Indian J. pure appl. Math., 27(5) : 475-486, May 1996

PROPERTIES OF QUASI-PRECONTINUOUS FUNCTIONS

RAMPRASAD PAUL AND P. BHATTACHARYYA

Department of Mathematics, University of Kalyani, Kalyani 741 235, *West Bengal*

(*Received* 10 *February* 1995; *after revision* 13 *November* 1995; *accepted* 22 *November* 1995)

Quasi-precontinuity introduced by the present authors[31] has been further investigated in this paper. The relationship between this continuity and some separation axioms along with the interconnection with other weakened forms of continuity have also been studied.

1. INTRODUCTION

Mashhour *et al.*[16] introduced the notions of preopen sets and precontinuity in a topological space and obtained their various properties. Recently the present authors[31] introduced, with the aid of preopen sets, a new weakened form of continuity, called quasi-precontinuity and showed that the class of quasi-precontinuous functions includes in it the class of precontinuous functions. In that paper a set of conditions characterising quasi-precontinuous functions were investigated. The purpose of the present paper is to give a set of further characterisations for quasi-precontinuous functions, to study a few invariant properties under this function along with some other basic properties, to investigate the relationship between quasi-precontinuity and some separation axioms and to deal with the inter-relationship between this function and other weak forms of continuity. Recently Popa and Noiri[29] (Theorem 3.1) showed that quasi-precontinuity is equivalent to almost weak continuity due to Janković[8]. As a result of this, various properties of quasi-precontinuous functions studied in this paper are equally true for almost weak continuous functions introduced by Janković[8].

2. PRELIMINARIES

Throughout the paper, (X, τ) and (Y, σ) etc. (or simply X and Y etc.) will always denote topological spaces. If A be a subset of a space (X, τ), then the closure of A (resp. interior of A) in (X, τ) is denoted by $Cl_\tau(A)$ (resp. $Int_\tau(A)$) or simply by $Cl\ A$ (resp. $Int\ A$) if there is no possibility of confusion. The following known definitions and results will be required in this paper.

Definition 2.1 — A subset A of (X, τ) is called

(i) preopen[16] if $A \subset Int\,(Cl\,A)$ (briefly p.o.);

(ii) semi-open[13] if $A \subset Cl\,(Int\,A)$ (briefly s.o.);

(iii) an α-set[19] if $A \subset Int\,(Cl\,(Int\,A))$;

(iv) regular open[35] if $A = Int\,(Cl\,A)$ (briefly r.o.).

The family of all p.o. sets (resp. s.o. sets, r.o. sets) is denoted by $PO(X, \tau)$ (resp. $SO\,(X, \tau)$, $RO(X, \tau)$). The family of α-sets is denoted by τ^{α} which is a topology on X (Njåstad[19]). It is also known[24] that $\tau^{\alpha} = PO(X, \tau) \cap SO(X, \tau)$. The complement of a p.o. set (resp. s.o. set) is called preclosed[16] (resp. semi-closed[3]).

Definition 2.2 — For $A \subset X$, the preclosure[4] (resp. semi-closure[3]) of A, denoted by $pclA$ (resp. $scl\,A$) is defined by $pcl\,A = \bigcap \{B : B$ is preclosed and $B \supset A\}$ (resp. $scl\,A = \bigcap \{B : B$ is semi-closed and $B \supset A)\}$. Preinterior of A, denoted by $pint\,A$, is defined by $pint\,A = \bigcup \{B : B \in PO(X, \tau)$ and $B \subset A\}$[2].

Definition 2.3 — A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called

(i) quasi-precontinuous[31] (resp. weakly semi-continuous[9], weakly α-continuous[25], weakly continuous[12]), briefly q.p.c. (resp. w.s.c., w.α.c., w.c) if for each $x \in X$ and each $V \in \sigma$ containing $f(x)$ there exists a $U \in PO(X, \tau)$ (resp. $U \in SO(X, \tau)$, $U \in \tau^{\alpha}$, $U \in \tau$) containing x such that $f[U] \subset Cl\,V$;

(ii) precontinuous[16] (resp. semi-continuous[13], α-continuous[17]) briefly p.c. (resp. s.c., α.c.) if for each $V \in \sigma$, $f^{-1}[V] \in PO(X, \tau)$ (resp. $f^{-1}[V] \in SO(X, \tau)$, $f^{-1}[V] \in \tau^{\alpha}$);

(iii) almost α-continuous[26] (resp. almost continuous in the sense of Singal[33]), briefly a.α.c. (resp. a.c.S) if for each $V \in RO(Y, \sigma)$, $f^{-1}[V] \in \tau^{\alpha}$ (resp. $f^{-1}[V] \in \tau$);

(iv) almost continuous in the sense of Husain[6], briefly a.c. H, if for each $x \in X$ and each $V \in \sigma$ containing $f(x)$, $Cl(f^{-1}[V])$ is a neighbourhood in X.

(v) θ-continuous[5], if for each $x \in X$ and each $V \in \sigma$ containing $f(x)$ there exists a $U \in \tau$ such that $f[Cl\,U] \subset Cl\,V$.

Definition 2.4 — Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be any function. Then the subset $G(f) = \{(x, f(x)) : x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the graph of f (Husain[7]).

$G(f)$ is said to be strongly closed if for each $(x, y) \in X \times Y - G(f)$, there exist open sets $U \subset X$ and $V \subset Y$ containing x and y, respectively, such that $[U \times Cl_{\sigma}(V)] \cap G(f) = \phi$ (Long and Herrington[15]).

