Indian J. pure appl. Math., 27(4) : 393-405, April 1996

A BITOPOLOGICAL VIEW OF QUASI-TOPOLOGICAL GROUPS

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(*Received* 23 *December* 1994; *accepted* 3 *November* 1995)

A quasi-bitopological group is a triple (G, T, T^{-1}) such that (G, T) is a quasi-topological group and T^{-1} is the conjugate topology of T, i.e., $T^{-1} = \{A \subseteq G : A^{-1} \in T\}$. We prove, among other results, the following : (1) Each quasi-bitopological group is quasi-uniformizable; (2) Each first countable quasi-bitopological group admits a compatible left invariant quasi-pseudometric; (3) If (G, T, T^{-1}) is a 2-Hausdorff quasi-bitopological group, there is a 2-Hausdorff quasi-bitopological group which is bicomplete in its two-sided quasi-uniformity and has G as a 2-dense quasi-bitopological subgroup.

1. INTRODUCTION

Quasi-topological groups [paratopological groups in the terminology of Bourbaki[1], p. 297] have been investigated by several authors. They were mainly interested in obtaining conditions under which a quasi-topological group is a topological group. Relevant contributions in this direction may be found in the works of Numakura[14], Wallace[20], Ellis[4, 5], Zelazko[22], Mukherjea and Tserpes[13], Fletcher and Lindgren[6], Raghavan and Reilly[16, 17], Brand[2], Pfister[15], Grant[8], etc.

However, the fact that the topology T of a quasi-topological group (G, T) generates a conjugate topology $T^{-1} = \{A \subseteq G : A^{-1} \in T\}$ such that the map $x \rightarrow x^{-1}$ is a homeomorphism of (G, T) to (G, T^{-1}) (see Raghavan and Reilly[17]) suggests the question of investigating quasi-topological groups from "a bitopological view point". The purpose of this note is to start a systematized study of quasi-bitopological groups. We will show that this class of structures provides several satisfactory results and permits us to obtain appropriate extensions of classical theorems on topological groups. For instance we prove that every quasi-bitopological group is quasi-uniformizable and that every first countable quasi-bitopological group admits a compatible left invariant quasi-pseudometric. We also show that for each 2-Hausdorff

The authors thank the financial support of the Conselleria de Educació i Ciencia de la Generalitat Valenciana, grant GV2223/94.

quasi-bitopological group (G, T, T^{-1}) there is a 2-Hausdorff quasi-bitopological group which is bicomplete in its two-sided quasi-uniformity and has G as a 2-dense quasi-bitopological subgroup; an analogous result is also obtained for quasi-pseudometric quasi-bitopological groups. Finally we show that a 2-Hausdorff quasi-pseudometrizable quasi-bitopological group is bicompletely quasi-pseudometrizable if and only if its two-sided quasi-uniformity is bicomplete.

2. Definitions and Basic Properties

Let $(G, \cdot)$ be a group. As usual, e denotes the identity element of G and, for each $x \in G$, x^{-1} denotes the inverse of x. If $x, y \in G$ we will write xy instead of $x \cdot y$ and G instead of $(G, \cdot)$ if no confusion results. For $A, B \subseteq G$ we write $AB = \{ab : a \in A,\ b \in B\}$ and $A^{-1} = \{a^{-1} : a \in A\}$.

A quasi-topological group[6, 16, 17] is a pair (G, T) such that G is a group and T is a topology on G such that the function $\phi : (G \times G, T \times T) \to (G, T)$ defined by $\phi(x, y) = xy$, is continuous. If in addition the function $\theta : (G, T) \to (G, T)$ defined by $\theta(x) = x^{-1}$, is continuous, then (G, T) is called a topological group (Good references to the study of topological groups are Comfort[3], Hewitt and Ross[9] and Wilansky[21]).

If (G, T) is a quasi-topological group, then so is (G, T^{-1}) where $T^{-1} = \{A \subseteq G : A^{-1} \in T\}$ is called the conjugate topology of T. Clearly, the map $x \to x^{-1}$ is a homeomorphism of (G, T) to (G, T^{-1}) (Raghavan and Reilly[17], p. 748).

Definition 1 — A quasi-bitopological group is an ordered triple (G, T, T^{-1}) such that (G, T) is a quasi-topological group and T^{-1} is the conjugate topology of T.

It easily follows from this definition that for any quasi-bitopological group (G, T, T^{-1}) the functions

(i) $\phi_1 : (G \times G, T^{-1} \times T^{-1}) \to (G, T^{-1})$ defined by $\phi_1(x, y) = xy$,

(ii) $\phi_2 : (G \times G, T \times T^{-1}) \to (G, T)$ defined by $\phi_2(x, y) = xy^{-1}$

and

(iii) $\phi_3 : (G \times G, T \times T^{-1}) \to (G, T^{-1})$ defined by $\phi_3(x, y) = x^{-1}y$

are continuous.

Example 1 — Let + be the usual additive law on $\mathbb{R}$ and let $u = \{]-\infty, a[: a \in \mathbb{R}\}$ be the so called uppertopology on $\mathbb{R}$. Then $(\mathbb{R}, +, u, u^{-1})$ is a quasi-bitopological group such that $(\mathbb{R}, +, u \vee u^{-1})$ is the usual topological group on $\mathbb{R}$.

Example 2 — Let + as in Example 1 and let $(\mathbb{R}, S)$ be the Sorgenfrey line (basic open sets of S are of the form $[x, a[$, $x < a$). Then $(\mathbb{R}, +, S, S^{-1})$ is a quasi-bitopological group).

Example 3 — Let $\cdot$ be the usual multiplicative law on $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and S^+ the restriction of the Sorgenfrey line to $\mathbb{R}^+$. Then $(\mathbb{R}^+, \cdot, S^+, (S^+)^{-1})$ is a quasi-bitopological group.

