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# TWO ALMOST FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN TOPOLOGICAL VECTOR SPACES

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Using the KKM principle of Ky Fan we prove two almost fixed point theorems in not necessarily locally convex topological vector spaces.

## 1. INTRODUCTION

Some fixed point and almost fixed point theorems in not necessarily locally convex topological vector spaces are proved in Hadžić[3-6].

If $X$ is a topological vector space with the fundamental system of neighbourhoods of zero $\mathcal{V}$, $\phi \neq D \subset X$ and $f : D \to \mathcal{P}(X)$, we say that $f$ has a $V$-almost fixed point $(V \in \mathcal{V})$ if and only if there exists $y \in D$ such that $f(y) \cap (y + V) \neq \phi$.

An almost fixed point theorem in not necessarily locally convex topological vector spaces is proved in Hadžić[3], where we assumed that $f : D \to \mathcal{P}(D)$ is uniformly $u$-continuous and that the following condition is satisfied :

For every neighbourhood of zero $V$ in $X$ there exists a neighbourhood of zero $U_V$ in $X$ such that

$$co(U_V \cap (f(D) - f(D))) \subseteq V \qquad \text{... (1)}$$

and that $D$ is a paracompact convex subset of a topological vector space $X$ ($co$ is the convex hull operation). If (1) holds we say that $f(D)$ is of Zima's type Hadžić[6].

Let $(X, \|\cdot\|)$ be a paranormed space i.e. $X$ is a vector space over $\mathbf{R}$ and $\|\cdot\| : X \to [0, \infty)$ such that

1. $\|x\| = 0 \iff x = 0.$
2. For every $x \in X$, $\|-x\| = \|x\|$.
3. For every $x, y \in X$, $\|x + y\| \leq \|x\| + \|y\|$.

4. If $\lambda_n, \lambda \in \mathbf{R}$ $(n \in \mathbf{N})$ and $x_n, x \in X (n \in \mathbf{N})$ such that $|\lambda_n \to \lambda| \to 0$, $\| x_n - x \| \to 0$, $n \to \infty$ then $\| \lambda_n x_n - \lambda x \| \to 0$, $n \to \infty$.

It is well known that $(X, \| \cdot \|)$ is a metrizable topological vector space, where $\mathcal{V} = \{V_\varepsilon\}_{\varepsilon > 0}$, $V_\varepsilon = \{x;\ x \in X,\ \| x \| < \varepsilon\}$, is the fundamental system of neighbourhoods of zero in $X$.

If $\phi \neq K \subset X$, and $(X, \| \cdot \|)$ is a paranormed space then a sufficient condition for :

$$co\,(U_V \bigcap (K - K)) \subseteq V,$$

where $V \in \mathcal{V}$, $U_V \in \mathcal{V}$, is the following condition :

There exists $C(K) > 0$ such that :

$$\| \lambda(u - v) \| \leq C(K)\, \lambda \, \| u - v \|,$$

for every $\lambda \in [0, 1]$ and every $u, v \in K$.

*Example* — Let $S(0, 1)$ be the space of all the equivalence classes of measurable functions $\tilde{x} : [0, 1] \to \mathbf{R}$ and :

$$\| \tilde{x} \| = \int_0^1 \frac{|x(t)|}{1 + |x(t)|}\, \mu(dt),\ \tilde{x} \in S(0, 1),\ \{x(t)\} \in \tilde{x}.$$

Then $(S(0, 1), \| \cdot \|)$ is a paranormed space and the convergence in $S(0, 1)$ is equivalent to the convergence in the measure. If $\alpha > 0$ and

$$K_\alpha = \{\tilde{x};\ \tilde{x} \in S(0, 1),\ |x(t)| \leq \alpha,\ t \in [0, 1]\}$$

then for every $\tilde{x}, \tilde{y} \in K_\alpha$ and every $s \in [0, 1]$

$$\| s(\tilde{x} - \tilde{y}) \| \leq (1 + 2\alpha)\, s \, \| \tilde{x} - \tilde{y} \|.$$

Hence, $C(K_\alpha) = 1 + 2\alpha$.

In this paper we shall use the following two theorems on nonempty intersections. Theorem A is proved by Fan[2] and Theorem B by Horvath[7].