Definition 2.5 — X is said to be a

(i) Urysohn space if for $x, y \in X, x \neq y$, there exist neighbourhoods U of x and V of y with $Cl\ U \bigcap Cl\ V = \phi$ (Willard[35]);

(ii) rim-compact, if each point of X has a base of neighbourhoods with compact frontiers[35];

(iii) strongly compact if every preopen cover of X admits a finite subcover[18];

(iv) extremally disconnected if the closure of every open set in X is open[35];

(v) pre-T_2 iff to each pair of distinct points x, y of X, there exists a pair of disjoint preopen sets, one containing x and the other containing y (Kar and Bhattacharyya[10]).

Definition 2.6 — $A \subset X$ is said to be quasi-H closed relative to X if for every open cover $\{V_\alpha : \alpha \in \Lambda\}$ of A, there exists a subfamily Λ_0 of Λ such that $A \subset U$ $\{Cl\ V_\alpha : \alpha \in \Lambda_0\}$ (Porter *et al.*[30]).

Definition 2.7 — For $A \subset X$, the θ-closure of A, denoted by $Cl_\theta (A)$, is defined by $Cl_\theta (A) = \{x \in X : Cl\ V \bigcap A \neq \phi$ for each open V containing $x\}$. A is called θ-closed if $Cl_\theta (A) = A$ (Veličko[34]).

The present authors[31] establish the following characterization of q.p.c. functions.

Theorem 2.8 — For $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent :

(a) f is q.p.c.

(b) For each open $E \subset Y$, $f^{-1} [E] \subset pint_\tau (f^{-1} [Cl_\sigma (E)])$.

(c) For each open $B \subset Y$, $pcl_\tau (f^{-1} [Int_\sigma (Cl_\sigma (B))]) \subset f^{-1} [Cl_\sigma (B)]$.

Preclosure and preinterior have been expressed in terms of closure and interior in the following theorem by Andrijević[1] and they will be utilised in the sequel.

Theorem 2.9 — For $A \subset X, pcl\ A = A \bigcup Cl\ (Int\ A)$ and $pint\ A = A \bigcap Int\ (Cl\ A)$.

3. New Characterizations

Theorem 3.1 — For a function $f : (X, \tau) \to (Y, \sigma)$, the following are equivalent :

(a) f is q.p.c.

(b) For each open $A \subset Y$, $f^{-1} [A] \subset Int_\tau (Cl_\tau (f^{-1} [Cl_\sigma (A)]))$.

(c) For each open $B \subset Y$, $Cl_\tau (Int_\tau (f^{-1} [B])) \subset f^{-1} [Cl_\sigma (B)]$.

Proof : (a) $\to$ (b). Let $A \subset Y$ be open. Since f is q.p.c., by Theorem 2.8, $f^{-1} [A] \subset pint_\tau (f^{-1} [Cl_\sigma (A)])$. So, by Theorem 2.9, $f^{-1} [A] \subset f^{-1} [Cl_\sigma (A)] \bigcap Int_\tau [Cl_\tau (f^{-1} [Cl_\sigma (A)]))$. Hence $f^{-1} [A] \subset Int_\tau (Cl_\tau (f^{-1} [Cl_\sigma (A)]))$.

(b) → (a). Let $A \subset Y$ be open. Then $f^{-1}[A] \subset Int_\tau(Cl_\tau(f^{-1}[Cl_\sigma(A)]))$. Also $f^{-1}(A) \subset f^{-1}[Cl_\sigma(A)]$. So, by Theorem 2.9, $f^{-1}[A] \subset f^{-1}[Cl_\sigma(A)] \bigcap Int_\tau(Cl_\tau(f^{-1}[Cl_\sigma(A)])) = pint_\tau(f^{-1}[Cl_\sigma(A)])$ and therefore, by Theorem 2.8, f is q.p.c.

(a) → (c). Let $B \subset Y$ be open. Since f is q.p.c., by Theorem 2.8, $pcl_\tau(f^{-1}[B]) \subset f^{-1}[Cl_\sigma(B)]$. So, by Theorem 2.9, $f^{-1}[B] \bigcup Cl_\tau(Int_\tau(f^{-1}[B])) \subset f^{-1}[Cl_\sigma(B)]$. Hence $Cl_\tau(Int_\tau(f^{-1}[B])) \subset f^{-1}[Cl_\sigma(B)]$.

(c) → (a). Let $B \subset Y$ be open. Now $Int_\sigma(Cl_\sigma(B))$ is open in Y. Hence by (c), $Cl_\tau(Int_\tau(f^{-1}[Int_\sigma(Cl_\sigma(B))])) \subset f^{-1}[Cl_\sigma(Int_\sigma(Cl_\sigma(B)))] \subset f^{-1}[Cl_\sigma(B)]$. Hence by Theorem 2.9, $pcl_\tau(f^{-1}[Int_\sigma(Cl_\sigma(B))]) = f^{-1}[Int_\sigma(Cl_\sigma(B))] \bigcup Cl_\tau(Int_\tau(f^{-1}[Int_\sigma(Cl_\sigma(B))])) \subset f^{-1}[Int_\sigma(Cl_\sigma(B))] \bigcup f^{-1}[Cl_\sigma(B)] = f^{-1}[Cl_\sigma(B)]$ so that f is q.p.c. by Theorem 2.8.

4. Some Basic Properties of q.p.c. Functions

4.1. *q.p.c. in Subspaces*

If a function is q.p.c. in a topological space, the present authors (Paul and Bhattacharyya[31], Theorem 6) showed that the function is, then, q.p.c. in any open subspace. The next theorem provides a sufficient condition for a function to be q.p.c. when it is given to be so in some subspaces. In fact we have

Theorem 4.1 — Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $A \in PO(X)$. If the restriction $f_A : A \to Y$ is q.p.c. at $x \in A$, then f is q.p.c. at x.

