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0-ARCHIMEDEAN SEMIGROUPS

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In this paper we consider (completely) 0-Archimedean semigroups as a generalization of (completely) 0-simple and (completely) Archimedean semigroups. We describe nil-extensions of (completely) 0-simple semigroups.

Throughout this paper, $\mathbf{Z}^+$ will denote the set of all positive integers and $S = S^0$ means that S is a semigroup with the zero 0. If $S = S^0$, we will write 0 instead of $\{0\}$ and if A is a subset of S, we will write $A^\bullet = A - 0$, $A^0 = A \cup 0$. For an element a of a semigroup S, $J(a)$ will denote the principal ideal of S generated by a.

Let $S = S^0$. An element a of S is a nilpotent if there exists $n \in \mathbf{Z}^+$ such that $a^n = 0$. The set of all nilpotents of S is denoted by $Nil(S)$. S is a nil-semigroup if $S = Nil(S)$, otherwise it is non-nil. An ideal I of S is a nil-ideal of S if I is a nil-semigroup. By $\mathcal{R}(S)$ we denote Clifford's radical of a semigroup $S = S^0$, i.e. the union of all nil-ideals of S (it is the greatest nil-ideal of S). An ideal extension S of a semigroup K is a nil-extension of K if S/K is a nil-semigroup.

A semigroup S is intra-π-regular if for every $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in Sa^{2n}S$. A semigroup S is completely π-regular if for every $a \in S$ there exists $n \in \mathbf{Z}^+$ and $x \in S$ such that $a^n = a^n x a^n$, $a^n x = x a^n$.

Let S be a semigroup. For $a, b \in S$, $a \mid b$ if $b \in J(a)$ and $a \to b$ if $a \mid b^n$, for some $n \in \mathbf{Z}^+$. For $a \in S, \Sigma_1(a) = \{x \in S \mid a \to x\}$ and an equivalence σ_1 on S is defined by : $a \, \sigma_1 \, b$ if and only if $\Sigma_1(a) = \Sigma_1(b)$, $a, b \in S$ (Ćirić and Bogdanović[3]).

An ideal I of a semigroup S is prime if for all $a, b \in S$, $aSb \subseteq I$ implies that either $a \in I$ or $b \in I$, or, equivalently, if for all ideals A, B of S, $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$.

For undefined notions and notations we refer to Bogdanović[1], Bogdanović and Ćirić[2] and Clifford and Preston[4, 5].

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The purpose of this paper is to give some generalizations of (completely) 0-simple semigroups and of (completely) Archimedean semigroups and to describe some characteristics of these.

First we will give a connection between the Clifford's radical of a semigroup with zero and the relation σ_1.

Lemma 1 — The Clifford's radical $\mathcal{R}(S)$ of a semigroup $S = S^0$ is equal to the σ_1-class containing 0.

PROOF : Let C be the σ_1-class of S containing 0, and let $a \in C, x \in S$. Then $\Sigma_1(a) = \Sigma_1(0) = Nil(S)$. By Lemma 10 (Ćirić and Bogdanović[3]) it follows that

$$\Sigma_1(ax) \subseteq \Sigma_1(a) = Nil(S), \quad \Sigma_1(xa) \subseteq \Sigma_1(a) = Nil(S).$$

Since $Nil(S) \subseteq \Sigma_1(u)$ for all $u \in S$, then $\Sigma_1(ax) \subseteq \Sigma_1(xa) = Nil(S) = \Sigma_1(0)$, so $ax, xa \in C$. Hence, C is an ideal of S. It is clear that $C \subseteq Nil(S)$, so C is a nil-ideal, whence $C \subseteq \mathcal{R}(S)$.

