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LINEAR AND NONLINEAR PENETRATIVE CONVECTION IN A POROUS LAYER

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The local nonlinear stability of penetrative convection in a horizontal quiescent porous layer is investigated by applying the modified power integral technique. It is found that the effect of penetration is to contract the cells and this makes the system more unstable. From the finite amplitude analysis it is found that the bifurcation of conduction into convection is supercritical rather than two sided which is of special interest. The result is justified.

1. INTRODUCTION

In recent years, the study of natural convection in fluid – saturated porous layer has become one of the important subjects of scientific research especially in the heat transfer literature owing to its several Geophysical, Industrial and Technical applications [see Rudraiah and Srimani[19] (hereafter referred to as I), Srimani[18], Srimani and Sudhakar[21] (hereafter referred to as II)].

Convective instabilities arise as a result of unstable equilibrium in a region of the fluid/porous layer. When the region of unstable equilibrium is bounded by fluid in stable equilibrium the convective motion in most of the cases penetrates into the neighbouring regions of stable equilibrium. The causes for the penetration are (i) the velocities and (ii) the non-vanishing of the tangential stresses of the perturbed motion at the region of static stability.

Penetrative convection arises in many geophysical and astrophysical situations. In the atmosphere, a statically unstable layer is always surmounted by a stable region. For example, a low-level inversion or the stratosphere in the case of deep convection. Another example is, in the atmosphere solar radiations can heat air near the surface of the earth or ocean and generate a gravitationally unstable layer beneath a stable stratified environment. When convective motion occurs in the lower layer it mixes with the overlying stable air and thus convective motions penetrate into the stable fluid. The reciprocal situations of convection penetrating downwards from above can occur in lakes and oceans, though in the ocean upper mixed layer is typically formed by turbulence generated by surface wind. In other words convective circulations in the well-mixed surface layer penetrate into the stable thermocline region. These examples of penetrative convection are principally unsteady and transient.

Statistically stationary penetrative convection may occur in stars, where large changes in the mean free path of photons cause large changes in the diffusion of heat with temperature as a result of which convective motions result.

Another important example of stationary penetrative convection is that of convection in water of temperature near 4^0C. In the case of water driven by buoyancy force through a porous structure, the flow pattern is influenced dramatically by the occurrence of a density maximum at 3.98^0C if the pressure is atmospheric.

In most of the laboratory experiments pertaining to convection phenomenon, the unstable layer is sandwiched between rigid boundaries, but the stellar convection zones are bounded by stably stratified regions and the study of the penetrative convection across the interface between stable and unstable layers is of astrophysical importance. Actually in the stably stratified photosphere solar granulation is observed and may excite the oscillations detected in the upper atmosphere.

Penetrative convection may also affect nuclear abundances. For example, the apparent shortage of lithium in the sun and other late-type stars may be due to the slow mixing of material into the stable radiative zone and penetration can no longer be ignored in the physics of stellar interiors.

A thorough understanding of geophysical, astrophysical or meteorological convection requires a good knowledge of the qualitative and quantitative features of penetrative convection in a fluid/porous layer.

The earliest theoretical treatment of penetrative convection is the work of Veronis[26] (hereafter referred to as III). He has considered a simple model of an infinite layer of fluid of finite depth with a linear temperature profile across it. The density temperature relationship is assumed to be quadratic and the maximum density occurs some where in the interior. He has found the critical Rayleigh number to be dependent on the position of maximum density in the layer and his limited nonlinear analysis reveals that the layer could become unstable to finite amplitude perturbations at $R < R_c$. Musman[12] confirmed the results of Veronis[26] on the subcritical instability by employing mean filed approximation for solving the nonlinear governing equations. Encouraged by the above results many authors Sun *et al.*[22], Spigel[20], Rintel[15], Debler[7], Watson[27], Deardorff *et al.*[6], Moore and Weiss[11], Merker *et al.*[10], Roberts[16] and Richard *et al.*[14] have made theoretical/experimental study of different penetrative models for a fluid layer.

Very sparse literature is available for the case of penetrative convection in porous media. Sun *et al.* studied the thermal instability of a horizontal layer of liquid with maximum density by using a cubic density temperature relationship. Sun *et al.* extended the above work to include the effect of density maximum on the onset of convection in a porous medium. Only linear stability analysis is carried out by using the empirical expression of Daroy's law. He found the Rayleigh number to depend on two parameters λ_1 and λ_2 respectively. The experimental results of Yen and

Tien *et al.*[22-24] show that the effect of density inversion on heat transfer rate is quite significant and to decrease it as the temperature difference across the layer increases and the heat transfer was found to be sufficiently small for small ΔT. Encouraged by the earlier results, Bejan[1] made two studies pertaining to penetrative convection in porous media, one with application to grain storage problem and the other to the natural convection cooling of rotating electric windings. Blake[2] has made a numerical study of two dimensional natural convection in a horizontal porous layer heated from below and saturated with cold water with the objective to document numerically the characteristics of natural circulation in a porous layer heated from below and to illustrate the high number characteristics of the phenomenon. Later Poulikakos *et al.*[13] studied penetrative convection in porous medium bounded by a horizontal wall with hot and cold spots numerically. Their study reports a series of numerical simulations and a scale analysis of the penetrative convection occurring along the unevenly heated horizontal wall of a semi-infinite porous medium.

In this paper, the local nonlinear penetrative convection in a horizontal, quiescent porous layer is investigated by maintaining the bottom temperature of the water layer at 0°C and the top at a temperature greater than 4°C. Thus a gravitationally unstable layer of fluid lies below a stably stratified layer. The motions resulting at the onset of convection in the lower layer penetrates into the upper layer. The method employed for this model (Rudraiah and Srimani[19]) is the modified power integral technique which is well documented in the works of Veronis[26] and Srimani and Sudhakar[21]. This paper aims at answering the following questions using the Iterative technique mentioned above :

(i) What is the critical condition for the onset of Penetrative convection ?

(ii) What is the nature of the vertical cells associated with the motion ?

(iii) What is the effect of Penetration Parameter on the critical conditions ? Whether the unstable layer will be restricted to a small depth near the bottom or not.