Proof : Let V be any open set containing $f(x)$. Since f_A is q.p.c. at x, there exists $U \in PO(A)$ such that $x \in U$ and $f[U] \subset Cl_\sigma V$. This gives that $f[U] \subset Cl_\sigma V$. Since $A \in PO(X)$ and $U \in PO(A)$, by Lemma 1 (Kar and Bhattacharyya[10]), $U \in PO(X)$. From this it follows that f is q.p.c. at x.

4.2. *Graph Function Associated with q.p.c.*

Theorem 4.2 — A function $f : (X, \tau) \to (Y, \sigma)$ is q.p.c. if and only if the graph function $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ defined by $g(x) = (x, f(x))$ is q.p.c. at every $x \in X$.

Proof : *Necessity* — Suppose f is q.p.c. Let $x \in X$ and $g(x) \in W \in \tau \times \sigma$. Then there exist $U_1 \in \tau$ and $V \in \sigma$ such that $(x, f(x)) \in U_1 \times V \subset W$. Since f is q.p.c. there exists $U_2 \in PO(X, \tau)$ such that $x \in U_2$ and $f[U_2] \subset Cl_\sigma(V)$. Let $U = U_1 \bigcap U_2$. Since the intersection of an open set with a p.o. set is p.o.[16], $x \in U \in PO(X, \tau)$ and hence $f[U] \subset f[U_2] \subset Cl_\sigma(V)$. Now we observe that $g(U) \subset U \times Cl_\sigma(V) \subset U_1 \times Cl_\sigma(V) \subset Cl_{\tau\times\sigma}(U_1 \times V) \subset Cl_{\tau\times\sigma}(W)$. This shows that g is q.p.c. at x.

Sufficiency — Suppose g is q.p.c. Let $x \in X$ and $f(x) \in V \in \sigma$. Then $g(x) \in X \times V \in \tau \times \sigma$ and there exists $U \in PO(X, \tau)$ containing x such that $g(U) \subset Cl_{\tau\times\sigma}(X \times V) = X \times Cl_\sigma(V)$. Hence we obtain $f[U] \subset Cl_\sigma(V)$ which shows that f is q.p.c. at x.

4.3. *Invariant Properties under q.p.c.*

Definition 4.3 — X is said to be preconnected[27] if it is not the union of two non-empty disjoint preopen sets. A space which is not preconnected is called predisconnected.

Noiri[21] showed that connectedness is preserved under w.c. Example 4.4 substantiates the fact that connectedness is not preserved under p.c. and hence not under q.p.c. That preconnectedness is not preserved under w.c. and hence not under q.p.c. is evident from Example 4.5.

Example 4.4 — Let $X = \{a, b, c\}$, $Y = \{a^*, b^*, c^*\}$ be two sets endowed with the topologies $\tau_X = \{\phi, X\}$ and $\tau_Y = \{\phi, Y, \{a^*\}, \{b^*, c^*\}\}$ respectively. Let $f : X \rightarrow Y$ be defined by $f(x) = x^*$ for all $x \in X$. Then X is connected, f is a p.c. surjection but Y is disconnected.

Example 4.5 — Let $\tau_X = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$ and $\tau_Y = \{\phi, Y\}$ be the respective topologies on X and Y of Example 4.4. Let $f : X \rightarrow Y$ be the same function of Example 4.4. Then Y is not preconnected though f is a w.c. surjection and X is preconnected.

However, the following is true.

Theorem 4.6 — If X is preconnected, $f : X \rightarrow Y$ is p.c. and open then Y is preconnected.

PROOF : Suppose Y is not preconnected. Then there exist $\phi \neq V_1, V_2 \in PO(Y)$, $V_1 \cap V_2 = \phi$ such that $Y = V_1 \cup V_2$. Since f is p.c. and open, by Theorem 2[16], we observe that $f^{-1}[V_1], f^{-1}[V_2] \in PO(X)$. Also $f^{-1}[V_1] \cup f^{-1}[V_2] = X$ and $f^{-1}[V_1] \cap f^{-1}[V_2] = \phi$. This indicates that X is not preconnected, a contradiction.

It is not difficult to check that every preconnected space is connected and the converse of it may not hold. On the other hand, though preconnectedness is not invariant under q.p.c. (see Example 4.5) the following is valid.

Theorem 4.7 — If (X, τ) is preconnected and $f : (X, \tau) \rightarrow (Y, \sigma)$ is q.p.c. surjection, then (Y, σ) is connected.

PROOF : Suppose Y is not connected. Then there exist $\phi \neq V_1, V_2 \in \sigma$ such that $Y = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$. Hence $f^{-1}[V_1] \cup f^{-1}[V_2] = X$ and $f^{-1}[V_1] \cap f^{-1}[V_2] = \phi$. Since f is q.p.c. and surjective, $\phi \neq f^{-1}[V_i] \subset Int_\tau(Cl_\tau(f^{-1}[Cl_\sigma(V_i)]))$, $i = 1, 2$ by Theorem 3.1. But V_i is both open and closed and so $f^{-1}[V_i] \subset Int_\tau(Cl_\tau(f^{-1}[V_i]))$ for $i = 1, 2$. Hence $f^{-1}[V_i]$ is p.o., $i = 1, 2$. This then contradicts the hypothesis that (X, τ) is preconnected. Therefore, (Y, σ) is connected.

Theorem 4.8 — If $f : (X, \tau) \rightarrow (Y, \sigma)$ be q.p.c. and K be a strongly compact subset of (X, τ), then $f[K]$ is quasi H-closed relative to (Y, σ).