Let G be a group and let T be a topology on G. Then the T-neighbourhood system of each $x \in G$ will be denoted by $\eta(x)$ and the T^{-1}-neighbourhood system by

$\eta^{-1}(x)$. By T^* we will denote the coarsest topology finer than T and T^{-1}, i.e., $T^* = T \vee T^{-1}$. The T^*-neighbourhood system of each $x \in G$ will be denoted by $\eta^*(x)$. In particular, we write η, η^{-1} and η^* instead of $\eta(e)$, $\eta^{-1}(e)$ and $\eta^*(e)$, respectively.

Let (G, T, T^{-1}) be a quasi-bitopological group and $a \in G$. Consider the map $L_a : G \to G$ defined by $L_a(x) = ax$. Similarly to the classical case (Wilansky[21], p. 241), L_a is called left translation by a. The map $R_a : G \to G$ defined by $R_a(x) = xa$ is then called right translation by a.

It is easy to see that for each $a \in G$, both L_a and R_a are homeomorphisms of the topological space (G, T) onto itself. Clearly, they are also homeomorphisms of (G, T^{-1}) onto itself.

From these observations one obtains, similarly to the classical case, the following.

Proposition 1 — Let (G, T, T^{-1}) be a quasi-bitopological group, $x \in G$ and $U \subseteq G$. Then :

(i) $U \in \eta(x) \Leftrightarrow xV \subseteq U$ for some $V \in \eta \Leftrightarrow Wx \subseteq U$ for some $W \in \eta$

(ii) $U \in \eta^{-1}(x) \Leftrightarrow xV^{-1} \subseteq U$ for some $V \in \eta \Leftrightarrow W^{-1}x \subseteq U$ for some $W \in \eta$.

The next result will be useful later on.

Proposition 2 — Let (G, T, T^{-1}) be a quasi-bitopological group. Then (G, T^*) is a topological group.

PROOF : Let $x, y \in G$ and let A be a T^*-neighbourhood of xy. Then there exists a T-neighbourhood U of xy that $U \cap U^{-1} \subseteq A$. Therefore there exists a $T \times T$-neighbourhood (V_1, W_1) of (x, y) and a $T^{-1} \times T^{-1}$-neighbourhood (V_2, W_2) of (x, y) such that $V_1 W_1 \subseteq U$ and $V_2 W_2 \subseteq U^{-1}$. Put $V = V_1 \cap V_2$ and $W = W_1 \cap W_2$. Then $VW \subseteq U \cap U^{-1}$. We have shown that the map $(x, y) \to xy$ is continuous from $(G \times G, T^* \times T^*)$ to (G, T^*).

Finally, we will prove that the function θ given by $\theta(x) = x^{-1}$ in continuous from (G, T^*) to (G, T^*). Let $x \in G$ and $A \in \eta^*(x^{-1})$. Then there is $U \in \eta$ such that $x^{-1} U \cap x^{-1} U^{-1} \subseteq A$. Put $V = Ux \cap U^{-1}x$. Then $\theta(V) \subseteq A$. The proof is complete. ∎

The notion of an absolute-value function on a group G (see, for instance, Wilansky[21], p. 238) permits us to construct a pseudometric d on G for which $(G, T(d))$ is a topological group. Several interesting examples of topological groups can then easily be obtained and some proofs simplified. Similarly, we here introduce the notion of an absolute quasi-valued function and establish connections between absolute quasi-valued functions, quasi-bitopological groups and quasi-pseudometrics.

Let us recall that a quasi-pseudometric on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, x) = 0$ and (ii) $d(x, y) \leq d(x, z) + d(z, y)$. If in addition d satisfies (iii) $d(x, y) = 0$ if and only if $x = y$, then d is called a quasi-metric on X.

The topology $T(d)$ induced by a quasi-pseudometric d on X has basic open neighbourhoods of $x \in X$ of the form $B_d(x, r) = \{y \in X : d(x, y) < r\}$.

A topological space (X, τ) is called quasi-(pseudo)metrizable if there is a quasi-(pseudo)metric d on X compatible with τ, where d is compatible with τ provided that $\tau = T(d)$.

Each quasi-(pseudo)metric d on X induces a conjugate quasi-(pseudo)metric d^{-1}, given by $d^{-1}(x,y) = d(y,x)$. Thus the pair of topologies induced by a quasi-(pseudo)metric and its conjugate originate the following notion[11] : A bitopological space is an ordered triple (X, τ_1, τ_2) such that X is a nonempty set and τ_1 and τ_2 are topologies on X. (X, τ_1, τ_2) is called quasi-(pseudo)metrizable if there is a quasi-pseudometric d on X compatible with (X, τ_1, τ_2), where d is compatible with (X, τ_1, τ_2) provided that $\tau_1 = T(d)$ and $\tau_2 = T(d^{-1})$.

If d is a quasi-(pseudo)metric on X we will denote by d^* the (pseudo)metric on X given by $d^* = d \vee d^{-1}$.

Definition 2 — Let G be a group. An absolute quasi-valued function for G is a nonnegative real-valued function ρ on G such that (i) $\rho(e) = 0$; (ii) $\rho(xy) \leq \rho(x) + \rho(y)$ for all $x, y \in G$ and (iii) $\rho(x_n) \to 0$ implies $\rho(a x_n a^{-1}) \to 0$ for all $a \in G$.

As an immediate consequence of this definition we obtain the following

Proposition 3 — Let ρ be an absolute quasi-valued function on the group G. Then the function d defined on $G \times G$ by $d(x,y) = \rho(x^{-1}y)$ (or by $d(x, y) = \rho(yx^{-1})$) is a quasi-pseudometric on G such that $(G, T(d), T(d^{-1}))$ is a quasi-bitopological group.