*Theorem* A — Let $X$ be a topological vector space, $\phi \neq D \subset X$ and $G : D \to \mathcal{P}(X)$ such that $G(x)$ is closed for every $x \in D$. If $G$ is a KKM mapping i.e. for every subset $\{x_1, x_2, ..., x_n\} \subseteq D$

$$co\{x_1, x_2, ..., x_n\} \subseteq \bigcup_{i=1}^{n} G(x_i),$$

then the family $\{G(x)\}_{x \in D}$ has the finite intersection property. If there exists $x_0 \in D$ such that $G(x_0)$ is compact then

$$\bigcap_{x \in D} G(x) \neq \phi.$$

*Theorem* B — Let $(X, d)$ be a complete metric space and $\{F_i\}_{i \in I}$ be a family of closed subsets of $X$ having the finite intersection property. If $\inf_{i \in I} \alpha(F_i) = 0$, where $\alpha$ is the Kuratowski measure of noncompactness, then

$$\bigcap_{i \in I} F_i$$

is nonempty and compact.

Let $X$ be a topological space, $\phi \neq D \subset X$ and $f : D \to \mathcal{P}(X)$. For every $U \subseteq X$ let

$$f^{+}(U) = \{x;\ x \in D,\ f(x) \subseteq U\}$$

$$f^{-}(U) = \{x;\ x \in D,\ f(x) \bigcap U \neq \phi\}.$$

A mapping $f : D \to \mathcal{P}(X)$ is upper semicontinuous on $D$ iff for every open $U$, $f^{+}(U)$ is open and lower semicontinuous iff for every open $U$, $f^{-}(U)$ is open.

A mapping $f : D \to \mathcal{P}(X)$ is continuous iff it is upper and lower semicontinuous.

By $\mathcal{P}_{co}(X)$ we shall denote the family of all nonempty, convex subsets of $X$.

*Theorem* 1 — Let $X$ be a complete metrizable topological vector space, $K$ a nonempty convex and closed subset of $X$ and $f : K \to \mathcal{P}_{co}(X)$ a lower semicontinuous mapping such that $f(x) \bigcap K \neq \phi$, for every $x \in K$. If for every $V \in \mathcal{V}$ (the family of all neighbourhoods of zero in $X$) there exists $U_V \in \mathcal{V}$ such that

$$co(U_V \bigcap (K - f(K))) \subseteq V$$

and for every $V \in \mathcal{V}$

$$\inf_{x \in K} \alpha\,[f^{+}((x + V)^c)] = 0$$

then $f$ has a $V$-almost fixed point for every $V \in \mathcal{V}$.

PROOF : We shall assume that every $V$ in $\mathcal{V}$ is open and symmetric. It will be proved that for every $V \in \mathcal{U}$ there exist $x_V \in K$ and $y_V \in f(x_V)$ such that

$$x_V - y_V \in V.$$

Suppose, on the contrary, that there exists $V \in \mathcal{V}$ such that

$$x - z \notin V, \text{ for every } x \in K \text{ and every } z \in f(x). \qquad \text{... (2)}$$

Let $G_U : K \to \mathcal{P}(K)$ be defined in the following way.

$$G_U(x) = \{y;\ y \in K,\ x - z \notin U, \text{ for every } z \in f(y)\},$$

where $U \in \mathcal{V}$ is such that

$$co(U \cap (K - f(K))) \subseteq V.$$

Then $G_U(x)$ is closed in $K$, for every $x \in K$, since

$$G_U^c(x) = \{y;\ y \in K,\ f(y) \cap (x + U) \neq \phi\} = f^-(x + U)$$

and $f$ is lower semicontinuous. We shall prove that $G_U$ is a KKM mapping. If it is not the case, there exists $\{x_1, x_2, \ldots, x_n\} \subseteq K$ such that

$$co\{x_1, x_2, \ldots, x_n\} \nsubseteq \bigcup_{i=1}^{n} G_U(x_i).$$

Then there exists $u \in co\{x_1, x_2, \ldots, x_n\}$ such that $u \notin G_U(x_i)$, for every $i \in \{1, 2, \ldots, n\}$. Then

$$u = \sum_{i=1}^{n} \lambda_i x_i,\ \lambda_i \geq 0,\ i \in \{1, 2, \ldots, n\},\ \sum_{i=1}^{n} \lambda_i = 1$$

and $x_i - z_i \in U$, for some $z_i \in f(u), i \in \{1, 2, \ldots, n\}$. Then

$$\sum_{i=1}^{n} \lambda_i (x_i - z_i) \in co(U \cap (K - f(K))) \subseteq V$$

which implies that $u - z \in V,\ z = \sum_{i=1}^{n} \lambda_i z_i \in f(u)$.

But, this contradicts to (2).

