Indian J. pure appl. Math., 27(6) : 527-536, June 1996

HIGHLY IRREGULAR BIPARTITE GRAPHS

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(*Received* 16 *September* 1994; *after revision* 10 *July* 1995; *accepted* 4 *January* 1996)

A connected graph is said to be highly irregular if each of its vertices is adjacent only to vertices with distinct degrees. The bound given in Alavi *et al.*[1] for the number of edges in a highly irregular graph is improved for the case of graphs with an odd number of vertices. Further, we prove that a path on four vertices is the unique highly irregular graph whose complement is also highly irregular, and we establish some results on highly irregular bipartite graphs.

1. INTRODUCTION

In this paper we consider only finite, simple, connected graphs. Our notations and terminology are as in Bondy and Murty[6]. The concept of highly irregular graphs was introduced and studied by Alavi *et al.*[1]. A connected graph G is called highly irregular if every vertex of G is adjacent only to vertices having distinct degrees, i.e., if two vertices u and v of G are both adjacent to a vertex w of G, then $d(u) \neq d(v)$ in G. For example, the graphs G and H of Fig. 1 are highly irregular.

The following facts on highly irregular graphs are known[1] :

(1) If v is a vertex of maximum degree Δ in a highly irregular graph H, then v is adjacent to exactly one vertex of degree k, for each k, $1 \le k \le \Delta$.

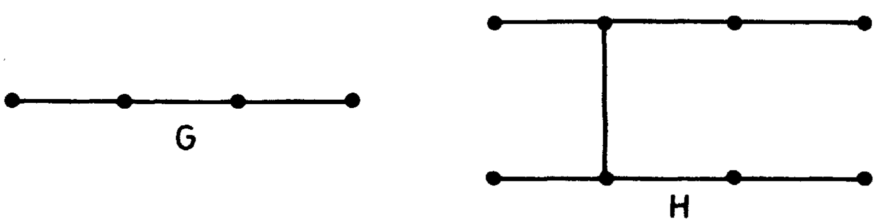

FIG. 1. Examples of highly irregular graphs.

(2) A highly irregular graph H with maximum degree Δ has at least 2Δ vertices.

(3) There is no highly irregular graph of order 3, 5 and 7.

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(4) For $n \neq 3$, 5 or 7, there is a highly irregular graph of order n.

(5) The size of a highly irregular graph of order n is at most $n(n + 2)/8$ with equality possible for even n.

(6) Every graph of order $n \geq 2$ is an induced subgraph of a highly irregular graph of order $4n - 4$.

The last fact implies that there can be no forbidden subgraph criterion for highly irregular graphs.

In this paper we give an upper bound for the size of a highly irregular graph of order $2n + 1$ which is better than the bound given in Alavi *et al.*[1]. Further we characterize those highly irregular graphs whose complements are also highly irregular. Also we construct for $n \neq 3$, 5 or 7, a highly irregular bipartite graph of order n. In addition, we show that for $n \geq m \geq 3$, there is a highly irregular bipartite graph with bipartite sizes (n, n) and with maximum degree m. Throughout this paper $H_{n,n}$ denotes the highly irregular bipartite graph with bipartite sets $X = \{v_1, v_2, ..., v_n\}$ and $Y = \{u_1, u_2, ..., u_n\}$ and with the edge set $E = \{v_i u_j : 1 \leq i \leq n, n - i + 1 \leq j \leq n\}$. Clearly in $H_{n,n}$, $d(u_i) = d(v_i) = i$ for $1 \leq i \leq n$. For example, $H_{4,4}$ is shown in Fig. 2.

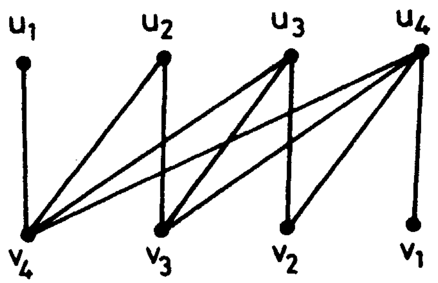

FIG. 2. $H_{4,4}$.

Many results on highly irregular trees and highly irregular graphs which are m-chromatic were obtained in Alavi *et al.*[2]. The concept of highly irregular digraphs was introduced by Alavi *et al.*[3]. They investigated some problems concerning the existence of highly irregular digraphs with special properties. Also they discussed on highly irregular directed trees as well as their independence number. The result which states that any finite group is the automorphism group of a highly irregular graph is proved in Alavi and Ruiz[4]. Balconi[5] provides some results on $H_{n,n}$.

