiom (C-2), the approximation scheme $(\mathcal{P}_n, P_n)_n$ verifies condition (AS-3) and the sequence $(C_k)_k$ is bounded.

(ii) The semigroup $T(\cdot)$ is compact and the function $F(t, \varphi)$ is ω-periodic in t.

(iii) Conditions (F-1) and (F-3) hold and for each $k \in \mathbb{N}$ the function $F_k(t, \varphi^k)$ is ω-periodic in t and takes bounded sets into bounded sets, uniformly on k.

Then there exists a ω-periodic solution of eqn. (4.3).

PROOF : We begin by showing that the set $\{u^k(\cdot) : k \in \mathbb{N}\}$ is relatively compact in $C([0, \omega]; X)$, where $u^k(\cdot)$ is the ω-periodic solution of $\dot{x}(t) = Ax(t) + F_k(t, x_t)$. By the Ascoli-Arzela's theorem, it is sufficient to prove that $\{u^k(t) : k \in \mathbb{N}\}$ is relatively compact in X for each $0 \leq t \leq \omega$ and that the functions $u^k(\cdot)$ are equicontinuous. Since u^k is ω-periodic, it follows that φ^k is continuous and ω-periodic and, employing condition (AS-3) with $\varphi := 0$, we infer that

$$\| u_s^k \|_{\mathcal{P}_k} \leq C_k \max_{0 \leq \xi \leq \omega} \| u^k(\xi) \| .$$

Consequently, from hypotheses (i) and (iii), we obtain that the sets

$\{\| u_s^k(\cdot) \|_{\mathcal{P}_k} : k \in \mathbb{N},\ 0 \le s \le \omega\}$ and $\{F_k(s, u_s^k) : k \in \mathbb{N},\ 0 \le s \le \omega\}$ are bounded. In order to prove the first assertion, we consider initially $t > 0$. If we take $\varepsilon > 0$ enough small, the boundedness properties already established, the compactness of $T(\cdot)$ and the following expression

$$u^k(t) = T(t)\,\varphi^k(0) + T(\varepsilon) \int_0^{t-\varepsilon} T(t-\varepsilon-s)\,F_k(s, u_s^k)\,ds$$

$$+ \int_{t-\varepsilon}^{t} T(t-s)\,F_k(s, u_s^k)\,ds$$

show that $\{u^k(t) : k \in \mathbb{N}\}$ is included in a compact set W_t. In view of $u^k(0) = u^k(\omega)$ the property also holds for $t = 0$. We turn now our attention to the equi-continuity of functions $u^k(\cdot)$. For this purpose, for $h > 0$, we write

$$u^k(t+h) - u^k(t) = (T(h) - I)\,[\,T(t)\,\varphi^k(0) + \int_0^t T(t-s)\,F_k(s, u_s^k)\,ds]$$

$$+ \int_t^{t+h} T(t+h-s)\,F_k(s, u_s^k)\,ds$$

$$= (T(h) - I)\,u^k(t) + \int_t^{t+h} T(t+h-s)\,F_k(s, u_s^k)\,ds.$$

Since we know that $u^k(t), k \in \mathbb{N}$, are included in the compact set W_t then $(T(h) - I)\,u^k(t) \to 0$, as $h \to 0$, uniformly on k. Furthermore, the second term of the right hand side of the above expression is of order of h, independent of k. Using a similar argument for $h < 0$, we can conclude that the functions $u^k(\cdot)$ are equi-continuous on $[0, \omega]$, which completes the proof of our initial assertion.

Consequently there exists a subsequence of $(u^k(\cdot))_k$, which be indicated by the same index, that converges uniformly to some continuous function $u(\cdot)$. Let us denote by φ and ψ^k the ω-periodic extension of $u(\cdot)$ and $u^k(\cdot)$ to $(-\infty, 0]$, respectively. It is clear that $\psi^k|_{[-k\omega, 0]} = \varphi^k$. Besides, in view of φ and ψ^k are continuous and bounded, from axiom (C-2), we infer that they are included in $\mathcal{B}$ and, since $\psi^k \to \varphi$ as $k \to \infty$, uniformly on $(-\infty, 0]$, the axiom (C-2) also implies that $\psi^k \to \varphi$ as $k \to \infty$ in the space $\mathcal{B}$.