PROOF : Let $\{V_\alpha : \alpha \in \Lambda\}$ be any cover of $f[K]$ by open sets in Y. Then $f[K] \subset \bigcup_{\alpha \in \Lambda} V_\alpha$ and so $K \subset \bigcup_{\alpha \in \Lambda} f^{-1}[V_\alpha]$. Since f is q.p.c., $f^{-1}[V_\alpha] \subset pint_\tau(f^{-1}[Cl_\sigma

$(V_\alpha)])$ for each $\alpha \in \Lambda$ by Theorem 2.8. Therefore, $K \subset \bigcup_{\alpha \in \Lambda} pint_\tau\,(f^{-1}[Cl_\sigma(V_\alpha)])$. Strong compactness of K indicates that there exists a finite subfamily Λ_0 of Λ such that $K \subset \bigcup_{\alpha \in \Lambda_0} pint_\tau\,(f^{-1}[Cl_\sigma(V_\alpha)])$. This gives $f[K] \subset f\left[\bigcup_{\alpha \in \Lambda_0} pint_\tau\,(f^{-1}[Cl_\sigma(V_\alpha)])\right] \subset f\left[\bigcup_{\alpha \in \Lambda_0} f^{-1}[Cl_\sigma(V_\alpha)]\right] \subset \bigcup_{\alpha \in \Lambda_0} Cl_\sigma(V_\alpha)$. Hence $f[K]$ is quasi H-closed relative to Y.

5. Some Separation Axioms and q.p.c. Functions

Noiri[21] established that domain space is T_2 if the range space is Urysohn under a w.c. injection. Analogous result is the following :

Theorem 5.1 — If Y is a Urysohn space and $f : (X, \tau) \to (Y, \sigma)$ is a q.p.c. injection then X is pre-T_2.

Proof : Since f is injective, for any pair of distinct points $x_1, x_2 \in X$ $f(x_1) \neq f(x_2)$. Since Y is Urysohn, there exist $V_1, V_2 \in \sigma$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $Cl_\sigma(V_1) \bigcap Cl_\sigma(V_2) = \phi$. This gives $f^{-1}[Cl_\sigma(V_1)] \bigcap f^{-1}[Cl_\sigma(V_2)] = \phi$ and hence $pint_\tau\,(f^{-1}[Cl_\sigma(V_1)]) \bigcap pint_\tau\,(f^{-1}[Cl_\sigma(V_2)]) = \phi$. Since f is q.p.c. $x_i \in f^{-1}[V_i] \subset pint_\tau\,(f^{-1}[Cl_\sigma(V_i)])$, $i = 1, 2$ by Theorem 2.8 and this indicates that (X, τ) is pre-T_2.

Remark 5.2 : Pre-T_2 on (X, τ) cannot be replaced by T_2 in Theorem 5.1 as shown by the following.

Example 5.3 — Let $X = [0, 1]$ and τ_X be the family of subsets of X such that (i) $\phi \in \tau_X, X \in \tau_X$ and (ii) a non-empty proper subset A of X belongs to τ_X if and only if for every $x \in A$, there exists a $r > 0$ such that $I_r(x) = (x - r, x + r) \subset A$. Then τ_X is a topology for X. Let $Y = [0, 1]$ be equipped with the usual topology τ_Y and $f : (X, \tau_X) \to (Y, \tau_Y)$ be the identity mapping. Then f is a q.p.c. injection and Y is Urysohn but X is not T_2.

Definition 5.4 — If $A \subset X$, then a mapping $f : X \to A$ is termed as q.p.c. retraction if f is q.p.c. and f_A is the identity mapping on A.

Theorem 5.5 — Let $A \subset X$ and $f : X \to A$ be a q.p.c. retraction. If X is T_2, then A is a preclosed subset of X.

Proof : Suppose A is not preclosed. Then $pclA - A \neq \phi$. Let $x \in pclA - A$. Since f is a q.p.c. retraction, $f(x) \neq x$. Since X is T_2 there exist disjoint open sets U and V such that $x \in U$, $f(x) \in V$ and this implies that $U \bigcap Cl\,V = \phi$. Let $W \subset X$ be any p.o. set containing x. Then $U \bigcap W \in PO(X)$ (Mashhour *et al.*[16]) and $x \in U \bigcap W$. Since $x \in pcl\,A$, $(U \bigcap W) \bigcap A \neq \phi$ (Lemma 2.2, El-Deeb *et al.*[4]). Let $y \in U \bigcap W \bigcap A$. Then $y \in A$ and so $f(y) = y \in U \bigcap W \bigcap A \subset U$. Hence $f(y) \notin Cl\,V$. This indicates that $f[W] \not\subset Cl\,V$, a contradiction to the fact that f is q.p.c. Hence A is preclosed.

Noiri[21] showed that for a w.c. mapping $f : X \to Y$, the graph $G(f)$ is closed if Y is T_2. Analogous result to this is obtained here where "w.c." and "closed" are replaced by "q.p.c." and "preclosed" respectively. We prove first the parallel of Theorem 1 of Noiri[21] by utilising the notions of q.p.c. and p.o. set.

Theorem 5.6 — If $f : (X, \tau) \to (Y, \sigma)$ is q.p.c. and Y is T_2 then for each $(x, y) \notin G(f)$ there exist $U \in PO(X, \tau)$ and $V \in \sigma$ such that $x \in U$, $y \in V$ and $f[U] \bigcap Int_\sigma(Cl_\sigma(V)) = \phi$.

PROOF : Suppose $(x, y) \notin G(f)$. Then $y \neq f(x)$. Since Y is T_2, there exist $V, W \in \sigma$ such that $y \in V$, $f(x) \in W$ and $V \bigcap W = \phi$. It is easy to verify that $Int_\sigma(Cl_\sigma(V)) \bigcap Cl_\sigma(W) = \phi$. The q.p.c. of f gives a $U \in PO(X, \tau)$ containing x such that $f[U] \subset Cl_\sigma(W)$ and hence $f[U] \bigcap Int_\sigma(Cl_\sigma(V)) = \phi$.

Corollary 5.7 — If $f : (X, \tau) \to (Y, \sigma)$ is q.p.c. and Y and is T_2, then $G(f)$ is preclosed.