2. Highly Irregular Graphs

By Fact 5 (Alavi *et al.*[1]), the size of a highly irregular graph of order n is at most $n(n + 2)/8$. For even n, the upper bound can be attained. In this section we establish a sharper inequality for odd n.

Lemma 1 — The size of highly irregular graph of order $2n + 1$, $n \geq 4$, with maximum degree n is at most $n(n+1)/2 + \lfloor (n+1)/5 \rfloor$ where $\lfloor x \rfloor$ denotes the greatest integer which is less than or equal to x.

PROOF : Let G be a highly irregular graph of order $2n + 1$ with maximum degree n. Further, let us denote the neighbours of v_n by $u_1, u_2, ..., u_n$ so that $d(u_i) = i$, for $1 \leq i \leq n$. Since $d(u_n) = n$, there exists another set of $n-1$ vertices $v_1, v_2, ..., v_{n-1}$ which, as G is highly irregular, are distinct from $u_1, u_2, ..., u_n$ such that $u_n v_i \in E(G)$ and $d(v_i) = i$ for $1 \leq i \leq n$.

So far we have labelled $2n$ vertices of G and for each k, $1 \leq k \leq n$, there are exactly two vertices of degree k. Now label the remaining vertex by v and let $d(v) = m$. Then m is even. Let A_i, $1 \leq i \leq n$ denote the set of all vertices of degree i. Then $|A_i| = 2$ for $1 \leq i \leq n$, $i \neq m$ and $|A_m| = 3$.

Claim : Assume $m \geq (n/2)$. Then for $i = 1, 2, ..., n-m$, we have

$$N(A_i) = \{u_n, u_{n-1}, ..., u_{n-i+1}, v_n, v_{n-1}, ..., v_{n-i+1}\}.$$

PROOF OF THE CLAIM (by induction on i) : When $i = 1$, $A_n = \{u_n, v_n\}$ has $2n$ neighbours and for each $j = 1, 2, ..., n$, A_j contains at most two neighbours to A_n. Thus A_n must have exactly two neighbours in each of $A_1, A_2, ..., A_n$ and $N(A_1) = \{u_n, v_n\}$. Let i be an integer, $1 < i \leq n-m$. Assume by induction that $N(A_{i-1}) = \{u_n, u_{n-1}, ..., u_{n-i+2}, v_n, v_{n-1}, ..., v_{n-i+2}\}$. Then A_{n-i+1} can have no neighbour in $A_1, A_2, ..., A_{i-1}$ and can have at most two neighbours in each of $A_i, A_{i+1}, ..., A_n$. Since $n - i + 1 > m$ we have $|A_{n-i+1}| = 2$ and thus A_{n-i+1} has exactly two neighbours in each of $A_i, A_{i+1}, ..., A_n$ and $N(A_i) = \{u_n, u_{n-1}, ..., u_{n-i+1}, v_n, v_{n-1}, ..., v_{n-i+1}\}$. This proves the claim.

It follows from the claim that if $m \geq n/2$, then $N(A_1 \bigcup A_2 \bigcup ... \bigcup A_{n-m}) = \{u_n, u_{n-1}, ..., u_{m+1}, v_n, v_{n-1}, ..., v_{m+1}\}$. This implies that $u_1, u_2, ..., u_{n-m}, v_1, v_2, ..., v_{n-m} \notin N(A_m)$ and hence $N(A_m) \subseteq \{u_n, u_{n-1}, ..., u_{n-m+1}, v_n, v_{n-1}, ..., v_{n-m+1}, v\}$ so that $3m \leq 2m + 1$. This implies $m \leq 1$ in contradiction to $m \geq 2$. Thus we have proved that $m < (n/2)$. As before, for $j = 1, 2, ..., m-1$, the induction argument gives that $N(A_j) = \{u_n, u_{n-1}, ..., u_{n-j+1}, v_n, v_{n-1}, ..., v_{n-j+1}\}$ so that $A_1, A_2, ..., A_{m-1}$ have all their neighbours in $A_{n-m+2} \bigcup A_{n-m+3} \bigcup ... \bigcup A_n$, therefore A_m must have its $3m$ neighbours in $A_m \bigcup A_{m+1} \bigcup ... \bigcup A_n$ which gives that $3m \leq 2n - 2m + 3$. Therefore $m \leq (2n+3)/5$. Since m is even we have $m \leq \lfloor (2n+2)/5 \rfloor$ and hence the size of G is at most $(1/2)\ (2(1 + 2 + ... + n) + \lfloor (2n+2)/5 \rfloor)$. As $\varepsilon(G)$ is an integer, we have $\varepsilon(G) \leq n(n+1)/2 + \lfloor (n+1)/5 \rfloor$. ■