(iv) What is the role of the Porous parameter on the linear stability predictions ?

(v) Which order of approximation provides an adequate description of the field of motion ?

(vi) Does the nonlinear relationship between the water density and temperature (near 4°C) facilitate finite amplitude motions ?

If so, How do the porous and penetrative parameters influence the heat transport ?

2. Mathematical Formulation of the Problem

The physical configuration considered is shown schematically in Fig. 1 under the approximations cited in Rudraiah and Srimani[19] the basic equations of the problem

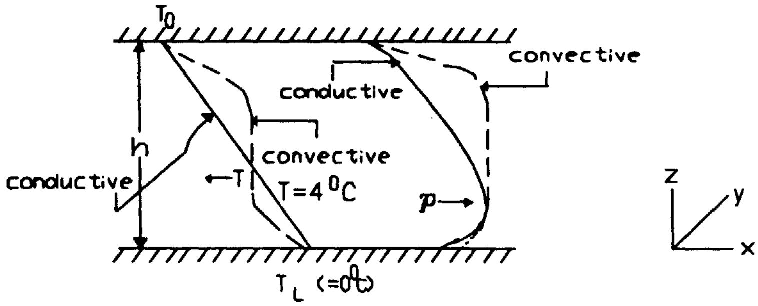

FIG. 1. A layer of water is bounded by two surfaces with temperature 0°C at the lower boundary $z = 0$ and > 4°C at the upper boundary ($z = h$). The depth of the 4°C layer in the conductive state is denoted by d (< h).

Conservation of momentum : $$\frac{\partial \vec{q}}{\partial t} = -\frac{1}{\rho_m}\nabla p - \frac{\rho}{\rho_m} g\hat{k} - \frac{\nu}{k}\vec{q} \quad \text{... (2.1)}$$

Conservation of energy : $$\frac{\partial T}{\partial t} + M(\vec{q}\cdot\nabla)T = \kappa\nabla^2 T \quad \text{... (2.2)}$$

Conservation of mass : $$\nabla\cdot\vec{q} = 0. \quad \text{... (2.3)}$$

As mentioned in Rudraiah and Srimani[19], in the case of water driven by buoyancy through a porous structure, the flow pattern is influenced dramatically by the occurrence of a density maximum near 4°C if the pressure is atmospheric. Due to this nonlinear relationship, the linear Boussinesq[3] approximation commonly used in natural convection analysis is no longer valid. Hence the nonlinear equation of state is :

$$\rho = \rho_m[1 - \alpha_T(T - T_m)^2] \quad \text{... (2.4)}$$

where $T_m = 4$°C and ρ_m is the density at 4°C, and all the quantities have their predefined meanings. Substitution of (2.4) into (2.1) yields

$$\frac{\partial \vec{q}}{\partial t} = -\frac{1}{\rho_m}\nabla p - g[1 - \alpha_T(T - T_m)^2]\hat{k} - \frac{\nu}{k}\vec{q}. \quad \text{... (2.5)}$$

The temperature is expressed in terms of a mean and a fluctuating value in the subsequent analysis, i.e,

$$T = \overline{T}(z) + T(x, y, z, t) \quad \text{... (2.6)}$$

where an over bar or angular brackets denotes horizontal average. Clearly by definition $\overline{T}(x, y, z, t) = 0$. This is meaningful while studying the finite amplitude process because the results of such a study lead to a better understanding of turbulent flow characteristics in which the average properties are regarded as functionals of the turbulent velocity field (Joseph[9]).

Taking the horizontal average and integration w.r.t. z of (2.2) results in the equation for the mean temperature

$$-\kappa \frac{d\overline{T}}{dz} + \langle WT \rangle = H_T = \text{const.} \qquad \dots (2.7)$$

where H_T is the vertical kinematic heat flux. The vertical average of (2.7) yields

$$-\kappa \frac{\Delta T}{d} + \langle WT \rangle_m = H_T \qquad \dots (2.8)$$

where $\langle WT \rangle_m$ is the ensemble average over the entire fluid. In (2.8), ΔT = 4°C and d is the depth of the 4°C layer above the lower boundary so that $\frac{\Delta T}{d}$ is the constant temperature gradient in the conduction range.

From (2.7) and (2.8) we get the important relation connecting the mean gradient and the fluctuation quantities W and T, i.e,

$$-\kappa \frac{d\overline{T}}{dz} = \kappa \frac{\Delta T}{d} + \langle WT \rangle - \langle WT \rangle_m \qquad \dots (2.9)$$

which when integrated yields

$$\frac{\overline{T} - T_m}{\Delta T} = -1 + \left(\frac{1}{d} - \frac{\langle WT \rangle_m}{\kappa \, \Delta T} \right) z + \frac{1}{\kappa \, \Delta T} \int_0^z \langle WT \rangle \, dz. \qquad \dots (2.10)$$

Substituting the mean part of the temperature in (2.6) into (2.5), we get

$$\frac{\partial \vec{q}}{\partial t} = -\frac{1}{\rho_m} \nabla \tilde{p} + \left[-g \hat{k} + 2 \alpha_T g (\overline{T} - T_m) T + \alpha_T g T^2 \right] \hat{k} - \frac{\nu}{k} \vec{q} \qquad \dots (2.11)$$

where $\tilde{p}$ denotes the total pressure.

By operating curl twice on (2.11) we get on simplification, the equation for the vertical component of velocity as

$$\left(\frac{\partial}{\partial t} + \frac{\nu}{k} \right) \nabla^2 W = 2 \alpha_T g (\overline{T} - T_m) \nabla_1^2 T + \alpha_T g \nabla_1^2 T^2 \qquad \dots (2.12)$$

which is a coupled equation for the vertical velocity w and the temperature T and is very useful in discussing the stability of the system.