PROOF : If $(x, y) \in X \times Y - G(f)$, then there exists a$U \in PO(X, \tau)$, a $V \in \sigma$ such that $x \in U$, $y \in V$ and $f[U] \bigcap Int_\sigma(Cl_\sigma(V)) = \phi$. Hence $f[U] \bigcap V = \phi$ so that $(U \times V) \bigcap G(f) = \phi$. Thus $(x, y) \in U \times V \subset X \times Y - G(f)$ where $U \times V$ is a p.o. set in $X \times Y$ (Mashhour *et al.*[16]). This shows that $X \times Y - G(f)$ is p.o. and hence $G(f)$ is preclosed.

Definition 5.8 — For a function $f : X \to Y$, the graph $G(f)$ is said to be strongly preclosed if for each $(x, y) \in X \times Y - G(f)$ there exist $U \in PO(X)$, $V \in PO(Y)$ such that $x \in U$, $y \in V$ and $[U \times pcl\, V] \bigcap G(f) = \phi$.

A useful characterization of functions with strongly preclosed graph is the following lemma.

Lemma 5.9 — The function $f : X \to Y$ has a strongly preclosed graph if and only if for $(x, y) \in X \times Y - G(f)$ there exist $U \in PO(X)$, $V \in PO(Y)$ such that $x \in U$, $y \in V$ and $f[U] \bigcap pcl\, V = \phi$.

PROOF : Follows from Definition 5.8 and is omitted.

Theorem 5.10 — If $f : (X, \tau) \to (Y, \sigma)$ is a q.p.c. function and Y is Urysohn, then $G(f)$ is strongly preclosed.

PROOF : Let $(x, y) \in X \times Y - G(f)$. Then $y \neq f(x)$. Since Y is Urysohn, there exist $V, W \in \sigma$ such that $f(x) \in V, y \in W$ and $Cl_\sigma(V) \bigcap Cl_\sigma(W) = \phi$. The openness of W gives $pcl_\sigma(W) = Cl_\sigma(W)$, by Theorem 2.4 of El-Deeb *et al.*[4]. Since f is q.p.c. there exists $U \in PO(X, \tau)$ such that $x \in U$ and $f[U] \subset Cl_\sigma(V)$. This, therefore, implies that $f[U] \bigcap pcl_\sigma(W) = \phi$. So, by Lemma 5.9, $G(f)$ is strongly preclosed.

The study of q.p.c. functions now raises the following natural question : When do two functions — one of which is q.p.c.-coincide on a topological space ? The outcome of our endeavour to answer the above question forms the material of the rest of this section. In the process of our answer, we come across some sets which, though utilised to obtain the desired result, are interesting in their own right.

Theorem 5.11 — Let f, g : $(X, \tau) \to (Y, \sigma)$ be functions and Y be Urysohn. If f is w.c., g is q.p.c., then the set $\{x \in X : f(x) = g(x)\}$ is preclosed.

PROOF : Let $A = \{x \in X : f(x) = g(x)\}$. If $x \in X - A$, then $f(x) \neq g(x)$. Since Y is Urysohn there exists $V_1, V_2 \in \sigma$ such that $f(x) \in V_1, g(x) \in V_2$ and $Cl_\sigma(V_1) \cap Cl_\sigma(V_2) = \phi$. The w.c. of f gives $x \in f^{-1}[V_1] \subset Int_\tau(f^{-1}[Cl_\sigma(V_1)])$, by Theorem 1 of Levine[12]. Also the q.p.c. of g gives $x \in g^{-1}[V_2] \subset pint_\tau(g^{-1}[Cl_\sigma(V_2)])$ by Theorem 2.8.

Let $U = Int_\tau(f^{-1}[Cl_\sigma(V_1)]) \cap pint_\tau(g^{-1}[Cl_\sigma(V_2)])$. Since the intersection of an open set with a p.o. set is p.o. (Mashhour *et al.*[16]), $U \in PO(X, \tau)$ and $x \in U$. Again disjointness of $Cl_\sigma(V_i)$ for $i = 1, 2$, implies that $U \cap A = \phi$ and hence $x \in U \subset X - A$. This indicates that $X - A$ is a union of p.o. sets. Therefore, $X - A \in PO(X, \tau)$ (Mashhour *et al.*[16]). Consequently, A is preclosed.

Note : The referee raises the natural question : Whether is it possible to replace the Urysohn hypothesis by Hausdorff hypothesis in Theorem 5.10 and 5.11 ? As to Theorem 5.10 we like to point out that if Y is Hausdorff then $G(f)$ is preclosed but not strongly preclosed (see Corollary 5.7). So, to achieve the desired result, the Urysohn hypothesis is irreplaceable. The method of proof exhibited in Theorem 5.11 fails if the assumption "Urysohn" is replaced by the assumption "Hausdorff". Whether it holds under Hausdroff hypothesis remains open.

Corollary 5.12 — Under the hypotheses of the preceding theorem, if $f = g$ on a dense open set D on X, then $f = g$ on X.

PROOF : Since $f = g$ on D, $D \subset A$ and hence $pcl_\tau(D) \subset pcl_\tau(A) = A$, by the above Theorem. Since D is open and dense, $pcl_\tau(D) = Cl_\tau(D) = X$, by Theorem 2.4 of El-Deeb *et al.*[4]. Consequently $X \subset A$. Therefore $f = g$ on X.

Theorem 5.13 — If g : $(X, \tau) \to (Y, \sigma)$ is q.p.c. and S is a θ-closed subset of $(X \times Y, \tau \times \sigma)$, then $p_1(S \cap G(g))$ is preclosed in X, where p_1 represents the projection of $X \times Y$ onto X.