Definition 3 — Let G be a group and d a quasi-pseudometric on G. Then d is called left invariant if for all $a, x, y \in G$, $d(ax, ay) = d(x, y)$; right invariant if $d(xa, ya) = d(x, y)$; and two-sided invariant if it is both left and right invariant.

It is obvious that if d is left (right) invariant then so is its conjugate quasi-pseudometric d^{-1}.

The following result should be considered as a converse of Proposition 3 (we omit its easy proof).

Proposition 4 — Let G be a group and d a left (right) invariant quasi-pseudometric on G such that $(G, T(d), T(d^{-1}))$ is a quasi-bitopological group. Then the function ρ defined on G by $\rho(x) = d(e, x)$, is an absolute quasi-valued function.

Example 4 — Let $I = [0, 1]$ and let $G = \{f : I \to I$, such that f is continuous, bijective and increasing$\}$. Let the group operation be composition, $f \circ g = f(g)$. Then the function ρ defined on G by

$$\rho(f) = \begin{cases} \max\{f(x) - x : x \in I\}, & \text{if for each } x \in I,\ f(x) \geq x \\ 1, & \text{if for some } x \in I,\ f(x) < x \end{cases}$$

is an absolute quasi-valued function for G, that generates a quasi-bitopological group by Proposition 3.

Remark 1 : (a) Let d be the quasi-pseudometric defined on $\mathbb{R}$ by $d(x, y) = \max\{y - x, 0\}$. Then $T(d) = u$ and $T(d^{-1}) = u^{-1}$ where u is the upper topology on $\mathbb{R}$ (Example 1). Since for all $a, x, y \in \mathbb{R}$, $d(a + x, a + y) = d(x + a, y + a) = d(x, y)$, the quasi-bitopological group $(\mathbb{R}, +, u, u^{-1})$ is quasi-pseudometrizable via a two-sided invariant quasi-pseudometric.

(b) Let d be the quasi-metric defined on $\mathbb{R}$ by $d(x, y) = y - x$ if $x \leq y$ and $d(x, y) = 1$ if $x > y$. Then $T(d) = S$ where S is the topology of the Sorgenfrey line (Example 2). Similarly to (a), $(\mathbb{R}, +, S, S^{-1})$ is quasi-metrizable via a two-sided invariant quasi-metric.

3. Quasi-Uniformities on Quasi-Bitopological Groups and Quasi-Metrization

A quasi-uniformity[7] on a set X is a filter $\mathcal{U}$ on $X \times X$ such that (i) for each $U \in \mathcal{U}$, $\Delta = \{(x, x) : x \in X\} \subseteq U$; and (ii) for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$, where, as usual, $V^2 = \{(x, y) :$ there is $z \in X$ such that $(x, z) \in V$ and $(z, y) \in V\}$.

A quasi-uniform space is a pair $(X, \mathcal{U})$ such that X is a nonempty set and $\mathcal{U}$ is a quasi-uniformity on X.

The topology $T(\mathcal{U})$ induced by a quasi-uniformity $\mathcal{U}$ on X has basic neighbourhoods of $x \in X$ of the form $U[x] = \{y \in X : (x, y) \in U \in \mathcal{U}\}$.

Each quasi-uniformity $\mathcal{U}$ on X induces a conjugate quasi-uniformity $\mathcal{U}^{-1}$, given by $\mathcal{U}^{-1} = \{U \subseteq X \times X : U^{-1} \in \mathcal{U}\}$, where as usual $U^{-1} = \{(x, y) : (y, x) \in U\}$. Thus a bitopological space (X, τ_1, τ_2) is called quasi-uniformizable if there is a quasi-uniformity $\mathcal{U}$ on X compatible with (X, τ_1, τ_2), where $\mathcal{U}$ is called compatible with (X, τ_1, τ_2) provided that $\tau_1 = T(\mathcal{U})$ and $\tau_2 = T(\mathcal{U}^{-1})$.

If $\mathcal{U}$ is a quasi-uniformity on X we will denote by $\mathcal{U}^*$ the uniformity given on X by $\mathcal{U}^* = \mathcal{U} \vee \mathcal{U}^{-1}$.

Each quasi-pseudometric d on X induces a quasi-uniformity $\mathcal{U}(d)$ on X which has as a base the family of all sets of the form $\{(x, y) : d(x, y) < 2^{-n}\}$, $n \in \mathbb{N}$ (Fletcher and Lindgren[7], p.3).

Similarly to Lemma 12.1.1 of Wilansky[21] we obtain the following easy but crucial result.

Lemma 1 — Let G be a group and T a topology on G. Then the map $\phi : (G \times G, T \times T) \to (G, T)$ given by $\phi(x, y) = xy$, is continuous at (e, e) if and only if for each $U \in \eta$ there is $V \in \eta$ such that $V^2 \subseteq U$.

Following Fletcher and Lindgren[6] (p. 98) we shall introduce three interesting different quasi-uniformities on a quasi-bitopological group (G, T, T^{-1}). For each $U \in \eta$ put

$$U_L = \{(x, y) : x^{-1} y \in U\}.$$

From Lemma 1 it follows that $\{U_L : U \in \eta\}$ is a base for a quasi-uniformity $\mathcal{L}$ on G. Now put for each $U \in \eta$,

$$U_R = \{(x, y) : yx^{-1} \in U\}.$$

Then $\{U_R : U \in \eta\}$ is also a base for a quasi-uniformity $\mathcal{R}$ on G.

Theorem 1 — Each quasi-bitopological group is quasi-uniformizable.