The two equations which relate the horizontal components of the velocity u and v are

$$\left(\frac{\partial}{\partial t} + \frac{\nu}{k} \right) \left(\nabla_1^2 u + \frac{\partial^2 W}{\partial x \, \partial z} \right) = 0 \qquad \dots (2.13)$$

$$\left(\frac{\partial}{\partial t} + \frac{\nu}{k} \right) \left(\nabla_1^2 v + \frac{\partial^2 W}{\partial y \, \partial z} \right) = 0. \qquad \dots (2.14)$$

By subtracting the horizontal average of (2.2) from (2.7) we get

$$\frac{\partial T}{\partial t} - \kappa \nabla^2 T = -\frac{\partial \overline{T}}{\partial z} W - h \qquad \dots (2.15)$$

where $h = (\vec{q} \cdot \nabla T) - \frac{\partial}{\partial z} \langle WT \rangle$ is the heat advection term which plays an important role in the determination of the convective heat transfer.

The boundaries are assumed to be stress-free and perfect conductors of heat, so that

$$W = 0, \; T = 0 \quad \text{at} \quad z = 0, \, h. \qquad \text{... (2.16)}$$

The fundamental set of equations (2.12) to (2.15) are made dimensionless using the following scales :

$$(x, y, z) = h(x', y', z') \; ; \; \vec{q} = \frac{\kappa}{h} \vec{q}' \; ;$$

$$T = \frac{\nu\kappa}{2h^3 \, \alpha_T \, g\Delta T} T' \; ; \; t = \frac{h^2}{\kappa} t' \; ; \; \sigma = \frac{\nu}{\kappa}, \; \lambda = \frac{h}{d} \; ;$$

$$R = \frac{2d^3 \, \alpha_T \, g \, (\Delta T)^2}{\kappa\nu} \cdot \lambda^4. \qquad \text{... (2.17)}$$

In the remainder of this work all the unprimed quantities are dimensionless unless otherwise stated. The dimensionless forms of eqns. (2.12) to (2.16) are :

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial t} + \frac{1}{P_L} \right) \nabla^2 W = - \left[(1 - \lambda z) + \lambda z \frac{\langle WT \rangle_m}{R} - \frac{\lambda}{R} \int_0^z \langle WT \rangle \, dz \right] \nabla_1^2 T + \frac{\lambda}{2R} \nabla_1^2 T^2 \qquad \text{... (2.18)}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2 \right) T = - W [R + \langle WT \rangle - \langle WT \rangle_m] - h \qquad \text{... (2.19)}$$

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial t} + \frac{1}{P_L} \right) \left(\nabla_1^2 u + \frac{\partial^2 W}{\partial x \, \partial z} \right) = 0 \qquad \text{... (2.20)}$$

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial t} + \frac{1}{P_L} \right) \left(\nabla_1^2 u + \frac{\partial^2 W}{\partial y \, \partial z} \right) = 0. \qquad \text{... (2.21)}$$

Further the dimensionless form of (2.10) is given by

$$\frac{\bar{T} - T_m}{\Delta T} = - 1 + \lambda \left[1 - \frac{1}{R} \langle WT \rangle_m \right] z + \frac{\lambda}{R} \int_0^z \langle WT \rangle \, dz. \qquad \text{... (2.22)}$$

In deriving (2.18), the use of (2.22) is made.

It is interesting to note that the Rayleigh number of the present investigation contains a potential energy term $\frac{\alpha_T \, g(\Delta T)^2}{d}$ in the numerator and product of viscous and thermal dissipation terms κ/d^2 and ν/d^2 respectively. This suggests that an increase in R always corresponds to an increase in the rate of potential energy or a decrease in the rate of dissipation resulting in more destabilization.

To study the local nonlinear stability analysis, the velocity and temperature fluctuations are expanded in terms of a small parameter ε (amplitude factor) as proposed in Rudraiah and Srimani[19]. Thus

$$\vec{q} = \varepsilon \vec{q_0} + \varepsilon \vec{q_1} + \varepsilon^2 \vec{q_2} + \ldots ,$$

$$T = \varepsilon T_0 + \varepsilon^2 T_1 + \varepsilon^3 T_2 + \ldots ,$$

$$R = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \ldots \qquad \ldots (2.23)$$

where $R_1, R_2, \ldots$ are integral functions of W_i and T_i. Substitution of (2.23) into the set of dimensionless equations yields a set of differential equations corresponding to different powers of ε. Linear stability analysis is associated with the first order equations only and the necessary condition for the first order solutions to be complete is that the parameter ε must be proportional to the amplitude of the disturbance and this amplitude must be infinitesimal.

3. Linear Stability Analysis

Since the finite amplitude analysis is based on the results of the linear theory, the stability analysis pertaining to infintesimal perturbations is first carried out.

3.1. *Formulation of First Order Equations*

By substituting (2.23) into (2.18) and (2.19) we get the first order equations for the vertical velocity component W_0 and temperature fluctuations T_0 as,

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} + \frac{1}{P_L}\right)\nabla^2 w_0 = -(1 - \lambda z)\,\nabla_1^2 T_0 \qquad \ldots (3.1)$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right) T_0 = -W_0 R_0 \qquad \ldots (3.2)$$

$$\nabla_1^2 u_0 = -\frac{\partial^2 W_0}{\partial x\, \partial z}. \qquad \ldots (3.3)$$

The corresponding boundary conditions from (2.16) are

$$W_0 = T_0 = 0 \text{ at } z = 0,\ 1. \qquad \ldots (3.3a)$$

A comprehensive study of both linear and nonlinear analysis is made by Rudraiah and Srimani[19] and Srimani and Sudhakar[21] in the absence of penetration. However, in the present case first the principle of exchange of stabilities is carried out and then by the Fourier series method the eigenvalue is determined.

For the sake of convenience one of the variables say W_0 or T_0 can be eliminated from (3.1) and (3.2). The equations and the boundary conditions can be expressed in terms of a single variable (say T_0). Thus, we have

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t}+\frac{1}{P_L}\right)\left(\frac{\partial}{\partial t}-\nabla^2\right)\nabla^2 T_0 = R_0\,(1-\lambda z)\,\nabla_1^2 T_0 \qquad \text{... (3.4)}$$

and

$$T_0 = \frac{\partial^2 T_0}{\partial z^2} = 0 \quad \text{on} \quad z = 0,\ 1. \qquad \text{... (3.5)}$$

Since (3.4) and (3.5) are homogeneous equations in T_0, the solution of the linear stability problem turns out to be the soltuion of the eigenvalue problem.