PROOF : Let $x \in pcl(p_1(S \cap G(g)))$, where S is a θ-closed subset of $X \times Y$. Let U be an arbitrary open set containing x and V be an arbitrary open set containing $g(x)$. Since g is q.p.c., by Theorem 2.8, $x \in g^{-1}[V] \subset pint_\tau(g^{-1}[Cl_\sigma(V)])$. Then $U \cap pint_\tau(g^{-1}[Cl_\sigma(V)])$ is a p.o. set (Mashhour *et al.*[16]) containing x. Since $x \in pcl_\tau(p_1(S \cap G(g)))$, $[U \cap pint_\tau(g^{-1}[Cl_\sigma(V)])] \cap p_1(S \cap G(g))) \neq \phi$ (El-Deeb *et al.*[4]). Let $x_0 \in [U \cap pint_\tau(g^{-1}[Cl_\sigma(V)])] \cap p_1(S \cap G(g))$. This implies that $(x_0, g(x_0)) \in S$ and $g(x_0) \in Cl_\sigma(V)$. Therefore, $\phi \neq [U \times Cl_\sigma(V)] \cap S \subset Cl_{\tau \times \sigma}(U \times V) \cap S$ and, consequently $(x, g(x)) \in Cl_\theta(S)$. Since S is θ-closed, $(x, g(x)) \in (S) \cap G(g)$. Hence $x \in p_1(S \cap G(g))$. This shows that $p_1(S \cap G(g))$ is preclosed in X.

Corollary 5.14 — If $f : X \times Y$ has a θ-closed graph and g : $(X, \tau) \to (Y, \sigma)$ is q.p.c. then the following hold :

(a) The set $\{x \in X : f(x) = g(x)\}$ is preclosed.

(b) If $f = g$ on a dense open set D on X, then $f = g$ on X.

PROOF : Since $\{x \in X : f(x) = g(x)\} = p_1(G(f) \cap G(g))$ and $G(f)$ is a θ-closed subset of $X \times Y$, it follows from the preceding theorem that $\{x \in X : f(x) = g(x)\}$ is preclosed. The second assertion follows from the first one and from the fact that $X = Cl_\tau(D) = pcl_\tau(D) \subset pcl_\tau(A)$.

Corollary 5.15 — If $f : X \to Y$ is θ-continuous, $g : X \to Y$ is q.p.c. and Y is Urysohn, then the following hold :

(a) The set $\{x \in X : f(x) = g(x)\}$ is preclosed.

(b) If $f = g$ on a dense open set D on X, then $f = g$ on X.

PROOF : Noiri[22] (Theorem 7) showed that for a Urysohn space Y, if $f : X \to Y$ is θ-continuous, then $G(f)$ is θ-closed in $X \times Y$. The results, now, follow from Corollary 5.14.

Theorem 5.16 — Let $f, g : (X, \tau) \to (Y, \sigma)$ be functions and Y be Hausdorff. If f is q.p.c., g is a.α.c. and $f = g$ on a dense open set D on X, then $f = g$ on X.

PROOF : Let $A = \{x \in X : f(x) = g(x)\}$. Suppose $x \in X - A$. Then $f(x) \neq g(x)$. Since Y is Hausdorff, there exist open sets V and W such that $f(x) \in V$, $g(x) \in W$ and $V \cap W = \phi$. The disjointness of V and W gives $Cl_\sigma(V) \cap Int_\sigma(Cl_\sigma(W)) = \phi$. Since f is q.p.c. there exists $U_1 \in PO(X, \tau)$ containing x such that $f[U_1] \subset Cl_\sigma(V)$. Since g is a.α.c., there exists $U_2 \in \tau^\alpha$ containing x such that $g[U_2] \subset Int_\sigma(Cl_\sigma(W))$ (by Theorem 3.2 of Noiri[26]). Therefore we obtain $f[U_1] \cap g[U_2] = \phi$ and this implies that $(U_1 \cap U_2) \cap A = \phi$. Since the intersection of an α-set with a p.o. set is p.o. by Lemma 4.2 of El-Deeb *et al.*[4], $U_1 \cap U_2 \in PO(X, \tau)$ and $x \in U_1 \cap U_2$. So $x \notin pcl_\tau(A)$. This implies that A is preclosed. Again by hypothesis, $f = g$ on D and hence $D \subset A$. But $X = Cl_\tau(D) = pcl_\tau(D) = pcl_\tau(A) = A$, by Theorem 2.4 of El-Deeb *et al.*[4]. Consequently, $f = g$ on X.

6. INTERRELATIONSHIP

The study of q.p.c. functions now raises another natural question : What additional property will make quasi-precontinuity equivalent to continuity and to other weak forms of continuous functions such as p.c., w.c., a.c.H., a.c.S ? Our attempt to answer this question forms the material of this section.

I. *q.p.c. and p.c. Functions*

Though q.p.c. functions are not generally p.c. (Paul and Bhattacharyya[31]), the rim-compactness of the range space gives a condition for precontinuity of q.p.c. functions as shown by

Theorem 6.1 — If $f : X \to Y$ is a q.p.c. function with the closed graph $G(f)$ and Y is a rim-compact space, then f is p.c.