Let $a \in \mathcal{R}(S)$ and $x \in \Sigma_1(a)$, i.e., $x^n \in SaS$ for some $n \in \mathbf{Z}^+$. Since $SaS \subseteq S\mathcal{R}(S)S \subseteq \mathcal{R}(S) \subseteq Nil(S)$, then $x \in Nil(S) = \Sigma_1(0)$. Thus $\Sigma_1(a) \subseteq \Sigma_1(0)$. It is clear that $\Sigma_1(0) \subseteq \Sigma_1(a)$. Therefore, $a \in C$ so $\mathcal{R}(S) = C$. □

Note that a semigroup $S = S^0$ is 0-simple if and only if $a \mid b$, for all $a, b \in S^{\bullet}$. Using the relation $\rightarrow$, we can introduce a generalization of 0-simple semigroups. A semigroup $S = S^0$ is 0-Archimedean if $a \rightarrow b$, for all $a, b \in S^{\bullet}$. Also, we can introduce a more general notion : A semigroup $S = S^0$ is weakly 0-Archimedean if $a \rightarrow b$, for all $a, b \in S - \mathcal{R}(S)$.

A relationship between weakly 0-Archimedean and 0-Archimedean semigroups is given by the next theorem. Since every nil-semigroup is (weakly) 0-Archimedean, then a consideration of nil-semigroups will be omitted.

Theorem 1 — The following conditions on a non-nil semigroup $S = S^0$ are equivalent :

(i) S is weakly 0-Archimedean;

(ii) S is an ideal extension of a nil-semigroup by a 0-Archimedean semigroup;

(iii) S contains at most two σ_1-classes.

PROOF : (i) $\Rightarrow$ (ii). Let S be weakly 0-Archimedean. Then S is an ideal extension of a nil-semigroup $R = \mathcal{R}(S)$ by a semigroup Q. Assume $a, b \in Q^{\bullet}$. Then $a, b \in S - R$, so there exists $x, y \in S$ and $n \in \mathbf{Z}^+$ such that $b^n = xay$, since S is weakly 0-Archimedean. If $x \in R$ or $y \in R$, then $b^n \in R$, whence $b^n = 0 \in QaQ$ in Q, so $a \rightarrow b$ in Q. Assume that $x, y \in S - R = Q^{\bullet}$. Then $b^n = xay \in QaQ$ in Q, so $a \rightarrow b$ in Q. Thus, Q is 0-Archimedean.

(ii) $\Rightarrow$ (i). Let S be an ideal extension of a nil-semigroup R by a 0-Archimedean semigroup Q. Assume $a, b \in S - \mathcal{R}(S)$. Since $R \subseteq \mathcal{R}(S)$, then $a, b \in S - R = Q^{\bullet}$. Thus, there exist $x, y \in Q$ and $n \in \mathbf{Z}^+$ such that $b^n = xay$. If $x = 0$ or $y = 0$, then $b^n = 0$ in Q, whence $b^n \in R \subseteq Nil(S)$ in S, so $b^{nk} = (b^n)^k = 0 \in SaS$ in S, for some $k \in \mathbf{Z}^+$, i.e. $a \to b$ in S. Assume that $x, y \neq 0$ in Q. Then $x, y \in Q^{\bullet} = S - R$, so $b^n = xay \in SaS$ in S, whence $a \to b$ in S. Thus, S is weakly 0-Archimedean.

(i) $\Rightarrow$ (iii). Let S be weakly 0-Archimedean. By Lemma 1 we obtain that $\mathcal{R}(S)$ is equal to the σ_1-class containing 0. Assume $a, b \in S - \mathcal{R}(S)$. Let we prove that $a\sigma_1 b$. Let $x \in \Sigma_1(a)$, i.e. let $x^n = uav$ for some $n \in \mathbf{Z}^+$, $u, v \in S$. If $uav \in \mathcal{R}(S)$, then $x \in Nil(S)$, so $b \to x$, i.e. $x \in \Sigma_1(b)$. Let $uav \in S - \mathcal{R}(S)$. Then $(uav)^k \in SbS$ for some $k \in \mathbf{Z}^+$, whence $x^{nk} \in SbS$, i.e. $x \in \Sigma_1(b)$. Thus, $\Sigma_1(a) \subseteq \Sigma_1(b)$. Similarly we prove the opposite inclusion. Therefore, (iii) holds.