Next, we will prove that $u(\cdot)$ is the solution of equation (4.3) with initial condition φ. It is easy to see, from the construction of $u(\cdot)$ and condition (AS-3), that $\| u_s^k - P_k u_s \|_{\mathcal{P}_k} \to 0$ as $k \to \infty$. Thus, from (F-3), we can assert that $F_k(t, u_t^k) \to F(t, u_t)$ as $k \to \infty$ on $t \in [0, \omega]$. In view of $F_k(t, u_t^k)$ are uniformly bounded, taking limit as $k \to \infty$ in the expression

$$u^k(t) = T(t)\,\varphi^k(0) + \int_0^t T(t-s)\,F_k(s, u_s^k)\,ds$$

and applying the Lebesgue's dominated convergence theorem we obtain that

$$u(t) = T(t)\,\varphi(0) + \int_0^t T(t-s)\,F(s, u_s)\,ds,$$

which shows that $u(\cdot)$ is a solution of (4.3) on $[0, \omega]$. Finally, using the periodicity properties of F, F_k and u^k we can prove that the ω-periodic extension of u to $[0, \infty)$ is a solution of (4.3). ∎

Next, we will study some examples of approximation schemes.

Example 2 — Let us consider as phase space $\mathcal{B}$ the space denoted by C_g^0 in Hino *et al.*[7]. Specifically, we assume that $g : (-\infty, 0] \to [0, \infty)$ is a continuous positive function which satisfies the following two conditions :

(g-1) The function $G(t) := \sup_{-\infty < \theta \le -t} \dfrac{g(t+\theta)}{g(\theta)}$ is bounded for $t \ge 0$.

(g-2) $g(\theta) \to \infty$, as $\theta \to -\infty$.

From the properties of g it follows that $1/g$ is bounded on $(-\infty, 0]$ and that $\dfrac{g(-r)}{g(\theta)} \le \| G \|_\infty$ for all $r \ge 0$ and $\theta \le -r$, where we have denoted by $\| \cdot \|_\infty$ the sup norm on $[0, \infty)$.

We define the set C_g^0 formed by all continuous functions $\varphi : (-\infty, 0] \to X$ such that $\dfrac{\| \varphi(\theta) \|}{g(\theta)} \to 0$, as $\theta \to -\infty$. We consider C_g^0 endowed with the norm

$$\| \varphi \|_g := \sup_{-\infty < \theta \le 0} \frac{\| \varphi(\theta) \|}{g(\theta)}.$$

Then C_g^0 satisfies axioms (A), (A-1), (B) and (C-2) (see Hino *et al.*[7], Theorem 1.3.6).

On the other hand, we will introduce the spaces $\mathcal{P}_n := C([-r_n, 0]; X)$ provided with the norm

$$\| \psi \|_g := \sup_{-r_n \le \theta \le 0} \frac{\| \psi(\theta) \|}{g(\theta)}.$$

It is clear that $\mathcal{P}_n$ is a phase for ARFDE with finite delay r_n. Moreover, we can choose the constants $H^n := g(0)$; $K^n := \| 1/g \|_\infty$ and the function

$$M^n(t) := \begin{cases} \sup_{-r_n \le \theta \le -t} \dfrac{g(t+\theta)}{g(\theta)}, & t < r_n, \\ 0, & t \ge r_n. \end{cases}$$

We consider the operators $P_n : \mathcal{B} \to \mathcal{P}_n$ and $E_n : \mathcal{P}_n \to \mathcal{B}$ defined by the expressions

$$(P_n\varphi)(\theta) := \varphi(\theta), \quad -r_n \le \theta \le 0,$$

$$(E_n \psi)(\theta) := \begin{cases} \psi(\theta), & -r_n \le \theta \le 0, \\ \psi(-r_n), & \theta < -r_n. \end{cases}$$

