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D-SPACES AND THE ASSOCIATED TOPOLOGY

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This paper studies resolvability, irresolvability and open hereditary irresolvability in topological spaces from an algebraic standpoint.

INTRODUCTION

Hewitt[8] introduced the concepts of resolvability and irresolvability in topological spaces.

A topological space (X, τ) is called resolvable if there exists a pair of disjoint dense subsets of (X, τ); otherwise (X, τ) is called irresolvable.

After a major contribution owing to Hewitt, several authors found interest in the study of resolvable and irresolvable spaces. El'kin[4] studied irresolvable spaces using ultrafilters. Ganster[5, 6] with his coauthors has made connections among different types of generalized open sets with resolvability and irresolvability of spaces. Recently, maximal resolvable and minimal irresolvable spaces have been studied by us[1] and we also have extended the study of resolvable and irresolvable spaces to the bitopological setting[2]. A subclass of the class of irresolvable spaces, has been studied in Chattopadhyay and coworkers[3, 1] where a space is called open hereditarily irresolvable (in short o.h.i.) if each open set is irresolvable.

In this paper our aim is to show that resolvability and irresolvability of topological spaces can be viewed solely from an algebraic standpoint; in other words, our attempt here is to describe the topological properties like resolvability, irresolvability, open hereditary irresolvability etc. by means of a binary relation defined on the power set of a given set.

Now recall that for a space (X, τ), the topology τ is an α-topology if $\tau = \tau^{\alpha}$ where τ^{α} is the collection of all α-sets as introduced by Njastad[10], and two topologies α and σ are α-equivalent if $\tau^{\alpha} = \sigma^{\alpha}$.

We shall define a binary relation Θ (satisfying certain axioms) on the power set $\mathbb{P}(X)$ of a set X which will be called a *D*-relation and which induces a topology $\tau^*(\Theta)$ on X (see Definition 1). Given a space (X, τ) and a *D*-relation Θ on $\mathbb{P}(X)$ we shall say that Θ is weakly compatible with τ if $\tau^*(\Theta)$ and τ are α-equivalent.

We shall also observe that $\tau^*(\Theta)$ is an α-topology. Thus it follows that Θ is weakly compatible with τ if $\tau^*(\Theta) = \tau^\alpha$. Also it will be proved that given any topological space (X, τ), there exists such a relation Θ for which $\tau^*(\Theta) = \tau^\alpha$. Now noting the fact that the spaces (X, τ) and (X, τ^α) possess the same collection of dense subsets, the spaces satisfy the property that if one of the spaces has the property P, the other also has the property P, where P stands for resolvability, irresolvability or open hereditary irresolvability. Therefore, for any topological space (X, τ), it will be possible to define a weakly compatible relation Θ with restrictions imposed acordingly, so as to describe resolvability, irresolvability, open hereditary irresolvability of the topology τ.

For a subset A of a topological space (X, τ), $int_\tau A$ and $cl_\tau A$ will denote respectively the interior and closure of A with respect to τ.

D-Spaces and the α-Topology

Let X be a non-empty set.

Definition 1 — A binary relation Θ on $\mathbb{P}(X)$ will be called a D-relation if Θ satisfies the following axioms :

(D_1) $A \Theta B \Rightarrow A \cap B \neq \phi$.

(D_2) $A_\alpha \Theta B \;\; \forall \;\; \alpha \in \Lambda \Rightarrow (\bigcup A_\alpha) \Theta B$.

(D_3) $A \Theta B$ and $C \Theta D \Rightarrow A \Theta D$ and $C \Theta B$.

(D_4) $A \Theta X \Rightarrow$ there exists a non-empty subset $C \subseteq A$ satisfying the condition that whenever $P \Theta E$ and $P \cap C \neq \phi$ we have $(P \cap C) \Theta E$.

(D_5) $A \neq \phi$, $A \not\!\Theta X \Rightarrow$ there exists $O \subseteq X$ satisfying the following conditions :

(1) $A \cap O \neq \phi$ and if $P \Theta E$, $P \cap O \neq \phi$ then $(P \cap O) \Theta E$ and

(2) if $\phi \neq B \subseteq A \cap O$ then there exist subsets P and E such that $P \Theta E$, $P \cap B \neq \phi$ but $(P \cap B) \not\!\Theta E$.

(D_6) $X \not\!\Theta B \Rightarrow$ there exists $P \subseteq X - B$ such that $P \Theta X$.