The assumption that the field of motion is separable leads to,

$$T_0 = e^{st} f(x, y)\, T_0(z) \qquad \text{... (3.6)}$$

where $f(x, y)$ satisfies the membrane equation

$$\nabla_1^2 f(x, y) = -a^2 f(x, y). \qquad \text{... (3.7)}$$

The separation parameter a is the effective wavenumber of the disturbance in the horizontal plane and s is the frequency of oscillations, and $f(x, y)$ is treated as a normalized function. The assumption is justified only when horizontal plan-form of the motion consists of regular close-packed cells at the lateral boundaries of which the normal derivatives of velocity and temperature fluctuations vanish. The plan-forms satisfying these assumptions are two-dimensional roll, the equilateral triangle and the hexagon. But in the present investigation only $2D$-rolls are studied which predict the qualitative behaviour of the convective motion.

Here T_0 has a cellular structure in the horizontal direction. Under these assumptions (3.4) and (3.5) can be written as

$$(D^2-a^2)\left(\frac{s}{\sigma}+\frac{1}{P_L}\right)(s-(D^2-a^2))\,T_0 = -a^2 R_0\,(1-\lambda z)\,T_0 \qquad \text{... (3.8)}$$

$$T_0 = D^2 T_0 = 0 \quad \text{on} \quad z = 0,\ 1 \qquad \text{... (3.8a)}$$

where $D = d/dz$.

Thus we have a fourth-order ordinary differential equation with homogeneous boundary conditions.

4. Principle of Exchange of Stabilities

The Marginal stability is characterized by $s = 0$ and the principle of exchange of stabilities will be valid, if s is real for undamped disturbances. In order to establish this, multiply (3.8) by T_0^*, the complex conjugate of T_0 and integrate w.r.t. z between 0 and 1. We get after simplification the following :

$$\left(\frac{s}{\sigma}+\frac{1}{P_L}\right)(s^2+a^2)\int_0^1 (DT_0\,DT_0^*+a^2\,T_0\,T_0^*)\,dz$$

$$+\left(\frac{s}{\sigma}+\frac{1}{P_L}\right)\int_0^1 (D^2\,T_0\,D^2\,T_0^*+2a^2\,DT_0\,DT_0^*+a^4\,T_0\,T_0^*)\,dz$$

$$+\;a^2 s\left(\frac{s}{\sigma}+\frac{1}{P_L}\right)\int_0^1 T_0\,T_0^*\,dz=a^2 R_0\int_0^1 (1-\lambda z)\,T_0\,T_0^*\,dz \qquad \dots (4.1)$$

or
$$s^2 I_1 + s I_2 + I_3 = R_0 I_4 \qquad \dots (4.2)$$

where the I_i's (i = 1, 2, 3, 4) are the positive coefficients appearing in (4.1).

Again taking the complex conjugate of (4.2) and multiplying by T_0, we get after integration w.r.t. z

$$s^{*2} I_1 + s^* I_2 + I_3 = R_0 I_4. \qquad \dots (4.3)$$

Setting $s = s_1 + is_2$, $s^* = s_1 - is_2$ and subtracting (4.3) from (4.2), we get

$$is_2\,[4s_1 I_1 + 2I_2] = 0$$

which implies that either $s_2 = 0$ or

$$s_1 = -I_2/2I_1. \qquad \dots (4.4)$$

Thus the principle exchange of stabilities is established.

5. Solution to the Linear Stability Problem

The solution to the linear stability problem in the present case cannot be obtained as in the case of ordinary porous convection which is evident from eqn. (3.4). Therefore, we preceed as follows :

Equation (3.8) with $s = 0$ becomes

$$LT_0 = \frac{1}{P_L}(D^2-a^2)\,T_0 - a^2 R_0\,(1-\lambda z)\,T_0 = 0 \qquad \dots (5.1)$$

where L is a linear operator.

This equation is analogous to that of flow between rotating cylinders in the presence of porous material. Since the boundaries are assumed to be free, the Fourier series representation for T_0 is most suited and the boundary conditions are automatically satisfied. Introducing the relations

$$a^2 = \pi^2\alpha^2 \text{ and } \pi z = \zeta \qquad \dots (5.2)$$

eqn. (5.1) may be conveniently written as

$$\frac{1}{P_L}\left(\frac{d^2}{d\zeta^2}-\alpha^2\right)^2 T_0 = \alpha^2 R_0^1\left(1-\frac{\lambda\zeta}{\pi}\right)T_0 \qquad \text{... (5.3)}$$

where $R_0^1 = \dfrac{R_0}{\pi^2}$.

Consider the Fourier series representation

$$T_0 = \sum_1^{\infty} A_n \sin n\,\zeta. \qquad \text{... (5.4)}$$

Dividing (5.3) by R_0^1 and substituting for T_0 from (5.4) we get

$$\frac{1}{R_0^1 P_L}\sum_1^{\infty}(n^2+\alpha^2)^2 A_n \sin n\,\zeta = \left(1-\frac{\lambda\zeta}{\pi}\right)\alpha^2\sum_{n=1}^{\infty} A_n \sin n\,\zeta. \qquad \text{... (5.5)}$$

Multiplying (5.5) by sin $r\,\zeta$ and integrating w.r.t. ζ between the limits $\zeta = 0$ and $\zeta = \pi$ we get the following set of equations :

$$\text{odd } r: \ \frac{\pi^2}{2}\left[\alpha^2-\frac{(r^2+\alpha^2)^2}{R_0^1 P_L}\right]A_r = \alpha^2\lambda\left[\frac{\pi^2}{4}A_r-\sum_{n\text{ even}}\frac{4rn}{(r^2-n^2)}A_n\right]$$

$$\text{even } r: \ \frac{\pi^2}{2}\left[\alpha^2-\frac{(r^2+\alpha^2)^2}{R_0^1 P_L}\right]A_r = \alpha^2\lambda\left[\frac{\pi^2}{4}A_r-\sum_{n\text{ odd}}\frac{4rn}{(r^2-n^2)}A_n\right]. \qquad \text{... (5.6)}$$

A glance at (5.6) reveals that it represents an infinite matrix relating the various Fourier amplitudes. Further the set of equations is homogeneous and hence the equations will be consistent only when the determinant of the coefficients vanishes for particular values of P_L, α, R_0 and λ.