It is clear that $\| P_n \| \le 1$ and $\| E_n \| \le \| G \|_\infty$ for each $n \in \mathbb{N}$. On the other hand, $(P_n\varphi)(0) = \varphi(0)$ for all $n \in \mathbb{N}$ and, with the notations of Definition 1,

$$\| \tilde{v}_t - P_n \tilde{u}_t \|_{\mathcal{P}_n} = \sup_{-r_n \le \theta \le 0} \frac{\| \tilde{v}_t(\theta) - (P_n \tilde{u}_t)(\theta) \|}{g(\theta)}$$

so that, for $t \ge r_n$, we have

$$\| \tilde{v}_t - P_n \tilde{u}_t \|_{\mathcal{P}_n} = \sup_{-r_n \le \theta \le 0} \frac{\| v(t+\theta) - u(t+\theta) \|}{g(\theta)}$$

$$\le \| g^{-1} \|_\infty \sup_{t - r_n \le s \le t} \| v(s) - u(s) \|$$

whereas, if $t < r_n$, then

$$\| \tilde{v}_t - P_n \tilde{u}_t \|_{\mathcal{P}_n} = \max \left\{ \sup_{-t \le \theta \le 0} \frac{\| v(t+\theta) - u(t+\theta) \|}{g(\theta)}, \right.$$

$$\left. \sup_{-r_n \le \theta \le -t} \frac{\| \psi(t+\theta) - \psi(t+\theta) \|}{g(\theta)} \right\}.$$

Thus, we can write

$$\| \tilde{v}_t - P_n \tilde{u}_t \|_{\mathcal{P}_n} \le \| g^{-1} \|_\infty \max_{0 \le s \le t} \| v(s) - u(s) \| + M^n(t) \| \psi - P_n \varphi \|_{\mathcal{P}_n}$$

$$\le \| g^{-1} \|_\infty \max_{0 \le s \le t} \| v(s) - u(s) \| + G(t) \| \psi - P_n \varphi \|_{\mathcal{P}_n},$$

which implies that we can choose $\alpha_n(t) \equiv \| g^{-1} \|_\infty$ and $\beta_n(t) = G(t)$ for all $n \in \mathbb{N}$ and $t \ge 0$. This shows that $(\mathcal{P}_n, P_n)_n$ is an approximation scheme of $\mathcal{B}$. Furthermore, if $\varphi^n \in \mathcal{P}_n$ and $\varphi \in \mathcal{B}$ then

$$\| \varphi^n - P_n \varphi \|_{\mathcal{P}_n} \le \| g^{-1} \|_\infty \max_{-r_n \le \theta \le 0} \| \varphi^n(\theta) - \varphi(\theta) \|,$$

which in turn implies that $(\mathcal{P}_n, P_n)_n$ also satisfies axiom (AS-3). In addition, if $\varphi \in \mathcal{B}$ then

$$\| \varphi - E_n P_n \varphi \|_{\mathcal{B}} = \sup_{\theta \le 0} \frac{\| \varphi(\theta) - (E_n P_n \varphi)(\theta) \|}{g(\theta)}$$

$$= \sup_{\theta \le -r_n} \frac{\| \varphi(\theta) - \varphi(-r_n) \|}{g(\theta)}$$

$$\le \sup_{\theta \le -r_n} \frac{\| \varphi(\theta) \|}{g(\theta)} + \frac{\| \varphi(-r_n) \|}{g(-r_n)} \| G \|_\infty$$

converges to zero, as $n \to \infty$, so that the condition (E-2) is verified. Further, we mention that condition (E-3) is not satisfied in general. Nevertheless, if the function g possesses the following additional property :