(iii) $\Rightarrow$ (i). This follows by Lemma 1. □

Lemma 2 — Let $S = S^0$ be a nil-extension of a 0-simple semigroup K. Then

$$\mathcal{R}(S) = \{x \in S \mid SxS \cap K = 0\}.$$

PROOF : Let $A = \{x \in S \mid SxS \cap K = 0\}$. Assume $a \in A$, $x \in S$. Then $SaS \cap K = 0$ so

$$SaxS \cap K \subseteq SaS \cap K = 0, \quad SxaS \cap K \subseteq SaS \cap K = 0,$$

whence $ax, xa \in A$. Thus, A is an ideal of S. It is clear that A is a nil-semigroup. Assume a nil-ideal I of S. Then $I \cap K$ is an ideal of K, whence $I \cap K = 0$ or $I \cap K = K$. Since K contains a non-nilpotent element, then $I \cap K = 0$, so $SaS \cap K \subseteq SIS \cap K \subseteq I \cap K = 0$, for every $a \in I$. Therefore, $I \subseteq A$, whence $\mathcal{R}(S) = A$. □

Note that the smallest ideal, if it exists, of a semigroup S is called a kernel of S. But, in a semigroup with zero, this notion degenerates, since the zero ideal is the kernel, so we introduce the following notion : The smallest element of a set of all nonzero ideals of a semigroup $S = S^0$, if it exists, is called the 0-kernel of S. Note that Schein[6] used the notion core that unites the kernel of a semigroup without zero, if it exists, and the 0-kernel of a semigroup with zero, if it exists.

Let $S = S^0$ and K be the 0-kernel of S. By Theorem 2.29 (Clifford and Preston[4]), $K^2 = 0$, and then we say that K is a nilpotent 0-kernel, or K is 0-simple, and we call it a 0-simple 0-kernel.

If a semigroup S is an ideal extension of a semigroup T by a semigroup Q, then we usually identify partial semigroups $S - T$ and $Q^{\bullet}$ (see Clifford and Preston[4], §4.4). This fact we will use in the following :

Theorem 2 — A semigroup $S = S^0$ is a nil-extension of a 0-simple semigroup if and only if S is an ideal extension of a nil-semigroup R by a 0-Archimedean semigroup Q with a 0-simple 0-kernel K and the following conditions hold :

(a) for all $a \in K^{\bullet}$, $b \in S - R$

$$ab = 0 \text{ in } Q \Rightarrow ab = 0 \text{ in } S;$$

$$ba = 0 \text{ in } Q \Rightarrow ba = 0 \text{ in } S;$$

(b) $ab = ba = 0$, for all $a \in K^*$, $b \in R$.

PROOF : Let S be a nil-extension of a 0-simple semigroup T and let $R = \mathcal{R}(S)$. Then R is a nil-semigroup and S is an ideal extension of R by a semigroup Q. Since T is 0-simple, then $R \cap T = 0$.

Assume $a \in T^*$, $b \in S - R$. Then $ab \in T$, since T is an ideal of S. If $ab = 0$ in Q, then $ab \in R$ in S, so $ab = 0$ in S, since $R \cap T = 0$. Thus

$$ab = 0 \in Q \Rightarrow ab = 0 \in S.$$

Similarly we prove the second implication from (a).

Assume $a \in T^*$, $b \in R$. Then $ab = ba = 0$, since $ab,\, ba \in R \cap T = 0$.

Let $K = T^* \cup 0 \subseteq Q$. Then K is a subsemigroup of Q isomorphic to T, whence K is 0-simple. Therefore, by the previous facts we obtain (a) and (b).

Let I be an ideal of Q, $I \neq 0$. It is easy to verify that $I^* \cup R$ is an ideal of S and $I^* \cup R \neq 0$, whence $T \subseteq I^* \cup R$, so $K^* = T^* \subseteq I^*$, i.e. $K \subseteq I$. Thus K is a 0-simple 0-kernel of Q.

Assume $a, b \in S - R$. By Lemma 2 we obtain that $SaS \cap T \neq 0$, whence $T \subseteq SaS$. Thus, there exists $n \in \mathbf{Z}^+$ such that $b^n \in T \subseteq SaS$, i.e. $a \to b$. Hence, S is weakly 0-Archimedean, so by the proof of Theorem 1 we obtain that Q is 0-Archimedean.