If Θ is a D-relation on $\mathbb{P}(X)$ then (X, Θ) will be called a D-space.

Theorem 1 — If (X, Θ) is a D-space then

(a) $X \Theta X$

(b) $A \Theta B$ and $B \subseteq D \Rightarrow A \Theta D$.

Proof : (a) If $X \not\!\Theta X$ then by axiom (D_6) it follows that $\phi \Theta X$, which contradicts axiom (D_1).

(b) Let $A \not\!\Theta D$. Then $X \not\!\Theta D$, otherwise by axiom (D_3), $A \Theta B$ and $X \Theta D$ would imply $A \Theta D$. Hence by axiom (D_6), there exists $C \subseteq X - D$ such that $C \Theta X$. Thus by axiom (D_3), $C \Theta B$, which by axiom (D_1) implies, $C \cap B \neq \phi$. But $C \subseteq X - D$ whereas $B \subseteq D$, a contradiction. This completes the proof.

Let (X, Θ) be a *D*-space. If

$$\tau^*(\Theta) = \{A \subseteq X : P \,\Theta\, E \text{ and } P \cap A \neq \phi \;\Rightarrow\; (P \cap A)\,\Theta\, E\},$$

then it can be easily verified that $\tau^*(\Theta)$ is a topology on X.

Recall that a subset A of a topological space (X, τ) is called an α-set[10] if $A \subseteq int_\tau\, cl_\tau\, int_\tau A$ and a semi-open set[9] if $A \subseteq cl_\tau\, int_\tau A$. We now mention some facts which will be used in the sequel.

Proposition 1 — (1) If τ^α denotes the collection of all α-sets of (X, τ) then τ^α is a topology on X (Njastad[10]).

(2) A subset A of (X, τ) is an α-set if and only if $A \cap B$ is semi-open for each semi-open set B (Njastad[10]).

(3) A subset A of (X, τ) is semi-open if and only if $int_\tau A \cap O \neq \phi$ for each open set O with $A \cap O \neq \phi$.

(4) $\tau = \tau^\alpha$ if and only if every nowhere dense subset of (X, τ) is closed in (X, τ) (Njastad[10]).

Let us now review some of the axioms of a *D*-relation on X.

Axiom (D_4) means : if $A \,\Theta\, X$ then $\text{int}_{\tau^*(\Theta)} A \neq \phi$.

Axiom (D_5) (together with Proposition 1(3)) means : if A is nonempty and $A \not\Theta X$ then A is not semi-open in $(X, \tau^*(\Theta))$.

Axiom (D_6) means : if $X \not\Theta B$ then B is not dense in $(X, \tau^*(\Theta))$.

Proposition 2 — If $A \,\Theta\, X$ then A is nonempty and semi-open in $(X, \tau^*(\Theta))$.

PROOF : Suppose that $A \,\Theta\, X$. Let $x \in A$. We show that $x \in cl_{\tau^*(\Theta)}\, int_{\tau^*(\Theta)} A$. Let $H(\neq \phi) \in \tau^*(\Theta)$ and $x \in H$. Then from definition of $\tau^*(\Theta)$, it follows that $(A \cap H)\,\Theta\, X$ and by axiom (D_4), we get $int_{\tau^*(\Theta)}(A \cap H) \neq \phi$. Hence $x \in cl_{\tau^*(\Theta)}\, int_{\tau^*(\Theta)} A$. Thus A is semi-open in $\tau^*(\Theta)$. This completes the proof.

Proposition 3 — If $X \,\Theta\, B$ then B is dense in $(X, \tau^*(\Theta))$.

PROOF : Suppose $X \,\Theta\, B$. Let $H(\neq \phi) \in \tau^*(\Theta)$. Then by definition of $\tau^*(\Theta)$ it follows that $H \,\Theta\, B$. Hence by axiom (D_1), $(H \cap B) \neq \phi$. Consequently, B is dense in $\tau^*(\Theta)$.

So we have the following fundamental result.

Theorem 2 — Let Θ be a *D*-relation on X and $\tau^*(\Theta)$ be the associated topology. Then

(1) $A \,\Theta\, X$ if and only if A is nonempty and semi-open in $(X, \tau^*(\Theta))$.

(2) $X \,\Theta\, B$ if and only if B is dense in $(X, \tau^*(\Theta))$.