In deriving the solution to the present problem, the infinite matrix is truncated to different orders and the results are compared. The elements of the determinant are generated from (5.6) and the fourth-order determinant is of the form

$$\begin{vmatrix} \left(1-\frac{\lambda}{2}\right)\beta_1-\frac{1}{R_0^1 P_L} & \lambda\cdot\frac{16}{9\pi^2}\beta_1 & 0 & \lambda\cdot\frac{32}{225\pi^2}\beta_1 \\ \lambda\cdot\frac{16}{9\pi^2}\beta_2 & \left(1-\frac{\lambda}{2}\right)\beta_2-\frac{1}{R_0^1 P_L} & \lambda\cdot\frac{48}{25\pi^2}\beta_2 & 0 \\ 0 & \lambda\cdot\frac{48}{25\pi^2}\beta_3 & \left(1-\frac{\lambda}{2}\right)\beta_3-\frac{1}{R_0^1 P_L} & \lambda\cdot\frac{96}{49\pi^2}\beta_3 \\ \lambda\cdot\frac{32}{225\pi^2}\beta_4 & 0 & \lambda\cdot\frac{96}{49\pi^2}\beta_4 & \left(1-\frac{\lambda}{2}\right)\beta_4-\frac{1}{R_0^1 P_L} \end{vmatrix}$$

$$= 0 \qquad \text{... (5.7)}$$

where $$\beta_r = \frac{\alpha^2}{(r^2+\alpha^2)^2}\,.$$

In (5.7), the reciprocal of the modified Rayleigh number appears as the eigenvalue and is numerically computed for various values of the porous and penetration parameters P_L and λ respectively. The graphs of $R_0^{1\prime}$ versus α^2 are presented in Fig. 2. The critical wave number α_c is computed for different values of λ and it is found that the penetration parameter has a remarkable influence on $R = R(P_L, \lambda)$.

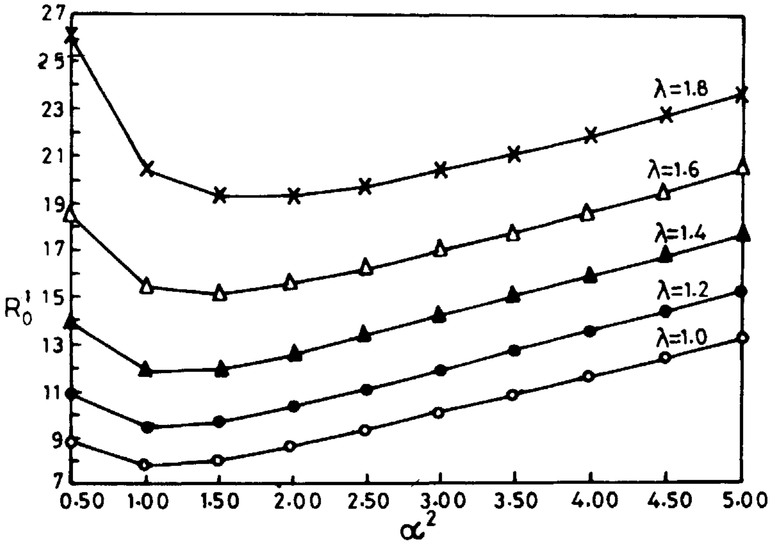

FIG. 2. R_0^1 versus α^2 for $1/P_L = 10^3$.

In the absence of penetration the results of the linear theory are in excellent agreement with the theoretical results of I and II and the experimental results of Buretta[4], Combarnous and Lefer[5] and Elder[8].

It is interesting to note that for $T = 4°C$ at $z = h$ and $\lambda = 1$, the first term in the determinant (5.7) yields the critical R_c as

$$\frac{R_0}{2\lambda^4} = \frac{4\pi^2}{P_L} \quad \text{for} \quad \alpha^2 = 1 \qquad \text{... (5.8)}$$

which corresponds exactly to the value of the critical Rayleigh number in the absence of penetration. In other words, the larger mass of dense water at the top of the layer tends to make the system more unstable gravitationally than the ordinary porous convection.

For $T = 8°C$ at $z = h$, $\lambda = 2$, the 2×2 determinant is sufficient to predict the result. In that case

$$R_0 = \frac{9\pi^4}{16\lambda} \cdot \frac{1}{P_L} \frac{\alpha^2}{(\alpha^2+1)(\alpha^2+4)} \qquad \text{... (5.9)}$$

the minimum value of which w.r.t. α^2 is given by

$$\frac{R_0}{2\lambda^4} = \frac{0.0013\,\pi^4}{P_L} \quad \text{for} \quad \frac{\alpha^2}{\lambda^2} = 0.5. \qquad \text{... (5.10)}$$

Thus the addition of the stable layer introduces an asymmetry in the problem and the effect is that the Rayleigh number is very much less when compared to the ordinary porous convection (5.8).

Further the scale is doubled which is evident from the results $\alpha^2 = 1$ at $T =$ 4°C, $\lambda = 1$ and $\alpha^2 = 2$ at $T = 8$°C, $\lambda = 2$ respectively. Also the results are very precise and exact when compared to the viscous case.

For the finite amplitude analysis, the values of the amplitude coefficients A_1 and A_2 of (5.4) calculated from (5.7) and (5.9) are utilized. Thus, for $T = 8$°C and $z = h$ and $\lambda = 2$ it is found from the 2×2 determinant that

$$A_1 = 1.7888, \; A_2 = 0.894, \; \alpha = 1.41 \qquad \text{... (5.11)}$$

and $A_1/A_2 = 2$ which is almost one half of the viscous case. Setting $\frac{\partial T_0}{\partial t} = 0$ in (3.2), W_0 can be computed.

In Table I, the critical Rayleigh numbers and the corresponding wave numbers for various values of λ and P_L are listed and several interesting features are evident for the present case.