Conversely, let S be an ideal extension of a nil-semigroup R by a 0-Archimedean semigroup Q with a 0-simple 0-kernel K and let (a) and (b) holds. By (a) it follows that $T = K^* \cup 0 \subseteq S$ is a subsemigroup of S isomorphic to K, so T is 0-simple. By (a) and (b) it follows that T is an ideal of S. By Theorem 1, S is weakly 0-Archimedean. Assume $x \in S$. If $x \in \mathcal{R}(S)$, then $x \in Nil(S)$, so $x^n = 0 \in T$ for some $n \in \mathbf{Z}^+$. Let $x \in S - \mathcal{R}(S)$ and assume $a \in T - Nil(S)$. Then $a \to x$, whence $x^n \in SaS \subseteq T$, for some $n \in \mathbf{Z}^+$. Therefore, S is a nil-extension of T. □

As we see, a 0-Archimedean semigroup is a generalization of a 0-simple semigroup. Similarly we generalize a notion of completely 0-simple semigroup : An idempotent e of a semigroup $S = S^0$ is a 0-primitive idempotent of S if it is a minimal element in the partially ordered set of all nonzero idempotents of S. A 0-Archimedean semigroup containing a 0-primitive idempotent is called completely 0-Archimedean semigroup.

It is easy to verify that every completely 0-Archimedean semigroup contains a (completely) 0-simple 0-kernel. By this and by Theorem 2 we obtain the following :

Corollary 1 — A semigroup $S = S^0$ is a nil-extension of a completely 0-simple semigroup if and only if S is an ideal extension of a nil-semigroup R by a completely

0-Archimedean semigroup Q, and the conditions (a) and (b) hold, where K is the 0-kernel of Q. □

Theorem 3 — The following conditions on a non-nil semigroup $S = S^0$ are equivalent :

(i) S is a 0-Archimedean semigroup with a 0-simple 0-kernel;

(ii) S is a 0-Archimedean semigroup with a 0-minimal ideal;

(iii) S is a weakly 0-Archimedean semigroup with a 0-simple 0-kernel;

(iv) S is a 0-Archimedean intra-π-regular semigroup;

(v) S is a nil-extension of a 0-simple semigroup and $\mathcal{R}(S) = 0$;

(vi) S is a nil-extension of a 0-simple semigroup and 0 is a prime ideal of S.

PROOF : (i) ⇒ (ii). This follows immediately.

(ii) ⇒ (i). Let S be a 0-Archimedean semigroup with a non-nil 0-minimal ideal M. Let $I \neq 0$ be an ideal of S, let $x \in I^{\bullet}$ and let $a \in M - Nil(S)$. Then $x \rightarrow a$, i.e. $a^n \in SxS \subseteq I$ for some $n \in \mathbf{Z}^+$, whence $a^n \in I \cap M$, $a^n \neq 0$. Thus $I \cap M \neq 0$ is an ideal of S contained in M, and since M is 0-minimal, we obtain that $I \cap M = M$, i.e. $M \subseteq I$. Hence, M is a 0-simple 0-kernel of S.

(i) ⇒ (v). Let S be a 0-Archimedean semigroup with a 0-simple 0-kernel K. Then K is 0-simple semigroup. Let $a \in K^{\bullet}$ and assume $x \in S^{\bullet}$. Then $a \rightarrow x$, i.e. $x^n \in SaS \subseteq SKS \subseteq K$, for some $n \in \mathbf{Z}^+$. Thus S is a nil-extension of K. If $\mathcal{R}(S) \neq 0$, then $K \subseteq \mathcal{R}(S)$, which is not possible, since $K \neq Nil(K)$. Thus $\mathcal{R}(S) = 0$, so (v) holds.

(v) ⇒ (iv). Let S be a nil-extension of a 0-simple semigroup K and let $\mathcal{R}(S) = 0$. Then it is clear that S is intra-π-regular and by the proof of Theorem 2 we obtain that S is 0-Archimedean.