Since $G_U(x) = f^+((x + U)^c)$ and

$$\inf_{x \in K} \alpha[G_U(x)] = \inf_{x \in K} \alpha[f^+((x + U)^c)] = 0$$

from Horvath's theorem it follows that

$$\bigcap_{x \in K} G_U(x) \neq \phi.$$

If $y_0 \in G_U(x)$, for every $x \in K$ then $x - z \notin U$, for every $x \in K$ and $z \in f(y_0)$.

Let $z_0 \in f(y_0) \cap K$ and $x = z_0$. Then

$$x - z_0 = z_0 - z_0 = 0 \notin U$$

which is a contradiction. Hence, for every $V \in \mathcal{V}$, there exists $x_V \in K$ and $y_V \in f(x_V)$ such that $x_V - y_V \in V$.

*Corollary* 1 — Let $(X, \|\ \|)$ be a complete paranormed space, $K$ a nonempty convex and closed subset of $E$ and $f : K \to \mathcal{P}_{co}(X)$ a lower semicontinuous mapping such that $f(x) \cap K \neq \phi$, for every $x \in K$.

If there exists $C(\tilde{K}), \tilde{K} = K \bigcup f(K)$, such that

$$\| \lambda(u-v) \| \leq C(\tilde{K}) \lambda \| u-v \|,$$

for every $\lambda \in [0, 1]$ and $u, v \in \tilde{K}$, and for every $\varepsilon > 0$ :

$$\inf_{x \in K} \alpha [f^{+} ((x + V_{\varepsilon})^{c})] = 0$$

then $f$ has a $V_{\varepsilon}$-almost fixed point, for every $\varepsilon > 0$.

*Theorem* 2 — Let $X$ be a topological vector space, $\mathcal{V}$ the family of open neighbourhoods of zero in $X$, $K$ a nonempty, convex precompact subset of $X$ and $f : K \rightarrow \mathcal{P}_{co}(X)$ a lower semicontinuous mapping such that for every $x \in K, f(x) \bigcap K \neq \phi$. If $K \bigcup f(K)$ is of Zima's type then $f$ has a $V$-almost fixed point, for every $V \in \mathcal{V}$.

PROOF : Let $V \in \mathcal{V}$. We shall prove that there exists $y \in K$ such that $f(y) \bigcap (y + V) \neq \phi$. Let $U \in \mathcal{V}$ be such that $co(U \bigcap (K - f(K))) \subseteq V$ and for every $x \in K$ :

$$G(x) = \{y;\ y \in K,\ f(y) \bigcap (x + U) = \phi\}.$$

Since $f$ is lower semicontinuous $G(x)$ is closed in $K$, for every $x \in K$. Since $K$ is precompact, there exists $\{x_1, x_2, ..., x_n\} \subset K$ such that

$$K \subseteq \bigcup_{i=1}^{n} \{x_i + U\}.$$

For every $y \in K$, $f(y) \bigcap K \subseteq f(y) \bigcap \left( \bigcup_{i=1}^{n} \{x_i + U\} \right) \neq \phi$ and so $\bigcap_{i=1}^{n} G(x_i) = \phi$, which implies the existence of a subset $\{z_1, z_2, ..., z_m\} \subset K$ such that

$$co\{z_1, z_2, ..., z_m\} \not\subset \bigcup_{i=1}^{m} G(z_i).$$

Hence, there exists an $y \in co\ \{z_1, z_2, ..., z_m\}$ such that $y \notin G(z_i)$, $i \in \{1, 2, ..., m\}$. Let $y = \Sigma_{i=1}^{m} \lambda_i z_i$, $\Sigma_{i=1}^{m} \lambda_i = 1$, $\lambda_i \geq 0$, $i \in \{1, 2, ..., m\}$. From relations $f(y) \bigcap \{z_i + U\} \neq \phi$, $i \in \{1, 2, ..., m\}$ it follows the existence of $u_i \in f(y) \bigcap \{z_i + U\}$, $i \in \{1, 2, ..., m\}$ and $w_i \in U$, $i \in \{1, 2, ..., m\}$ such that $u_i = z_i + w_i \in f(y)$. Then $w_i \in U \bigcap (f(K) - K)$, $i \in \{1, 2, ..., m\}$ and so

$$\sum_{i=1}^{m} \lambda_i u_i \in y + co(U \bigcap (f(K) - K)) \subseteq y + V.$$

Since $f(y)$ is convex it follows that $\Sigma_{i=1}^{m} \lambda_i u_i \in f(y) \cap (y + V)$.

***Remark*** : It is obvious that if $f$ is continuous and $K$ is closed then there exists at least one fixed point of $f$.

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