TABLE I

Critical wave number, critical penetration and Rayleigh numbers for different values of λ

λ	α^2	$\frac{R_0}{\pi^2} = R_0^1$	$\frac{\alpha^2}{\lambda^2}$	$\frac{R_0}{2\pi^2\lambda^4}$
1	1.00	7.812247	1	7.812
1.2	1.00	9.50	0.699	2.29
1.4	1.2	11.86	0.612	1.236
1.6	1.4	15.14	0.547	1.155
1.8	1.8	19.257	0.556	0.917

Based on the linear stability predictions, the following interesting interpretation can be made : If the temperature of the upper boundary is exactly 8°C, the entire region would be symmetric. On the other hand if the upper boundary temperature lies between 4°C and 8°C, the symmetric portion of the (density profile) lies in the range $2d - h \leq z \leq h$ [Fig. 3(a)]. Therefore in the configuration corresponding to the minimum potential energy, the layer of water below $z = 2d - h$ must be turned upside down and placed at the top while the 4°C layer must be placed at the bottom, such that the symmetric layers are on either side of the 4°C layers must be placed next to each other when the density decreases with height [Fig. 3(b)].

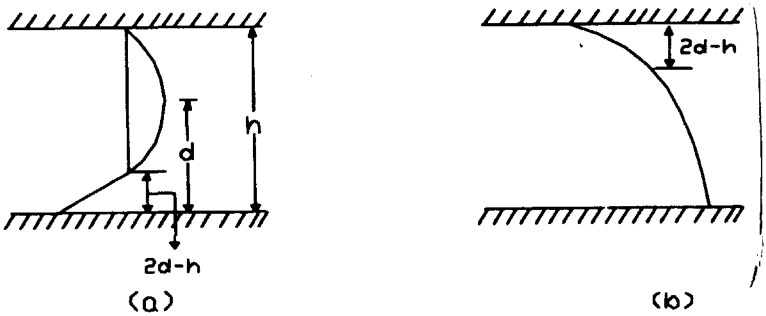

FIG. 3. (a) A schematic diagram showing the density profile associated with the pure conduction state for the case with the top temperature between 4°C and 8°C.
(b) The redistribution density profile has the densest fluid at the bottom.

The redistributed density profile can be written in the form

$$\rho = \rho_m \left[1 - \alpha_T' \left(\frac{z}{2} \right)^2 \right], \quad 0 \leq z \leq 2\,(h - d)$$

$$\rho = \rho_m \left[1 - \alpha_T' (z - h + d)^2 \right], \quad 2(h - d) \leq z \leq h$$

where $\qquad \alpha_T' = \alpha\,(\Delta\, T/d)^2.$ ... (5.12)

From Fig. 3(a), the vertically integrated potential energy per unit area is given by

$$E_1 = \int_0^h g\,\rho_m \left[1 - \alpha_T' (z - d)^2 \right] z\,dz. \qquad \text{... (5.13)}$$

From Fig. 3(b), the vertically integrated potential energy is given by

$$E_2 = \int_0^{2(h-d)} g\,\rho_m \left[1 - \alpha_T' \left(\frac{z}{2} \right)^2 \right] z\,dz$$

$$+ \int_{2(h-d)}^{h} g\,\rho_m \left[1 - \alpha_T' (z - h + d)^2 \right] z\,dz. \qquad \text{... (5.14)}$$

Hence from (5.13) and (5.14) we have after simplification,

$$E_2 - E_1 = g\,\rho_m\,\alpha_T'\left[\frac{(h-d)^4}{6} - \frac{d(h-d)^3}{3} + \frac{(h-d)\,d^3}{3} + \frac{1}{6}\right]\cdot$$

$$= \frac{g\,\rho_m\,\alpha_T'\,d^4}{3}\left[\frac{(\xi-1)}{2} - (\xi-1)^3 + (\xi-1) + 0.5\right] \qquad \text{... (5.15)}$$

where $\xi = \frac{h}{d}$.

The necessary condition for the existence of $(E_2 - E_1)\,\Big|$ opt. is given by

$$2(\xi-1)^3 - 3(\xi-1)^2 + 1 = 0 \qquad \text{... (5.16)}$$

so that $\xi = 2, 2, 1/2$. In particular, the maximum occurs at $\xi = 2$ i.e., $h = 2d$.

From the above results it is clear that the onset of Penetrative convection is affected by the stable layer (> 4°C) through three major physical features :

(1) The relaxation of the upper boundary condition results in a thick stable layer;

(2) The increase in the available potential energy increases the upper boundary temperature to 8°C and further deepening of the stable layer is reduced.

(3) Finally multicells will be formed only when there is the transfer of kinetic energy from the ustable layer and this suggests that the boundary temperature should be less than 8°C.

Thus the phenomenon of Penetrative Convection is associated with the complex interaction of the above mentioned features and the results are more elegant when compared to the viscous case.

6. Finite Amplitude Analysis ($\lambda = 2$)

In this section, the nonlinear stability of a system in which a gravitationally unstable porous layer lies below a stable porous layer is investigated by using the results of section (3.3). Two-dimensional analysis is carried out in order to facilitate the qualitative behaviour of the system with minimum mathematics. The higher order equations are derived from (2.18) to (2.21) by substituting expansions (2.23) for the variables.

ε^2-approximation

$$LT_1 = R_1\,(1-\lambda z)\,\nabla_1^2\,T_0 - \frac{1}{P_L}\,\nabla^2 h_{00} + \frac{\lambda}{2}\,\nabla_1^2\,T_0^2 \qquad \text{... (6.1)}$$

$$\nabla^2\,T_1 = R_0\,w_1 + R_1\,w_0 + h_{00} \qquad \text{... (6.2)}$$

$$\nabla_1^2\,u_1 = -\frac{\partial^2 w_1}{\partial x\,\partial z} \qquad \left(\because\ \frac{\partial}{\partial t} = 0;\ \frac{\partial}{\partial y} = 0\right). \qquad \text{... (6.3)}$$