PROOF : Let (G, T, T^{-1}) be a quasi-bitopological group. From Proposition 3.1 of Fletcher and Lindgren[6] it follows that $T(\mathcal{L}) = T$. Hence it suffices to prove that $T(\mathcal{L}^{-1}) = T^{-1}$. In fact for each $x \in G$ and each $U \in \eta$,

$$U_L^{-1}[x] = \{y : (y, x) \in U_L\} = \{y : y^{-1}x \in U\} = \{y : x^{-1}y \in U^{-1}\} = \{y : y \in xU^{-1}\} = xU^{-1}.$$

Therefore, $T(\mathcal{L}^{-1}) = T^{-1}$ by Proposition 1. Thus $\mathcal{L}$ is compatible with (G, T, T^{-1}). ■

Remark 2 : If we consider the quasi-uniformity $\mathcal{R}$ instead of $\mathcal{L}$, we obtain $T(\mathcal{R}) = T$ and $T(\mathcal{R}^{-1}) = T^{-1}$ since $U_R^{-1}[x] = U^{-1}x$ for each $x \in G$ and each $U \in \eta$. Therefore, $\mathcal{R}$ is also compatible with (G, T, T^{-1}).

According to Fletcher and Lindgren[6] (p. 98) the quasi-uniformities $\mathcal{L}$ and $\mathcal{R}$ are called the left quasi-uniformity and the right quasi-uniformity for (G, T, T^{-1}), respectively. The quasi-uniformity $\mathcal{B} = \mathcal{L} \vee \mathcal{R}$ is called the two-sided quasi-uniformity for (G, T, T^{-1}).

Note that, by Theorem 1 and Remark 2, $\mathcal{B}$ is also compatible with (G, T, T^{-1}).

We have shown in Proposition 2 that every quasi-bitopological group (G, T, T^{-1}) generates a topological group (G, T^*), where $T^* = T \vee T^{-1}$. If $\mathcal{L}^{\vee}$, $\mathcal{R}^{\vee}$ and $\mathcal{B}^{\vee}$ denote the left uniformity, the right uniformity and the two-sided uniformity for (G, T^*) respectively, we obtain the following proposition which plays a crucial role in establishing some results on completeness (see Theorem 5).

Proposition 5 — Let (G, T, T^{-1}) be a quasi-bitopological group. Then $\mathcal{L}^* = \mathcal{L}^{\vee}$, $\mathcal{R}^* = \mathcal{R}^{\vee}$ and $\mathcal{B}^* = \mathcal{B}^{\vee}$.

PROOF : Let $U \in \eta$. Then $(U \cap U^{-1})_L = U_L \cap U_L^{-1}$, so that $\mathcal{L}^{\vee} = \mathcal{L} \vee \mathcal{L}^{-1} = \mathcal{L}^*$ Similarly, $(U \cap U^{-1})_R = U_R \cap U_R^{-1}$, so that $\mathcal{R}^{\vee} = \mathcal{R}^*$. Thus $\mathcal{B}^{\vee} = \mathcal{B}^*$. ■

In Theorem 2 we will extend the classical result that each first countable topological group admits a compatible left invariant pseudometric to quasi-bitopological groups. We will need the following auxiliary result.

Proposition 6 — Let (G, T, T^{-1}) be a quasi-bitopological group and let d be a left (right) invariant quasi-pseudometric on G such that $T(d) = T$. Then $T(d^{-1}) = T^{-1}$.

PROOF : Suppose that d is left invariant. Since (G, T) is first countable, (G, T^{-1}) is also first countable. We then have :

$$x_n \to x \text{ (with respect to } T^{-1}) \Leftrightarrow x_n^{-1} \to x^{-1} \text{ (with respect to } T) \Leftrightarrow$$

$$\Leftrightarrow x_n^{-1} x \to e \text{ (with respect to } T) \Leftrightarrow d(e, x_n^{-1} x) \to 0 \Leftrightarrow d(x_n, x) \to 0.$$

Hence, $T(d^{-1}) = T^{-1}$. ■

We will say that a quasi-bitopological group (G, T, T^{-1}) is first countable if (G, T) (or equivalently, (G, T^{-1})) is first countable.

The following result extends a classical theorem due to Kakutani[10] (see Wilansky[21], Theorem 12.2.3 or Comfort[3], Theorem 1.8).

Theorem 2 — Each first countable quasi-bitopological group admits a compatible left invariant quasi-pseudometric.

PROOF : We sketch the proof since the argument needed to establish the result is standard (see for instance Wilansky[21], Theorem 12.2.3).

Let $\langle U_n \rangle$ be a countable base of T-neighbourhoods of e. From Lemma 1, there is a countable base $\langle V_n \rangle$ of T-neighbourhoods of e such that $V_0 = G$ and $V_n^3 \subseteq V_{n-1}$ for all $n \in \mathbb{N}$. Define a nonnegative real-valued function P on G by

$$P(x) = \begin{cases} 0, & \text{if } x \in V_n \text{ for all } n \in \mathbb{N} \bigcup \{0\} \\ 2^{-n}, & \text{if } x \in V_n \setminus V_{n+1}. \end{cases}$$

Then we have that : (i) $P(e) = 0$; and (ii) $x_n \to e$ (with respect to T) if and only if $P(x_n) \to 0$.