PROOF : Let $x \in X$ and V be any open set contianing $f(x)$. Since Y is rim-compact, there exists an open set $W \subset Y$ such that $f(x) \in W \subset V$ and $Fr(W)$ is compact. Clearly $f(x) \notin Fr(W)$. Thus for each $y \in Fr(W)$, $(x, y) \in X \times Y - G(f)$, which is open in $X \times Y$. Hence for each $y \in Fr(W)$, we can find sets U_y, V_y open in X and Y respectively such that $x \in U_y, y \in V_y$ and $f[U_y] \bigcap V_y = \phi$. Now, the family $\{V_y : y \in Fr(W)\}$ is an open cover of $Fr(W)$. Since $Fr(W)$ is compact, there exist finite number of points $y_1, y_2, ..., y_n$ in $Fr(W)$ such that $Fr(W) \subset \bigcup_{j=1}^{n} V_{y_j}$. Since f is q.p.c. there exists $U_0 \in PO(X)$ such that $x \in U_0$ and $f[U_0] \subset ClW$. We set $U = U_0 \bigcap \left[\bigcap_{j=1}^{n} U_{y_j}\right]$. Since the intersection of an open set with a p.o. set is p.o. (Mashhour *et al.*[16]), it, then, follows that $U \in PO(X)$, $x \in U$ and $f[U] \bigcap (Y - V) \subset f[U] \bigcap (Y - W) = f[U] \bigcap Fr(W) \subset f[U] \bigcap \left[\bigcup_{j=1}^{n} V_{y_j}\right] \subset \left(\bigcup_{j=1}^{n} f[U_{y_j}] \bigcap V_{y_j}\right) = \phi$. Hence $f[U] \subset V$ so that f is p.c.

II. *q.p.c. and a.c.H Functions*

Since a.c.H functions are the same as p.c. functions (Theorem 1 of Mashhour *et al.*[16]), the hypotheses of Theorem 6.1 provide us with a condition for a.c.H. of q.p.c. functions. Openness of f is another condition for a.c.H of q.p.c. functions as shown by :

Theorem 6.2 — An open q.p.c. function is always a.c.H.

PROOF : Let $f : (X, \tau) \to (Y, \sigma)$ be an open q.p.c. function and $x \in X$. Let $V \in \sigma$ containing $f(x)$. By Lemma of Long and Carnahan[14], $f^{-1}[Cl_\sigma(V)] \subset Cl_\tau(f^{-1}[V])$ and hence $x \in f^{-1}[V] \subset f^{-1}[Cl_\sigma(V)] \subset Cl_\tau(f^{-1}[V])$. Since f is q.p.c. $f^{-1}(V) \subset Int_\tau(Cl_\tau(f^{-1}[Cl_\sigma(V)]))$, by Theorem 3.1. Now we observe that $x \in Int_\tau(Cl_\tau(f^{-1}[Cl_\sigma(V)])) \subset Cl_\tau(f^{-1}[Cl_\sigma(V)]) \subset Cl_\tau(f^{-1}[V])$ by Lemma of Long and Carnahan[14]. Consequently, $Cl_\tau(f^{-1}[V])$ is a neighbourhood of x and hence f is a.c.H.

III. *q.p.c. and w.c. functions*

q.p.c. functions are not generally w.c. (Paul and Bhattacharyya[31]). On the other hand, semi-continuity is independent of weak continuity (Noiri[23]). But this semi-continuity gives a condition for w.c. of q.p.c. functions as shown by

Theorem 6.3 — If a q.p.c. function $f : (X, \tau) \to (Y, \sigma)$ is also s.c., then it is w.c.

PROOF : Suppose E be any open set in Y. Since f is q.p.c., $Cl_\tau(Int_\tau(f^{-1}[E])) \subset f^{-1}[Cl_\sigma(E)]$ by Theorem 3.1. Since f is s.c., $f^{-1}[E] \in SO(X, \tau)$ and hence $Cl_\tau(Int_\tau(f^{-1}[E])) = Cl_\tau(f^{-1}[E])$ by Lemma 1 of Noiri[20]. Consequently, $Cl_\tau(f^{-1}[E]) \subset f^{-1}[Cl_\sigma(E)]$ and so, by Theorem 7 of Rose[32], f is w.c.

Remark 6.4 : Theorem 6.3 shows that semi-continuity is independent of quasi-precontinuity.

Remark 6.5 : Popa[28] showed that if a function is both p.o. and s.c. then it is w.c. Theorem 6.3 is an improvement of this result of Popa.

IV. *q.p.c. and a.c.S. functions*

For a function $f: X \to Y$, the following implications are known[33, 31] a.c.S. $\Rightarrow$ w.c. $\Rightarrow$ q.p.c. But the reverse implications do not hold[33, 31]. The next two theorems give conditions under which q.p.c. and a.c.S. functions are equivalent.

Theorem 6.6 — If an open q.p.c. function $f: X \to Y$ is also s.c., then it is a.c.S.

PROOF : Follows from Theorem 6.3 and Theorem 2.3 of Singal and Singal[33].

Theorem 6.7 — If a semi-open q.p.c. function $f : (X, \tau) \to (Y, \sigma)$ is also s.c. and Y is extremally disconnected, then f is a.c.S.

PROOF : Suppose E be any open set in Y. Since f is s.c., $f^{-1}[E] \subset SO(X, \tau)$ so that $f^{-1}[E] \subset Cl_\tau(Int_\tau(f^{-1}[E]))$ and hence $Cl_\tau(f^{-1}[E]) \subset Cl_\tau(Int_\tau(f^{-1}[E]))$. Since f is q.p.c. $Cl_\tau(Int_\tau(f^{-1}[E])) \subset f^{-1}[Cl_\sigma(E)]$ by Theorem 3.1. Therefore, we obtain $Cl_\tau(f^{-1}[E]) \subset f^{-1}[Cl_\tau(E)]$. On the other hand, f being semi-open, $f^{-1}[scl_\sigma(E)] \subset Cl_\tau(f^{-1}[E])$ by Theorem 2 of Noiri[20]. Since Y is extremally disconnected and E is a s.o. set, $scl_\sigma(E) = Cl_\sigma(E)$ by Lemma 1.13 of Noiri[23] and therefore, $f^{-1}[Cl_\sigma(E)] \subset Cl_\tau(f^{-1}[E])$. Hence $Cl_\tau(f^{-1}[E]) = f^{-1}[Cl_\sigma(E)]$ and so, by Theorem 14 of Rose[32] f is a.c.S.