(g-3) $G(t) \to 0$ as $t \to \infty$,

then (E-3) also holds. In order to prove this statement, let $x(\cdot)$ be a function such that $x_0 = \varphi \in \mathcal{B}$ and $x|_{[0,\infty)}$ is continuous and bounded. We can write

$$\| x_s - E_n P_n x_s \|_{\mathcal{B}} = \sup_{\theta \le 0} \frac{\| x(s+\theta) - (E_n P_n x_s)(\theta) \|}{g(\theta)}$$

$$= \sup_{\theta < -r_n} \frac{\| x(s+\theta) - x(s-r_n) \|}{g(\theta)}$$

If $0 \le s \le r_n$ then

$$\| x_s - E_n P_n x_s \|_{\mathcal{B}} = \sup_{\theta < -r_n} \frac{\| \varphi(s+\theta) - \varphi(s-r_n) \|}{g(\theta)}$$

$$\le \sup_{\theta < -r_n} \frac{\| \varphi(s+\theta) \|}{g(s+\theta)} \frac{g(s+\theta)}{g(\theta)} + \frac{\| \varphi(s-r_n) \|}{g(s-r_n)} \frac{g(s-r_n)}{g(-r_n)} \| G \|_\infty$$

$$\le \sup_{\theta < -r_n} \frac{\| \varphi(s+\theta) \|}{g(s+\theta)} G(s) + \frac{\| \varphi(s-r_n) \|}{g(s-r_n)} \| G \|_\infty G(s).$$

Similarly, if $r_n < s$ then

$$\| x_s - E_n P_n x_s \|_{\mathcal{B}}$$

$$\le \max \left\{ \sup_{-s \le \theta < -r_n} 2 \frac{\| x \|_\infty}{g(\theta)}, \sup_{\theta < -s} \left(\| \varphi \|_{\mathcal{B}} G(s) + \frac{\| x \|_\infty}{g(\theta)} \right) \right\}$$

$$\le \| \varphi \|_{\mathcal{B}} G(s) + 2 \| x \|_\infty \sup_{\theta < -r_n} \frac{1}{g(\theta)}.$$

From these inequalities, and using conditions (g-2) and (g-3) as well as the properties of $x(\cdot)$, it follows that (E-3) holds.

Example 3 — We consider now the phase space $\mathcal{B}$ denoted by $C_r \times L^1(g)$, $r \ge 0$, in Hino *et al.*[7], which is formed by all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable and $g\varphi$ is Lebesgue integrable on $(-\infty, -r)$, where $g : (-\infty, -r) \to \mathbb{R}$ is a positive

Lebesgue integrable function. The seminorm in $\mathcal{B}$ is defined by

$$\| \varphi \| := \sup \{\| \varphi(\theta) \| : -r \le \theta \le 0\} + \int_{-\infty}^{-r} g(\theta) \| \varphi(\theta) \| \, d\theta.$$

We will assume that g satisfies conditions (g-6) and (g-7) in the terminology of Hino *et al.*[7]. This means that g is integrable on $(-\infty, -r)$ and that there exists a nonnegative and locally bounded function G on $(-\infty, 0]$ such that

$$g(\xi + \theta) \le G(\xi)\, g(\theta)$$

for all $\xi \le 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure 0. In this case, $\mathcal{B}$ is a phase space which verifies axioms (A), (A-1), (B) and (C-2) (see Hino *et al.*[7], Theorem 1.3.8).