ε^3-approximation

$$LT_2 = R_2 (1 - \lambda z) \nabla_1^2 T_0 + R_1 (1 - \lambda z) \nabla_1^2 T_1$$
$$+ \lambda \left(\langle w_0 T_0 \rangle_m z - \int_0^z \langle w_0 T_0 \rangle dz \right) \nabla_1^2 T_0$$
$$+ \nabla^4 ((\langle w_0 T_0 \rangle - \langle w_0 T_0 \rangle_m) w_0)$$
$$+ \frac{\lambda}{2} \nabla_1^2 T_1 T_0 - \frac{1}{P_L} \nabla^2 (h_{01} + h_{10}) \quad \text{... (6.4)}$$

$$\nabla^2 T_2 = R_0 w_2 + R_1 w_1 + R_2 w_0$$
$$+ ((\langle w_0 T_0 \rangle - \langle w_0 T_0 \rangle_m) w_0) + (h_{01} + h_{10}) \quad \text{... (6.5)}$$

$$\nabla_1^2 u_2 = - \frac{\partial^2 w_2}{\partial x \, \partial z} \quad \text{... (6.6)}$$

where $h_{ij} = q_i, \nabla T_j - \frac{\partial}{\partial z} \langle w_i T_j \rangle$ is the nonlinear heat advection term.

In obtaining the solution of the system of equations (6.1) to (6.3), first it is solved for T_1 and the remaining quantities w_1, u_1 etc. are determined. But the determination of T_1 poses a problem because (6.1) is an inhomogeneous equation with forcing terms. The necessary and sufficient condition for the solution of (6.1) to exist is that

$$\int_0^1 \tilde{T}_0 \left(R_1 (1 - \lambda z) \nabla_1^2 T_0 - \frac{1}{P_L} \nabla^2 h_{00} + \frac{\lambda}{2} \nabla_1^2 T_0^2 \right) dz = 0 \quad \text{... (6.7)}$$

where the adjoint function $\tilde{T}_0$ is defined by the conditions

$$\int_0^1 \tilde{T}_0 L \, T_0 \, dz = \int_0^1 T_0 \tilde{L} \tilde{T}_0 \, dz = 0. \quad \text{... (6.8)}$$

Here $\tilde{L}$ is the adjoint operator. From (6.7), we can write,

$$R_1 \int_0^1 \tilde{T}_0 (1 - \lambda z) \nabla_1^2 T_0 \, dz = \int_0^1 \left(\frac{1}{P_L} \nabla^2 h_{00} - \frac{\lambda}{2} \nabla_1^2 T_0^2 \right) dz. \quad \text{... (6.9)}$$

The determination of R_1 is straightforward since all the quantities except R_1 are functions of first-order solution. While determining T_1 the arbitrariness in $LT_i = 0$ can be removed by imposing the normalization condition i.e., $\left(T_0^2 \right)_m = 1$ such that $(T_0 T_i)_m = 0$ for $i > 0$. Once T_1 is determined W_1, U_1, V_1 can be easily determined.

Then the second-order solution will be complete. Further, we have a self-adjoint system i.e., $\tilde{T}_0 = T_0$ and $\tilde{L} = L$.

Second-order Solution

To determine the second-order solution the first-order solutions are needed. From section 2, the normalized approximate solution for T_0 for the case 0°-8°C is

$$T_0 = \left(A_1 \sin \pi z + A_2 \sin 2\pi z \right) \cos \pi \alpha x \quad \text{... (6.10)}$$

where A_1 and A_2 are given by (5.11) and $\left(T_0^2 \right)_m = 1$. The other quantities W_0 and U_0 are determined from (3.2) and (3.3) and are given by

$$w_0 = -\frac{\pi^2}{R_0} [(\alpha^2 + 1) A_1 \sin \pi z + (\alpha^2 + 4) A_2 \sin 2 \pi z] \cos \pi \alpha x \quad \text{... (6.11)}$$

$$u_0 = \frac{\pi^2}{\alpha R_0} [(\alpha^2 + 1) A_1 \cos \pi z + 2 (\alpha^2 + 4) A_2 \cos 2 \pi z] \sin \pi \alpha x \quad \text{... (6.12)}$$

$$v_0 = 0. \quad \text{... (6.13)}$$

Substituting these first order quantities in (6.1) and evaluating the relevant expressions as indicated earlier we find that

$$\nabla^2 h_{00} = \frac{-3 \pi^5 A_1 A_2}{4R_0} [(4\alpha^2 + 9) \sin 3\pi z - 3(4\alpha^2 + 1) \sin \pi z] \cos \pi \alpha x \quad \text{... (6.14)}$$

and

$$\nabla_1^2 T_0^2 = -\pi^2 \alpha^2 [A_1^2 (1 - \cos 2\pi z) + 2A_1 A_2 (\cos \pi z - \cos 3\pi z) + A_2^2 (1 - \cos 4\pi z)] \cos 2\pi \alpha x. \quad \text{... (6.15)}$$

Substituting these expressions on the right-hand side of (6.1), multiplying by T_0 and then integrating w.r.t. z, we find that (by orthogonality condition)

$$R_1 = 0. \quad \text{... (6.16)}$$

Now from (6.1),

$$LT_1 = -\frac{1}{P_L} \nabla^2 h_{00} + \frac{\lambda}{2} \nabla_1^2 T_0^2. \quad \text{... (6.17)}$$

In order to evaluate T_1, which has got a horizontal dependence $\cos 2\pi \alpha x$ in (6.17), ∇_1^2 is replaced by $-4\pi^2\alpha^2$ and again a Fourier series representation for T_1 is considered in the form

$$T_1 = \sum_{n=1}^{\infty} B_n \sin n\pi z \qquad \text{... (6.18)}$$

where the coefficients B_r's can be determined by multiplying (6.17) by sin $r\pi z$ and then integrating w.r.t. z. This procedure results in an infinite set of non-homogeneous equations. But the truncated 3 × 3 matrix gives the required result with an error of ∼ 0.09%. Therefore, (6.18) becomes

$$T_1 = \sum_{n=1}^{3} B_n \sin n\pi z. \qquad \text{... (6.19)}$$

Substituting (6.14), (6.15) and (6.19) into (6.17) we get

$$\left[\frac{1}{P_L}(D^2-\alpha^2)^2 - \alpha^2 R_0 (1-\lambda z)\right] [B_1 \sin \pi z + B_2 \sin 2\pi z + B_3 \sin 3\pi z]$$

$$= \frac{3}{4}\frac{\pi^5 A_1 A_2}{R_0 P_L}[(4\alpha^2+9)\sin 3\pi z - 3(4\alpha^2+1)\sin \pi z]\cos 2\pi\alpha x$$

$$-\frac{\lambda}{2}\pi^2\alpha^2 [A_1^2(1-\cos 2\pi z) + 2A_1 A_2(\cos \pi z - \cos 3\pi z)$$

$$+ A_2^2(1-\cos 4\pi z)]\cos 2\pi\alpha x. \qquad \text{... (6.20)}$$

Thus the values of the Coefficients B_i (i = 1, 2, 3) are given by

$$B_1 = -0.01933\, P_L,\; B_2 = -0.0265\, P_L,\; B_3 = 0.0729\, P_L. \qquad \text{... (6.21)}$$