Furthermore, $P(abc) \leq 2\max\{P(a), P(b), P(c)\}$ for all $a, b, c \in G$, so that

$$P(a_1 a_2 \ldots a_n) \leq 2 \sum_{i=1}^{n} P(a_i).$$

We then can define a function $\rho : G \to \mathbb{R}$ such that

$$\rho(x) = \inf \left\{ \sum_{i=1}^{k} P(a_i a_{i-1}^{-1}) : a_0 = e,\ a_k = x, k \text{ finite} \right\}.$$

It is easily seen that ρ is an absolute quasi-valued function on G. By Proposition 3 the function d defined on $G \times G$ by $d(x, y) = \rho(x^{-1}y)$ is a quasi-pseudometric on G which is, clearly, left invariant (see Definition 3). We will show that d is compatible with (G, T, T^{-1}). To this end, note firstly that $\rho(x) \leq P(x) \leq 2\rho(x)$ for all $x \in G$. Therefore

$$x_n \to x \text{ (with respect to } T) \Leftrightarrow x^{-1} x_n \to e \text{ (with respect to } T) \Leftrightarrow$$

$$\Leftrightarrow P(x^{-1} x_n) \to 0 \Leftrightarrow \rho(x^{-1} x_n) \to 0 \Leftrightarrow d(x, x_n) \to 0.$$

Consequently $T(d) = T$ and, by Proposition 6, $T(d^{-1}) = T^{-1}$. The proof is complete. ■

Remark 3 : It follows from Theorem 2 that every first countable quasi-bitopological group (G, T, T^{-1}) also admits a compatible right invariant quasi-pseudometric. In fact, let d be the left invariant quasi-pseudometric constructed in Theorem 2. Then the function d' defined on $G \times G$ by $d'(x, y) = d(y^{-1}, x^{-1})$ is a right invariant quasi-pseudometric compatible with (G, T, T^{-1}).

Proposition 7 — Let (G, T, T^{-1}) be a first countable quasi-bitopological group and let d be a left (right) invariant quasi-pseudometric on G such that $T = T(d)$. Then d induces the left (right) quasi-uniformity for (G, T, T^{-1}).

PROOF : Suppose that d is left invariant. For each $\varepsilon > 0$ set $U_L^{\varepsilon} = \{(x, y) : x^{-1}y \in B_d(e, \varepsilon)\}$ and $U_\varepsilon = \{(x, y) : d(x, y) < \varepsilon\}$. Then $U_L^{\varepsilon} = \{(x, y) : d(e, x^{-1}y) < \varepsilon\} = \{(x, y) : d(x, y) < \varepsilon\} = U_\varepsilon$, so that, $\mathcal{L} = \mathcal{U}(d)$. The parenthetical result follows similarly. ■

Definition 4 — Let G be a group and d a quasi-pseudometric on G. The quasi-pseudometric D defined on G by $D(x, y) = d(x, y) + d(y^{-1}, x^{-1})$, will be called the B-quasi-pseudometric associated to d.

Proposition 8 — Let (G, T, T^{-1}) be a first countable quasi-bitopological group and let d be a left (right) invariant quasi-pseudometric on G such that $T = T(d)$. Then the B-quasi-pseudometric associated to d induces the two-sided quasi-uniformity for the quasi-bitopological group (G, T, T^{-1}).

PROOF : Let D be the B-quasi-pseudometric associated to the left invariant quasi-pseudometric d. For each $\varepsilon > 0$ set $U_\varepsilon = \{(x, y) : D(x, y) < \varepsilon\}$, $U_L^{\varepsilon/2} = \{(x, y) : x^{-1}y \in B_d(e, \varepsilon/2)\}$ and $U_R^{\varepsilon/2} = \{(x, y) : yx^{-1} \in B_d(e, \varepsilon/2)\}$. Then

$$U_L^{\varepsilon/2} \cap U_R^{\varepsilon/2} = \{(x, y) : d(e, x^{-1}y) < \varepsilon/2\} \cap \{(x, y) : d(e, yx^{-1}) < \varepsilon/2\}$$

$$= \{(x, y) : d(x, y) < \varepsilon/2\} \cap \{(x, y) : d(y^{-1}, x^{-1}) < \varepsilon/2\}$$

$$\subseteq \{(x, y) : D(x, y) < \varepsilon\} = U_\varepsilon.$$

Hence, $\mathcal{U}(D) \subseteq \mathcal{L} \vee \mathcal{R} = \mathcal{B}$.

On the other hand, since $U_\varepsilon \subseteq U_L^{\varepsilon} \cap U_R^{\varepsilon}$, $\mathcal{B} \subseteq \mathcal{U}(D)$. Consequently, $\mathcal{B} = \mathcal{U}(D)$.

A similar argument proves the result when d is right-invariant. ■

The two final results in this section provide conditions under which a topology on a group generates a quasi-bitopological group and conditions under which one has a structure of quasi-bitopological group on a given group, respectively. We omit their standard proofs.

Proposition 9 — Let G be a group and T a topology on G. Then (G, T, T^{-1}) is a quasi-bitopological group if and only if the following conditions are satisfied :

(i) every left translate of a T-open set is T-open,

(ii) for every T-neighbourhood U of e, there is a T-neighbourhood V of e such that $V^2 \subseteq U$,

(iii) for every T-neighbourhood U of e and every $a \in G$, there is a T-neighbourhood V of e such that $aVa^{-1} \subseteq U$.

Proposition 10 — Let G be a group and $\mathcal{F}$ a collection of subsets of G such that

(i) $\mathcal{F}$ is filterbase,

(ii) for each $U \in \mathcal{F}$ there is $V \in \mathcal{F}$ such that $V^2 \subseteq U$,

(iii) for each $U \in \mathcal{F}$ and each $a \in G$, there is $V \in \mathcal{F}$ such that $aVa^{-1} \subseteq U$.

Then there is a unique topology T on G, such that $\mathcal{F}$ is a T-neighbourhood base at e and (G, T, T^{-1}) is a quasi-bitopological group.

4. Bicompletion of Quasi-bitopological Groups

A classical result (see for instance Wilansky[21]) states that for each Hausdorff topological group G there exists a Hausdorff topological group which is complete in its two-sided uniformity and has G as a dense topological subgroup. The main result of this section extends this theorem to quasi-bitopological groups. Our construction is based on the construction of the bicompletion of a 2-Hausdorff quasi-uniform space given by Fletcher and Lindgren[7].