We introduce spaces $\mathcal{P}_n := C_r \times L^1(g)([-r_n, 0]; X), r < r_n$, which consists of all classes of functions $\varphi : [-r_n, 0] \to X$ which are continuous on $[-r, 0]$, Lebesgue-measurables and $g\varphi$ is Lebesgue integrable on $[-r_n, -r)$. We consider $\mathcal{P}_n$ endowed with the seminorm.

$$\| \varphi \|_{\mathcal{P}_n} := \sup \{\| \varphi(\theta) \| : -r \le \theta \le 0\} + \int_{-r_n}^{-r} g(\theta) \| \varphi(\theta) \| \, d\theta.$$

It is easy to see that $\mathcal{P}_n$ is a phase space, and that we can choose $H^n := 1$, $K^n := 1 + \int_{-r_n}^{-r} g(\theta)\, d\theta$ and the function

$$M^n(t) := \begin{cases} 1 + \int_{-r_n}^{-r} g(\theta)\, d\theta, & r_n - r \le t < r, \\ \max\left\{ 1 + \int_{-r}^{t-r} g(\theta - t)\, d\theta,\ G(-t) \right\}, & t < r_n - r < r, \\ \max\left\{ \int_{-r}^{0} g(\theta - t)\, d\theta, G(-t) \right\}, & r \le t < r_n - r, \\ \int_{-r_n}^{-t} g(\theta)\, d\theta, & r < r_n - r \le t \le r_n, \\ 0, & t > r_n. \end{cases}$$

In this case we define the operators $P_n : \mathcal{B} \to \mathcal{P}_n$ and $E_n : \mathcal{P}_n \to \mathcal{B}$ by means of

$$P_n \varphi := \varphi|_{[-r_n, 0]},$$

$$(E_n \psi)(\theta) := \begin{cases} \psi(\theta) & -r_n \le \theta \le 0, \\ 0, & \theta < -r_n. \end{cases}$$

From these definitions, it follows clearly that conditions (AS-1), (AS-2) and (AS-3) hold, with functions $\alpha_n = K^n$, $\beta_n(t) = M^n(t)$ and constants $C_n = 1 + \int_{-r_n}^{-r} g(\theta)\, d\theta$. Furthermore, $\| E_n \| \le 1$ and $\| \varphi - E_n P_n \varphi \|_{\mathcal{B}} = \int_{-\infty}^{-r_n} g(\theta) \| \varphi(\theta) \| \, d\theta \to 0$ as $n \to \infty$, so that the operators E_n satisfy conditions (E-1) and (E-2). In addition, if $G(\xi) \to 0$ as $\xi \to -\infty$, then (E-3) also holds.

Example 4 — We consider now the abstract Volterra integro-differential equation

$$\dot{x}(t) = Ax(t) + h(t, x(t)) + \int_{-\infty}^{t} C(t, s, x(s))\, ds, \quad t \ge 0 \qquad \dots (4.9)$$

where $C(t, s, x)$ is a continuous X-valued function defined on $[0, \infty) \times \mathbb{R} \times X$ and $h : [0, \infty) \times X \to X$ is continuous. To study existence of solutions of (4.9) we must specify an initial condition φ included in some appropriated phase space $\mathcal{B}$. In this case, defining the function

$$F(t, \varphi) := h(t, \varphi(0)) + \int_{-\infty}^{0} C(t, t + \theta, \varphi(\theta))\, d\theta, \quad t \ge 0,$$

equation (4.9) can be interpreted as a particular case of eqn. (4.3). The continuity properties of the function $F(\cdot)$ depend on $C(\cdot)$ and the choice of $\mathcal{B}$. Following Burton[1] we will choose as phase space $\mathcal{B} := C^0_g$, where $g : (-\infty, 0] \to [1, \infty)$ is a continuous nonincreasing function such that $g(0) = 1$ and which satisfies conditions (g-1), (g-2) and (g-3) stated in Example 2. A case of special interest occurs when

$$C(t, s, x) := C(t - s)x$$

where $C(\cdot)$ is a strongly continuous function of bounded linear operators on X and $h(t, x) := h(t)$ is a function independent of x. In this case

$$F(t, \varphi) = h(t) + \int_{-\infty}^{0} C(-\theta)\, \varphi(\theta)\, d\theta.$$