(iv) ⇒ (i). Let S be a non-nil 0-Archimedean intra π-regular semigroup. Assume $a \in S - Nil(S)$. Then there exists $m \in \mathbf{Z}^+$ and $z, w \in S$ such that

$$a^m = za^{2m} w \in Sa^mS.$$

Let $K = Sa^mS$ and let $c, d \in K^{\bullet}$. Then $c = xa^my$ for some $x, y \in S$. On the other hand, by $a^m = za^{2m}w = za^m (a^mw)$ it follows that

$$a^m = z^n a^m (a^m w)^n, \qquad \text{... (1)}$$

for all $n \in \mathbf{Z}^+$. Since $d, a^mw \in S^{\bullet}$ and S is 0-Archimedean, then there exists $k \in \mathbf{Z}^+$ and $u, v \in S$ such that $(a^mw)^k = udv$. Now, by (1) we obtain that

$$c = xa^my = (xz^{k+1} a^m) (a^mw)^k (a^m wy) = (xz^{k+1} a^m) udv (a^m wy)$$
$$= (xz^{k+1} a^m u) d(va^m wy) \in KdK.$$

Thus, by Lemma 2.28 (Clifford and Preston[4]) we obtain that K is a 0-simple semigroup.

Let $I \neq 0$ be an ideal of S. Let $I \subseteq Nil(S)$. Assume $x \in I^{\bullet}$. Then $x \rightarrow a$, i.e., $a^n \in SxS \subseteq I$, for some $n \in \mathbf{Z}^+$, and since $I \subseteq Nil(S)$, then $a \in Nil(S)$, which gives the contradiction. Thus, there exists $b \in I - Nil(S) \subseteq S^{\bullet}$, so there exists $n \in \mathbf{Z}^{+}$ such that $b^n \in Sa^mS = K$, whence $b^n \in I \cap K$, $b^n \neq 0$, so $I \cap K \neq 0$. Now, since K is 0-simple, then $I \cap K = K$, so $K \subseteq I$. Thus, K is a 0-simple 0-kernel of S.

(iii) $\Rightarrow$ (i). Let S be a weakly 0-Archimedean semigroup with a 0-simple 0-kernel K. Since K is 0-simple, then $K \nsubseteq Nil(S)$ so $K \nsubseteq \mathcal{R}(S)$, whence $\mathcal{R}(S) = 0$, so by the proof of Theorem 1 we obtain that S is 0-Archimedean.

(i) $\Rightarrow$ (iii). This follows immediately.

(v) $\Rightarrow$ (vi). Let S be a nil-extension of a 0-simple semigroup K and let $\mathcal{R}(S) = 0$. Let A and B be nonzero ideals of S and let $a \in A^{\bullet}$, $b \in B^{\bullet}$. By Lemma 2 we obtain that $K \subseteq SaS \subseteq A$ and $K \subseteq SbS \subseteq B$, whence $K = K^2 \subseteq AB$. Thus $AB \neq 0$. Therefore, 0 is a prime ideal of S.

(vi) $\Rightarrow$ (v). Let S be a nil-extension of a 0-simple semigroup K and let 0 be a prime ideal of S. Let $R = \mathcal{R}(S)$. By the proof of Theorem 2 we obtain that $RK = 0$, whence $R = 0$ or $K = 0$. Since K is 0-simple, then $R = 0$, so (v) holds. □

In the following theorem a consideration of nil-semigroups will be omitted again.

Theorem 4 — The following conditions on a non-nil semigroup $S = S^0$ are equivalent :

(i) S is a completely 0-Archimedean semigroup;

(ii) S is 0-Archimedean and completely π-regular;

(iii) S is a nil-extension of a completely 0-simple semigroup and $\mathcal{R}(S) = 0$;

(iv) S is a nil-extension of a completely 0-simple semigroup and 0 is a prime ideal of S.

PROOF : (i) $\Rightarrow$ (iii). Let S be a completely 0-Archimedean semigroup, let $e \in S$ be a primitive idempotent and let $A \neq 0$ be an ideal of S. If $b \in A^{*}$, then $b \to e$, i.e. $e \in SbS \subseteq A$. Hence, if K is the intersection of all nonzero ideals of S, then $e \in K$, whence $K^2 \neq 0$. It is clear that K is 0-minimal ideal of S so by Theorem 2.29 (Clifford and Preston[4]) it follows that K is 0-simple, i.e. completely 0-simple, whence K is a 0-simple 0-kernel of S. Now, by the proof of Theorem 3, S is a nil-extension of a 0-simple semigroup K and $\mathcal{R}(S) = 0$. Thus, (iii) holds.

(ii) $\Rightarrow$ (iii). This follows by Theorem 3 and by Theorem 2.55 (Clifford and Preston[4]).

(iii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv). This follows by Theorem 3. □

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