(3) $A \,\Theta\, B$ if and only if A is nonempty and semi-open in $(X, \tau^*(\Theta))$ and B is dense in $(X, \tau^*(\Theta))$.

PROOF : It suffices to prove (3) only. Let $A \Theta B$. Then from Theorem 1(a) and axiom (D_3) it follows that $A \Theta X$ and $X \Theta B$. Hence by Proposition 2 A is nonempty and semi-open in $(X, \tau^*(\Theta))$ and by Proposition 3, B is dense in $(X, \tau^*(\Theta))$.

Next let A be nonempty and semi-open and B be dense in $(X, \tau^*(\Theta))$. Then we have $A \Theta X$ and $X \Theta B$. Hence by axiom (D_3), it follows that $A \Theta B$.

Let (X, τ) be a topological space and (X, Θ) be a D-space. Then :

Definition 2 — (i) Θ is compatible with τ if $\tau^*(\Theta) = \tau$

(ii) Θ is weakly compatible with τ if $\tau^*(\Theta)$ and τ are α-equivalent, i.e. $(\tau^*(\Theta))^\alpha = \tau^\alpha$.

Theorem 3 — Given any D-space (X, Θ), $\tau^*(\Theta)$ is an α-topology.

PROOF : Let $O \neq \phi$ be an α-set in $(X, \tau^*(\Theta))$. We have to show that $O \in \tau^*(\Theta)$. Suppose that $P \Theta E$ and $P \bigcap O \neq \phi$. Then P is nonempty and semi-open in $(X, \tau^*(\Theta))$, so by Proposition 1(2), $P \bigcap O$ is semi-open in $(X, \tau^*(\Theta))$ and hence $(P \bigcap O) \Theta X$ by Theorem 2. By axiom (D_3), we obtain $(P \bigcap O) \Theta E$ and so $O \in \tau^*(\Theta)$.

Theorem 4 — For any topological space (X, τ), there exists a D-space (X, Θ) such that Θ is weakly compatible with τ.

PROOF : Define Θ by $A \Theta B$ if and only if A is a nonempty semi-open set and B is a dense set in (X, τ). Then the axioms (D_1), (D_2), (D_3) hold readily. For (D_4), we take $C = int_\tau A$. For (D_5), we take $O = X - cl_\tau\, int_\tau A$, and, for ($D_6$), we take $P = int_\tau(X-B)$. Let us verify (D_5). Suppose $A \neq \phi, A \not\Theta X$. Then by definition, A is not semi-open in (X, τ). So $O \neq \phi, A \bigcap O \neq \phi$ and O is open in (X, τ). Also if $P \Theta E$ and $P \bigcap O \neq \phi$ then $P \bigcap O$ is semi-open and hence $(P \bigcap O) \Theta E$. Again if $\phi \neq B \subseteq A \bigcap O$, then $P = X = E$ will serve our purpose, because, otherwise, if $(P \bigcap B) \Theta E$ i.e., $B \Theta X$ then by definition of Θ, $int_\tau B \neq \phi$, a contradiction. Thus (D_5) is verified. Hence (X, Θ) becomes a D-space. Now we show that $\tau^*(\Theta) = \tau^\alpha$. Let $A \in \tau^\alpha$. Let $P \Theta E$ and $P \bigcap A \neq \phi$. Then by definition of Θ, P is semi-open in (X, τ) and since $A \in \tau^\alpha$, it follows that $P \bigcap A$ is semi-open in (X, τ). Hence $(P \bigcap A) \Theta E$. Thus $A \in \tau^*(\Theta)$, i.e., $\tau^\alpha \subset \tau^*(\Theta)$. For the reverse inclusion, let $A \in \tau^*(\Theta)$. To show $A \in \tau^\alpha$, it suffices to prove [by Proposition 1(2)] that whenever B is semi-open in (X, τ), $A \bigcap B$ is semi-open in (X, τ). If $A \bigcap B = \phi$, we are done. If $A \bigcap B \neq \phi$, then by definition of Θ, $B \Theta X$ and hence $(A \bigcap B) \Theta X$ (since $A \in \tau^*(\Theta)$). Thus $A \bigcap B$ is semi-open in (X, τ). Hence $A \in \tau^\alpha$ i.e. $\tau^*(\Theta) \subset \tau^\alpha$. Consequently $\tau^\alpha = \tau^*(\Theta)$.