Let us recall that a bitopological space (X, τ_1, τ_2) is 2-Hausdorff (Salbany[19]) if $\tau_1 \vee \tau_2$ is a Hausdorff topology. Thus a quasi-uniform space $(X, \mathcal{U})$ is said to be 2-Hausdorff provided that $(X, T(\mathcal{U}), T(\mathcal{U}^{-1}))$ is a 2-Hausdorff bitopological space.

Let (X, τ_1, τ_2) be a bitopological space. A subset $A \subseteq X$ is called 2-dense in X if it is $\tau_1 \vee \tau_2$-dense in X.

A filter $\mathcal{F}$ on a quasi-uniform space $(X, \mathcal{U})$ is called $\mathcal{U}^*$-Cauchy (Fletcher and Lindgren[7]) if for each $U \in \mathcal{U}$ there is $F \in \mathcal{F}$ such that $F \times F \subset U$.

The quasi-uniform space $(X, \mathcal{U})$ is called bicomplete (Fletcher and Lindgren[7]) if each $\mathcal{U}^*$-Cauchy filter converges with respect to $T(\mathcal{U}^*)$, i.e. if the uniform space $(X, \mathcal{U}^*)$ is complete.

Lemma 2 (Fletcher and Lindgren[7], Theorem 3.29) — Let $(X, \mathcal{U})$ be a quasi-uniform space, let $(Y, \mathcal{V})$ be a bicomplete 2-Hausdorff quasi-uniform space, let A be a 2-dense subset of $(X, T(\mathcal{U}), T(\mathcal{U}^{-1}))$ and let $f : (A, \mathcal{U}|_{A \times A}) \to (Y, \mathcal{V})$ be a quasi-uniformly continuous function. Then there exists a unique continuous extension $g : (X, T(\mathcal{U}^*)) \to (Y, T(\mathcal{V}^*))$ of f and g is quasi-uniformly continuous.

Lemma 3 (Fletcher and Lindgren[7], Proposition 3.32) — Let $(X, \mathcal{U})$ be a quasi-uniform space and let A be a 2-dense subset of $(X, T(\mathcal{U}), T(\mathcal{U}^{-1}))$. If every

Cauchy filter on the uniform space $(A, \mathcal{U}^* |_{A \times A})$ converges in $(X, T(\mathcal{U}^*))$, then $(X, \mathcal{U})$ is bicomplete.

A bicompletion of a quasi-uniform space $(X, \mathcal{U})$ is a bicomplete quasi-uniform space $(Y, \mathcal{V})$ that has a 2-dense subspace quasi-unimorphic to $(X, \mathcal{U})$ (Fletcher and Lindgren[7]).

A $\mathcal{U}^*$-Cauchy filter on a quasi-uniform space $(X, \mathcal{U})$ is said to be minimal provided that it contains no $\mathcal{U}^*$-Cauchy filter other than itself.

Lemma 4 (Fletcher and Lindgren[7], Theorems 3.33 and 3.34) — Let $(X, \mathcal{U})$ be a 2-Hausdorff quasi-uniform space. Let $Y = \{\mathcal{F} : \mathcal{F}$ is a minimal $\mathcal{U}^*$-Cauchy filter on $X\}$. For each $U \in \mathcal{U}$ let $V_U = \{(\mathcal{F}, \mathcal{G}) \in Y \times Y$: there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U$ and let $\mathcal{V} = \{V_U : U \in \mathcal{U}\}$. Then :

(a) $(Y, \mathcal{V})$ is a 2-Hausdorff bicomplete quasi-uniform space that has a 2-dense subspace quasi-unimorphic to $(X, \mathcal{U})$ (i.e. $(Y, \mathcal{V})$ is a 2-Hausdorff bicompletion of $(X, \mathcal{U})$).

(b) Any 2-Hausdorff bicompletion space of $(X, \mathcal{U})$ is quasi-unimorphic to $(Y, \mathcal{V})$.

The 2-Hausdorff bicomplete quasi-uniform space $(Y, \mathcal{V})$ of the preceding lemmas is called the bicompletion of $(X, \mathcal{U})$.

Theorem 3 — Let (G, T, T^{-1}) be a 2-Hausdorff quasi-bitopological group. Then there is a 2-Hausdorff quasi-bitopological group which is bicomplete in its two-sided quasi-uniformity and has G as a 2-dense quasi-bitopological subgroup.

PROOF : Let $\mathcal{B}$ be the two-sided quasi-uniformity for (G, T, T^{-1}) and let $(Y, \mathcal{V})$ be the bicompletion of $(G, \mathcal{B})$ (Lemma 4). Then the uniform space $(Y, \mathcal{V}^*)$ is the completion of the uniform space $(G, \mathcal{B}^*)$. By Proposition 5, $\mathcal{B}^*$ is the two sided uniformity of the uniform space (G, T^*). Hence, the Hausdorff topological space $(Y, T(\mathcal{V}^*))$ can be endowed of a structure of topological group. In fact, for $y_1; y_2 \in Y$, let

$$\mathcal{F}_i = \{W \cap G : W \text{ is a } T(\mathcal{V}^*)\text{-neighbourhood of } y_i\},\ i = 1, 2.$$

Then $\mathcal{F}_i$ is a $\mathcal{B}^*$-Cauchy filter on G, $i = 1, 2$, and, thus, $\mathcal{F}_1 \mathcal{F}_2$ is a $\mathcal{B}^*$-Cauchy filterbase on G. Denote by $y_1 y_2$ the $T(\mathcal{V}^*)$-limit point of $\mathcal{F}_1 \mathcal{F}_2$. Similarly, for $y_1 \in Y$ define y_1^{-1} as the $T(\mathcal{V}^*)$-limit point of the $\mathcal{B}^*$-Cauchy filter $\mathcal{F}_1^{-1}$. It is well-known that, then $(Y, T(\mathcal{V}^*))$ is a topological group and G is a subgroup of Y (see Wilansky[21], Theorem 12.2.4 for more details).