V. *q.p.c. and α-continuous functions*

For a function $f: X \to Y$ the following implications are known[17, 31, 9] q.p.c. $\Leftarrow$ p.c. $\Leftarrow$ α.c. $\Rightarrow$ s.c. $\Rightarrow$ w.s.c. But none of these implications can, in general, be reversed[17, 31, 9]. Rim-compactness and weak semi-continuity together with the closedness of $G(f)$ provide us with a condition for α-continuity of q.p.c. functions as shown by

Theorem 6.8 — If a q.p.c. function $f: X \to Y$ is also w.s.c. with the closed graph $G(f)$ and Y is rim-compact, then f is α.c.

PROOF : Follows from Theorem 6.1, Theorem 9 of Kar and Bhattacharyya[9] and Theorem 3.1 of Mashhour *et al.*[17].

VI. *q.p.c. and continuous functions*

It is obvious that the class of q.p.c. functions properly contains the class of continuous functions. The next three theorems give conditions under which q.p.c. and continuous functions are equivalent.

Theorem 6.9 — If $f: X \to Y$ is q.p.c. and w.s.c. and Y is regular, then f is continuous.

PROOF : Follows from Theorem 2 of Kar and Bhattacharyya[9], Theorem 2 of Paul and Bhattacharyya[31], Theorem 3.1 of Mashhour *et al.*[17] and the Remark.

Theorem 6.10 — If an open q.p.c. function is s.c. and Y is semi-regular, then f is continuous.

PROOF : Follows from Theorem 6.6 and Theorem 2.4 of Singal and Singal[33].

Theorem 6.11 — If a semi-open q.p.c. function $f: X \to Y$ is s.c. and Y both semi-regular and extremally disconnected, then f is continuous.

PROOF : Follows from Theorem 6.7 and Theorem 2.4 Singal and Singal[33].

ACKNOWLEDGEMENT

The authors are deeply grateful to the referee for his valuable suggestions and remarks.

REFERENCES

1. D. Andrijević, *Mat. Vesnik* **38** (1986), 24-32.
2. P. Bhattacharyya, *North Bengal University Review* 7(1) (1992), 46-52.
3. N. Biswas, *Atti. Accad. Naz. Lincei Rend. Cl. Sc. Fis. Mat. Natur.* **48** (1970), 399-402.
4. N. El-Deeb, I. A. Hasanein, A. S. Mashhour and T. Noiri, *Bull. Math. dela Soc. Sci. Math. dela R. S. de Roumanie Tome* **27** (75), nr. 4, (1983), 311-15.
5. S. V. Fomin, *Ann. Math.* **44** (1943), 471-80.
6. T. Husain, *Prace Mat.* **10** (1966), 1-7.
7. T. Husain, *Topology and Maps*, Plenum Press, New York, 1977.
8. D. S. Jancović, *Internat. J. Math. & Math. Sci.* **8** (1985), 615-19.
9. A. Kar and P. Bhattacharyya, *J. Indian Acad. Math.* **8**(2) (1986), 83-93.
10. A. Kar and P. Bhattacharyya, *Bull. Calcutta Math. Soc.* **82** (1990), 415-22.
11. A. Kar, *Soochow J. Math.* **15** (1) (1989), 65-77.
12. N. Levine, *Am. Math. Monthly* **68** (1961), 44-46.
13. N. Levine, *Am. Math. Monthly* **70** (1963), 36-41.
14. P. E. Long and D. A. Carnahan, *Proc. Am. Math. Soc.* **38** (1973), 413-18.
15. P. E. Long and L. L. Herrington, *Boll. Un. Mat. Ital.* (4) **12** (1975), 381-84.
16. A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, *Proc. Math. Phys. Soc. Egypt* **51** (1982), 47-53.
17. A. S. Mashhour, T. A. Hasanein and S. N. El-Deeb, *Acta Math. Hung.* **41**, (1983), 213-18.
18. A. S. Mashhour, M. E. Abd El-Monsef, I. A. Hasanein and T. Noiri, *Delta J. Sci.* **8** (1984), 30-46.
19. O. Njåstad, *Pacific J. Math.* **15** (1965), 961-70.
20. T. Noiri, *Bull. Calcutta Math. Sci.* **65** (1973), 197-201.
21. T. Noiri, *Proc. Am. Math. Soc.* **46** (1) (1974), 120-24.
22. T. Noiri, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **58** (1975), 887-91.
23. T. Noiri, *Czech. Math. J.* **31** (106), (1981), 314-21.
24. T. Noiri, *Časopis Pěst. Mat.* **109** (1984), 118-26.
25. T. Noiri, *Internat. J. Math. & Math. Soc.* **10** (3) (1987), 483-90.
26. T. Noiri, *Kyungpook Math. J.* **28** (1988), 71-77.
27. T. M. J. Nour, Ph.D. Thesis, University of Delhi, 1989.
28. V. Popa, *Studi Cerc. Mat.* **33** (1981), 543-46.
29. V. Popa and T. Noiri, *Demonstr. Math.* **25** (1992), 241-51.
30. J. Porter and J. Thomas, *Trans. Am. Math. Soc.* **138** (1969), 159-70.
31. Ramprasad Paul and P. Bhattacharyya, *J. Indian Acad. Math.* **14** (No. 2) (1992).
32. D. A. Rose, *Internat. J. Math. & Math. Sci.* **7** (2) (1984), 311-18.
33. M. K. Singal and A. R. Singal, *Yokohama Math. J.* **16** (1968), 63-73.
34. N. V. Veličko, *Am. Math. Soc. Trans.* **78** (2) (1968), 103-18.
35. S. Willard, *General Topology*, Addison-Wesley Publishing Company, 1970.