Theorem 5 — The following statements are equivalent for a topological space (X, τ) :

(i) τ is an α topology.

(ii) there exists a D-space (X, Θ) for which Θ is compatible with τ.

Proof follows directly from Theorem 3 and Theorem 4.

Definition 3 — In a D-space (X, Θ),

(i) Θ will be said to satisfy the condition (A) if for any two subsets A, B,

$$A \; \Theta \; B \Rightarrow A \not\Theta X - B.$$

(ii) Θ will be said to satisfy the condition (B) if for any two subsets A, B,

$$X \; \Theta \; A \text{ and } X \; \Theta \; B \;\; \Rightarrow X \; \Theta \; (A \cap B).$$

(iii) Θ will be said to satisfy the condition (C) if for any two subsets A, B,

$$A \; \Theta \; B \Rightarrow B \; \Theta \; A.$$

Theorem 6 — Given a D-space (X, Θ), $(X, \tau^*(\Theta))$ is irresolvable if and only if Θ satisfies the condition (A).

PROOF : If $(X, \tau^*(\Theta))$ is irresolvable then any two dense subsets in $(X, \tau^*(\Theta))$ intersect. Hence if $A \; \Theta \; B$ then by Theorem 2(3), $A \not\Theta X - B$. Conversely, let the condition (A) hold. Then for any dense subset D in $(X, \tau^*(\Theta))$, $X \; \Theta \; D$ and hence $X \not\Theta X - D$. Therefore by axioms (D_6) and (D_4) we have $int_{\tau^*(\Theta)} D \neq \phi$. i.e., $X - D$ is not dense in $(X, \tau^*(\Theta))$. Thus $(X, \tau^*(\Theta))$ is irresolvable.

Theorem 7 — The following statements are equivalent for a topological space (X, τ) :

(i) (X, τ) is irresolvable,

(ii) there exists a D-space (X, Θ) where Θ is weakly compatible with τ and Θ satisfies the condition (A).

PROOF : (i) $\Rightarrow$ (ii). Define Θ by $A \; \Theta \; B$ if and only if A is a nonempty semi-open set and B is a dense set in (X, τ). Then we have already seen in Theorem 4 that with this definition of Θ, (X, Θ) becomes a D-space such that Θ is weakly compatible with τ. Hence (X, τ) and $(X, \tau^*(\Theta))$ have the same collection of dense subsets. Since (X, τ) is irresolvable, $(X, \tau^*(\Theta))$ is so and hence by Theorem 6, Θ satisfies the condition (A).

(ii) $\Rightarrow$ (i). By Theorem 6, $(X, \tau^*(\Theta))$ is irresolvable and since Θ is weakly compatible with τ, (X, τ) is irresolvable.

The following Theorem is obvious from Theorem 7.

Theorem 8 — The following statements are equivalent for a topological space (X, τ) :

(i) (X, τ) is resolvable,

(ii) there exists a D-space (X, Θ) where Θ is weakly compatible with τ and Θ does not satisfy the condition (A).

Theorem 9 — Given a D-space (X, Θ), $(X, \tau^*(\Theta))$ is o.h.i. if and only if Θ satisfies the condition (B).

PROOF : We know from Theorem 1.4 of Ganster *et al.*[7] and Theorem 1.4 of Chattopadhyay and Roy[3] that a topological space (X, τ) is o.h.i., if and only if the collection of dense subsets forms a filter on X. Now if $(X, \tau^*(\Theta))$ is o.h.i. then using Theorem 2 we see that Θ satisfies the condition (B).

Conversely, if Θ satisfies the condition (B) then by Theorem 1(b) and Theorem 2(2) it follows that the collection of dense subsets in $(X, \tau^*(\Theta))$ forms a filter and hence $(X, \tau^*(\Theta))$ is o.h.i.

Theorem 10 — The following statements are equivalent for a topological space (X, τ) :

- (i) (X, τ) is o.h.i.,
- (ii) there exists a D-space (X, Θ) for which Θ is weakly compatible with τ and Θ satisfies the condition (B).

Proof follows from Theorem 9 and an argument similar to that in the proof of Theorem 7.

Theorem 11 — Given a D-space (X, Θ), $(X, \tau^*(\Theta))$ is hyperconnected and irresolvable if and only if Θ satisfies the condition (C).