Henceforth, we will assume that $h(\cdot)$ is ω-periodic and that g has been chosen so that the following condition holds

$$N_1 := \int_{-\infty}^{0} \| C(-\theta) \| \, g(\theta)\, d\theta < \infty.$$

These conditions imply that $F(\cdot)$ is continuous and ω-periodic. We consider the approximation scheme introduced in Example 2, with $r_n := n\omega$, and the function F_n defined on $[0, \infty) \times \mathcal{P}_n$ by

$$F_n(t, \varphi^n) := h(t) + \int_{-n\omega}^{0} C(-\theta)\,\varphi^n(\theta)\,d\theta.$$

It is easy to see that condition (F-1) and (F-3) hold. On the other hand, if we assume that $\| T(t) \| \leq \tilde{M} e^{-\mu t}$ for every $t \geq 0$ and for some constant $\mu > 0$ such that $e^{-\mu\theta} \in \mathcal{B}$ and $\tilde{M} N_1 \| e^{-\mu\theta} \|_{\mathcal{B}} < \mu$ then conditions (a), (b) and (c) of Example 1 are satisfied. Furthermore, since for each $r := n\omega$, and using the notations of Example 1, $e_r \leq \| e^{-\mu\theta} \|_{\mathcal{B}}$ and $G_r(t) \leq G(t)$, introducing $\nu := \tilde{M} N_1 \| e^{-\mu\theta} \|_{\mathcal{B}}$ we infer, from (4.8), that

$$\begin{aligned} \| x_t \|_g \leq G(t) \| \varphi \|_g &+ \tilde{M} \| \varphi(0) \| \, \| e^{-\mu\theta} \|_g \, e^{-(\mu-\nu)t} \\ &+ \frac{\tilde{M} N_2 \nu}{\mu} e^{-(\mu-\nu)t} \\ &+ \nu \| \varphi \|_g \int_0^t G(s)\, e^{-(\mu-\nu)(t-s)}\, ds + \frac{\tilde{M} N_2}{\mu - \nu}. \end{aligned}$$

Thus

$$\| x_t \|_g \leq C_t \| \varphi \|_g + C,$$

where C_t and C are independent of n. If we suppose that T is a compact semigroup, from Corollary 9, we infer that each equation

$$\dot{x}(t) = Ax(t) + F_n(t, x_t), \quad t \geq 0,$$

has a ω-periodic solution $x^n(\cdot, \varphi^n)$, which are uniformly bounded. Thus, from Theorem 6 we deduce that eqn. (4.9) has an ω-periodic solution.

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REFERENCES

1. T. A. Burton, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, Orlando, 1985.
2. C. Corduneanu and V. Lakshmikantham, *Nonlinear Anal.* **4** (1980), 831-77.
3. A. Grabosch and U. Moustakas, *One-parameter Semigroups of positive Operators*. Lect. Notes in Maths. **1184**, Springer-Verlag, Berlin, 1986, pp. 219-32.
4. J. K. Hale and J. Kato, *Funkcial. Ekvac.* **21** (1978), 11-41.
5. J. Hale, *Asymptotic Behaviour of Dissipative Systems*, American Mathematical Society, Providence, 1988.
6. H. R. Henríquez, *Funkcial. Ekvac.* **37** (2) (1994), 329-43.

7. Y. Hino, S. Murakami and T. Naito, *Functional Differential Equations with Infinite Delay*, Lecture Notes in Mathematics **1473**, Springer-Verlag, Berlin, 1991.
8. C. Marle, *Measures et Probabilités*. Hermann, Paris, 1974.
9. R. Nagel (Editor), *One-parameter Semigroups of Positive Operators*. Lect. Notes in Maths. **1184**. Springer-Verlag, Berlin, 1986.
10. B. N. Sadovskii, *Funct. Anal. Appl.* **1** (1967), 74-76.
11. J. S. Shin, *Funkcial. Ekvac.* **30** (1987), 19-29.
12. C. C. Travis and G. F. Webb, *Trans. Am. Math. Soc.* **200** (1974), 395-418.
13. M. Zima, *Bull. Austral. Math. Soc.* **46** (1992), 179-86.