Now we shall prove that the function $(y_1, y_2) \to y_1 y_2$ is continuous from $(Y \times Y, T(\mathcal{V}) \times T(\mathcal{V}))$ to $(Y, T(\mathcal{V}))$:

Let $y_1, y_2 \in Y$ and let $V \in \mathcal{V}$. Put $B = V \cap (G \times G)$. Then $B \in \mathcal{B}$, so that there exist two T-neighbourhoods U and W of e such that $U_L \cap U_R \subseteq B$ and $W^4 \subseteq U$. Since y_1 and y_2 are $\mathcal{B}^*$-Cauchy filters on G, there exist $F_1' \in y_1$ and $F_2' \in y_2$ such that $F_1' \times F_1' \subseteq W_L \cap W_R$ and $F_2' \times F_2' \subseteq W_L \cap W_R$. Fix $a_1 \in F_1'$ and $a_2 \in F_2'$. Then there exists a T-neighbourhood $H \subseteq W$ of e such that $a_1 H a_1^{-1} \subseteq W$ and $a_2^{-1} H a_2 \subseteq W$. Put $H_L \cap H_R = B_0$. Thus $B_0 \in \mathcal{B}$. Let $V_0 \in \mathcal{V}$ such that $V_0 \cap (G \times G) \subseteq B_0$. We shall show that $V_0(y_1)\, V_0(y_2) \subseteq V(y_1 y_2)$. In fact, if $y \in V_0(y_1)\, V_0(y_2)$, there exist $\mathcal{H}_1, \mathcal{H}_2 \in Y$ such that $y = \mathcal{H}_1 \mathcal{H}_2$. Since $\mathcal{H}_1 \in V_0(y_1)$ and $\mathcal{H}_2 \in V_0(y_2)$, there exist $F_1 \in y_1, F_2 \in y_2, H_1 \in \mathcal{H}_1$ and $H_2 \in \mathcal{H}_2$ such that $F_1 \times H_1 \subseteq B_0$ and $F_2 \times H_2 \subseteq B_0$ (we may assume that $F_1 \subseteq F_1'$, and $F_2 \subseteq F_2'$). We want to show that $F_1 F_2 \times H_1 H_2 \subseteq U_L \cap U_R$. In fact, for each pair $(f_1 f_2, h_1 h_2) \in F_1 F_2 \times H_1 H_2$ we have

$$(f_1 f_2)^{-1} h_1 h_2 = f_2^{-1} a_2 a_2^{-1} f_1^{-1} h_1 a_2 a_2^{-1} f_2 f_2^{-1} h_2 \in W^4$$

since $(f_2, a_2) \in F_2' \times F_2' \subseteq W_L$, $a_2^{-1} f_1^{-1} h_1 a_2 \in a_2^{-1} H a_2 \subseteq W$, $(a_2, f_2) \in W_L$ and $(f_2, h_2) \in B_0$. Similarly,

$$h_1 h_2 (f_1 f_2)^{-1} = h_1 f_1^{-1} f_1 a_1^{-1} a_1 h_2 f_2^{-1} a_1 a_1^{-1} f_1^{-1} \in W^4.$$

Thus, $(f_1 f_2, h_1 h_2) \in U_L \cap U_R$.

Consequently, $(Y, T(\mathcal{V}), T(\mathcal{V}^{-1}))$ is a 2-Hausdorff quasi-bitopological group that has G as a 2-dense quasi-bitopological subgroup, and such that $(Y, \mathcal{V})$ is bicomplete.

Now denote by $\hat{\mathcal{B}}$ the two-sided quasi-uniformity for the quasi-bitopological group $(Y, T(\mathcal{V}), T(\mathcal{V}^{-1}))$. It then remains to show that $\mathcal{V} = \hat{\mathcal{B}}$. To this end note firstly that $\hat{\mathcal{B}}|_{G \times G} = \mathcal{B}$. Hence, since every Cauchy filter on $(G, \hat{\mathcal{B}}^*|_{G \times G}) = (G, \mathcal{B}^*)$ converges in $(Y, T(\mathcal{V}^*)) = (Y, T(\hat{\mathcal{B}}^*))$, it follows Lemma 3 that $(Y, \hat{\mathcal{B}})$ is bicomplete. Finally, by Lemma 2, $\hat{\mathcal{B}} = \mathcal{V}$. The proof is complete. ■

Our two final results concern bicompletion and bicompleteness of quasi-metrizable quasi-bitopological groups, respectively.

Let G be a group and d a quasi-pseudometric on G. We will say that (G, d) is a quasi-pseudometric group if $(G, T(d), T(d^{-1}))$ is a quasi-bitopological group.

Let (G, d) be a quasi-pseudometric group. A subgroup A of G is called a 2-dense quasi-bitopological subgroup of (G, d) if $(A, T(d)|_A, T(d^{-1})|_A)$ is a quasi-bitopological subgroup such that A is 2-dense in $(G, T(d), T(d^{-1}))$.

Theorem 4 — Let (G, d) be a quasi-pseudometric group. Then there is a bicomplete quasi-pseudometric group (Y, D), with D a B-quasi-pseudometric, that has G as a 2-dense quasi-bitopological subgroup.