Using the above results W_1 and U_1 are computed from (6.2) and (6.3) and are given by

$$W_1 = (C_1 \sin \pi z + C_2 \sin 2\pi z + C_3 \sin 3\pi z)\cos 2\pi\alpha x \qquad \text{... (6.22)}$$

where

$$C_1 = -\frac{\pi^2}{R_0}\left[4\alpha^2 B_1 + \frac{9\pi A_1 A_2}{4R_0}\right]$$

$$C_2 = \frac{-4\pi^2\alpha^2}{R_0}B_2$$

$$C_3 = \frac{\pi^2}{R_0}\left[-4\alpha^2 B_3 + \frac{3\pi A_1 A_2}{4R_0}\right].$$

Similarly

$$u_1 = (D_3 \cos \pi z + D_4 \cos 2\pi z + D_5 \cos 3\pi z)\sin 2\pi\alpha x$$

where $D_3 = -C_1/2\alpha,\; D_4 = -C_2/\alpha,\; D_5 = -3C_3/2\alpha.$ ... (6.23)

Since $R_1 = 0$, the next higher order approximation will throw light on the amplitude and hence R_2 has to be evaluated as before. This is possible because U_i, W_i, T_i $(i = 0, 1)$ are all known quantities and the right-hand side of (6.4) is known. Thus following the procedure outlined earlier, we get the equation involving the definite integrals as

$$R_2 \int_0^1 \tilde{T}_0 (1 - \lambda z) \nabla_1^2 T_0 \, dz$$

$$= - \int_0^1 \left[\tilde{T}_0 \lambda \left(\langle W_0 T_0 \rangle_m z - \int_0^z \langle W_0 T_0 \rangle dz \right) \nabla_1^2 T_0 + \tilde{T}_0 \nabla^4 (\langle W_0 T_0 \rangle - \langle W_0 T_0 \rangle_m) W_0 + \tilde{T}_0 \frac{\lambda}{2} \nabla_1^2 T_1 T_0 - \tilde{T}_0 \frac{1}{P_L} \nabla^2 (h_{01} + h_{10}) \right] dz. \quad \text{... (6.24)}$$

The computation of R_2 is extremely tedious and hence the final value is presented by avoiding the details.

Thus

$$R_2 = 0.277 P_L. \quad \text{... (6.25)}$$

7. Heat Transport

The important contributions of the nonlinear analysis is the prediction of the amplitude and the associated heat transfer. The Nusselt number Nu which is the non-dimensional form of the heat flux, is computed as follows. From (2.23),

$$R = R_0 + \varepsilon^2 R_2 \text{ (since } R_1 = 0)$$

or

$$\varepsilon^2 = \frac{R - R_0}{R_2}. \quad \text{... (7.1)}$$

Therefore, since $\varepsilon^2 > 0$ and $R_2 > 0$ from (6.26), it implies that $R > R_0$. In other words, the solution of the present investigation is valid for supercritical motions only. The non-dimensional form of the heat flux to the second-order is given by

$$Nu = \frac{H_T}{-\kappa (\Delta T / d)} = 1 - \frac{\varepsilon^2}{R} \langle W_0 T_0 \rangle_m$$

$$= 1 + \frac{R - R_0}{R} (0.46119). \quad \text{... (7.2)}$$

This result is predicted in Fig. 4 for a particular set of the porous and penetration parameters. The result that the bifurcation is supercritical in the present model contrasts with that of the viscous, where a subcritical bifurcation for that model of penetrative convection is predicted. As described in earlier papers [17, 18, 21] this type of finite amplitude instability (i.e., subcritical motions) were observed for the values of the parameters which are practically impossible and meaningless and therefore the motions were considered to be supercritical unlike the viscous case (Veronis[25]). Therefore, the present result is justified and as in the case of ordinary porous convection, the motion is supercritical in the case of penetrative porous convection also. Hence 2*D* rolls are preferred. The result plotted in Fig. 4 reveals that the heat transport increases gradually with R and the maximum value is ~ 1.46.

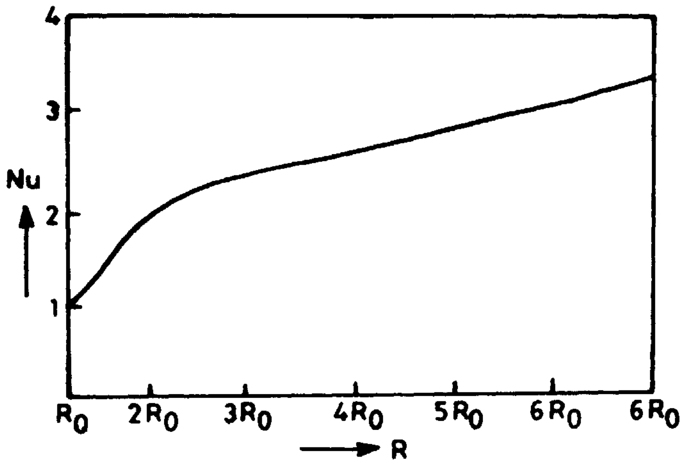

FIG. 4. Heat transport curve for different values of R ($\lambda = 2$, P_L^2 terms are neglected, $P_L = 10^{-5}$). The curve is independent of P_L.

These answer the questions posed in the beginning of this chapter.

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