PROOF : Suppose that $(X, \tau^*(\Theta))$ is hyperconnected and irresolvable. Also let $A \, \Theta \, B$. Then by Theorem 2(3), A is semi-open and B is dense in $(X, \tau^*(\Theta))$. Since $(X, \tau^*(\Theta))$ is hyperconnected and irresolvable, A is dense and B is semi-open and hence $B \, \Theta \, X$ and $E \, \Theta \, A$ for some $E \subseteq X$. Thus by axiom (D_3), $B \, \Theta \, A$. Hence Θ satisfies the condition (C). Conversely, suppose that the condition (C) holds for Θ. Let $A \; (\neq \phi) \in \tau^*(\Theta)$. Then $A \, \Theta \, X$ and hence $X \, \Theta \, A$, which implies that A is dense in $\tau^*(\Theta)$. Thus $(X, \tau^*(\Theta))$ is hyperconnected. Again if D is dense in $(X, \tau^*(\Theta))$, then $X \, \Theta \, D \Rightarrow D \, \Theta \, X \Rightarrow \; int_{\tau^*(\Theta)} D \neq \phi$ by axiom (D_4). Thus $(X, \tau^*(\Theta))$ is irresolvable also.

Theorem 12 — The following statements are equivalent for a topological space (X, τ) :

- (i) (X, τ) is hyperconnected and irresolvable,
- (ii) there exists a D-space (X, Θ) for which Θ is weakly compatible with τ and Θ satisfies the condition (C).

Proof follows from Theorem 11 and an argument similar to that in the proof of Theorem 7.

Observation 1 — It evidently follows that a hyperconnected irresolvable space is o.h.i. Then it is natural to ask whether in a D-space (X, Θ), if Θ satisfies the condition (C) then it must satisfy the condition (B). An affirmative answer to this question is given by the following argument.

Suppose Θ satisfies the condition (C). Let $X \, \Theta \, A$ and $X \, \Theta \, B$. Then $A \, \Theta \, X$ and $B \, \Theta \, X$. If possible let $X \not\Theta (A \cap B)$. Then by axiom D_6, there exists

$C \subseteq X-(A \cap B)$ such that $C \ \Theta \ X$. Therefore $X \ \Theta \ C$ and by axioms (D_3) and (D_1) we get $A \ \Theta \ C$, $B \ \Theta \ C$, $A \cap C \neq \phi$, $B \cap C \neq \phi$. Now by axiom (D_4), there exists a nonempty subset $D \subseteq C$ such that if $P \ \Theta \ E$ and $P \cap D \neq \phi$ then $(P \cap D) \ \Theta \ E$. We now contend that C satisfies the condition that if $P \ \Theta \ E$ and $P \cap C \neq \phi$ then $(P \cap C) \Theta E$. So let $P \ \Theta \ E$ and $P \cap C \neq \phi$. We claim that $P \cap D \neq \phi$. For, if $P \cap D = \phi$ then $P \subseteq X-D$. Now $P \ \Theta \ E \Rightarrow E \Theta P \Rightarrow E \Theta (X-D)$ (by Theorem 1(b)). Now $X \ \Theta \ X \Rightarrow D \Theta X$. Hence $E \ \Theta \ (X - D)$ and $D \ \Theta \ X$ imply $D \ \Theta \ (X - D)$, contradicting the axiom (D_1). Thus $P \cap D \neq \phi$. Now it follows that $(P \cap D) \ \Theta \ E$. Therefore $E \ \Theta \ (P \cap D) \Rightarrow E \ \Theta \ (P \cap C) \Rightarrow (P \cap C) \Theta E$. Hence our contention is true. Now

$$A \ \Theta \ X \text{ and } A \cap C \neq \phi \ \Rightarrow \ (A \cap C) \Theta X,$$

$$B \ \Theta \ X \text{ and } B \cap C \neq \phi \ \Rightarrow \ (B \cap C) \Theta X.$$

Therefore, $X \ \Theta \ (A \cap C)$ and $(B \cap C) \ \Theta X$

$$\Rightarrow \ (B \cap C) \Theta (A \cap C) \quad \text{[by axiom (D}_3\text{)]}$$

$$\Rightarrow \ B \cap C \cap A \neq \phi \quad \text{[by axiom (D}_1\text{)]}$$

which contradicts the fact that $C \subseteq X-(A \cap B)$. Hence $X \Theta (A \cap B)$. Thus Θ satisfies the condition (B).

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