PROOF : We sketch the proof. Suppose (G, d) a quasi-pseudometric group. By Theorem 2, the quasi-bitopological group $(G, T(d), T(d^{-1}))$ admits a compatible left invariant quasi-pseudometric d'. Let D be the B-quasi-pseudometric on G associated to d' (Definition 4). Then D is compatible with $(G, T(d), T(d^{-1}))$. Now let Y be the set of all D^*-Cauchy sequences in G. We shall define, as in the classical metric case, a bicomplete quasi-pseudometric $\hat{e}$ on Y as follows :

if $u = \langle u_n \rangle$ and $v = \langle v_n \rangle$ are D^*-Cauchy sequences on G put

$$\hat{e}(u, v) = \lim_{n \to \infty} D(u_n, v_n).$$

Then, it is easy to see that $(Y, \hat{e})$ is a bicomplete quasi-pseudometric space that has G as a 2-dense subset (Salbany[18]). Furthermore Y is a group if we define for $u = \langle u_n \rangle$ and $v = \langle v_n \rangle$ in Y, $uv = \langle u_n v_n \rangle$.

Now we will prove that $(Y, T(\hat{e}), T(\hat{e}^{-1}))$ is actually a quasi-bitopological group. To this end define for each $u \in Y$

$$\rho(u) = \frac{1}{2}\, e\,(e, u).$$

Then, one can show that ρ is an absolute quasi-valued function on Y. Let q be the left invariant quasi-pseudometric defined on Y by $q(x, y) = \rho(x^{-1} y)$ and let $\hat{D}$ be the B-quasi-pseudometric on Y associated to q. By Proposition 3, $(Y, \hat{D})$ is a quasi-pseudometric group. Since $q = \hat{e}/2$, $\hat{D}$ is a bicomplete quasi-pseudometric compatible with $(Y, T(\hat{e}), T(\hat{e}^{-1}))$ (and $\hat{D}|_{G \times G} = D$). The proof is complete. ■

We conclude the paper with a quasi-bitopological extension of a well-known theorem of Klee[12] which states that the two-sided uniformity of a metrizable topological group is complete if and only if it is completely metrizable.

Recall that a bitopological space is said to be bicompletely quasi-pseudometrizable if and only if it has a compatible bicomplete quasi-pseudometric.

Theorem 5 — The two-sided quasi-uniformity of a 2-Hausdorff quasi-pseudometrizable quasi-bitopological group (G, T, T^{-1}) is bicomplete if and only if $(G, T \vee T^{-1})$ is completely metrizable.

PROOF : Suppose that (G, T^*) is completely metrizable where $T^* = T \vee T^{-1}$. By Proposition 2, (G, T^*) is a topological group and by Klee's theorem cited above, the two-sided uniformity $\mathcal{B}^{\vee}$ for (G, T^*) is complete. If $\mathcal{B}$ denotes the two sided quasi-uniformity for (G, T, T^{-1}) it follows from Proposition 5 that $\mathcal{B}^* = \mathcal{B}^{\vee}$, so that $\mathcal{B}$ is bicomplete. Conversely, if $\mathcal{B}$ is bicomplete we obtain that $\mathcal{B}^*$ is complete and by Klee's theorem (G, T^*) is completely metrizable (It follows from Theorem 2 and Proposition 8 that actually (G, T, T^{-1}) is bicompletely quasi-pseudometrizable). ■

REFERENCES

1. N. Bourbaki, *General Topology*, Vol. I, Addison-Wesley, 1966.
2. N. Brand, *Archiv. Math. (Basel)* **39** (1982), 241-45.
3. W. W. Comfort, *Handbook of Set-Theoretic Topology* (Eds. K. Kunen and J. E. Vaughan), North-Holland, Amsterdam, 1984, pp. 1143-1263.
4. R. Ellis, *Proc. Am. Math. Soc.* **8** (1957), 372-73.
5. R. Ellis, *Duke Math. J.* **24** (1957), 119-25.
6. P. Fletcher and W. F. Lindgren, *Rocky Mountain J. Math.* **5** (1975), 95-106.
7. P. Fletcher and W. F. Lindgren, *Quasi Uniform Spaces*, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, 77, 1982.
8. D. L. Grant, *The Wallace problem and continuity of the inverse in pseudocompact groups*, General Topology and Applications, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, **134** (1991), pp. 93-114.
9. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Vol. I, Grundlehren der Math. Wissenschaffen, Springer-Verlag, Berlin, 115, 1963.
10. S. Kakutani, *Proc. Imperial Acad. Tokyo* **12** (1936), 82-84.
11. J. C. Kelly, *Proc. London Math. Soc.* **13** (1963), 71-89.
12. V. L. Klee Jr., *Trans Am. Math. Soc.* **13** (1962), 30-45.
13. A. Mukherjee and N. A. Tserpes, *J. Austral Math. Soc.* **13** (1972), 180-84.
14. K. Numakura, *Math. J. Okayama Univ.* **1** (1952), 99-108.
15. H. Pfister, *Proc. Am. Math. Soc.* **95** (1985), 312-14.
16. T. G. Raghavan and I. L. Reilly, *Colloquia Math. Soc. Janos Bolyai* **23** (1978), 1005-14.
17. T. G. Raghavan and I. L. Reilly, *Indian J. pure appl. Math.* **9** (1978), 747-52.
18. S. Salbany, *Bitopological Spaces Compactifications and Completions*, Math. Monographs. Univ. Cape Town, 1 (1974).
19. S. Salbany, *Categorical Topology* (Proc. Toledo, Ohio 1983). Heldemann, Berlin, (1984), 481-504.
20. A. D. Wallace, *Bull. Am. Math. Soc.* **61** (1955), 95-112.
21. A. Wilansky, *Topology for Analysis*, R. E. Krieger Publishing Company, Inc., Florida, 1983.
22. W. Zelazko, *Rozprawy Math.* **47** (1965), Warsaw.