Lemma 2 — The size of a highly irreguar graph of order $2n + 1$, $n \geq 4$, with maximum degree $\Delta \leq n-1$ is at most $n(n+1)/2 + \lfloor (n+1)/5 \rfloor$.

PROOF : Let u_Δ and v_Δ be two adjacent vertices of maximum degree Δ in G. Further, let $N(u_\Delta) = \{v_1, v_2, ..., v_\Delta\}$ and $N(v_\Delta) = \{u_1, u_2, ..., u_\Delta\}$. Take $V_1 = V(G) \setminus N(\{u_\Delta, v_\Delta\})$ and let $\delta_1 = \min_{v \in V_1} d_G(v)$ and $\Delta_1 = \max_{v \in V_1} d_G(v)$. If $\Delta_1 = \Delta$, then $\delta_1 = 1$, and we have

$$\varepsilon \le \frac{1}{2}(2(1+2+\ldots+\Delta) + (2n-2\Delta)\Delta + 1)$$

$$\le (1+2+\ldots+n) + \frac{1}{2} = n(n+1)/2 + \frac{1}{2} \le n(n+1)/2 + \lfloor (n+1)/5 \rfloor.$$

So assume that $\Delta_1 < \Delta$. Then $\Delta - \Delta_1 = i$, where $1 \le i \le \Delta - 1$.

Claim : $\delta_1 \le i$.

Suppose not, then $\delta_1 > i$. Let w be a vertex of degree Δ_1 in V_1. Set $A_m = \{u_m, v_m\}$ for $1 \le m \le \Delta$. As $\delta_1 > i$, we have for $1 \le k \le i$, $N(A_k) = A_{\Delta-k+1} \bigcup A_{\Delta-k+2} \bigcup \ldots \bigcup A_\Delta$ and hence $N(w) \bigcap A_k = \phi$, which implies that $N(w)$ contains the vertices having degree in $\{i+1, i+2, ..., \Delta-1\}$. Since G is highly irregular, $N(w)$ contains the vertices of distinct degrees. Thus $d(w) \le \Delta - i - 1$, a contradiction to the fact that $d(w) = \Delta - i$. Hence we have shown that $\delta_1 \le \Delta - \Delta_1 = i$.

Case 1 — Let $1 \le i \le \lfloor (2n+2)/5 \rfloor$.

Then $\delta_1 \le i \le \lfloor (2n+2)/5 \rfloor$, which implies that

$$\varepsilon \le \frac{1}{2}(2(1+2+\ldots\Delta) + (2n-2\Delta)\Delta + \delta_1)$$

$$\le \frac{1}{2}(2(1+2+\ldots+n) + \lfloor (2n+2)/5 \rfloor)$$

$$= \frac{1}{2}(n(n+1) + \lfloor (2n+2)/5 \rfloor).$$

Since ε is an integer, we have $\varepsilon \le \frac{1}{2}(n(n+1)) + \lfloor (n+1)/5 \rfloor$.

Case 2 — Let $i \ge \Delta - \lfloor (2n+2)/5 \rfloor$.

Then $\delta_1 \le \Delta_1 = \Delta - i \le \lfloor (2n+2)/5 \rfloor$. The result now follows as in Case 1.

Case 3 — Let $\lfloor (2n+2)/5 \rfloor < i < \Delta - \lfloor (2n+2)/5 \rfloor$.

Then $\lfloor (2n+2)/5 \rfloor + 1 \le \Delta_1 \le \Delta - \lfloor (2n+2)/5 \rfloor - 1$.

Now $\sum_{v \in V} d(v) \le 2(1+2+\ldots+\Delta) + (2n-2\Delta+1)\Delta_1$

$$\le 2(1+2+\ldots+\Delta) + (2n-2\Delta-2)\Delta + 3\Delta_1$$

$$\leq 2(1+2+\ldots+(n-1)) + 3\Delta_1 = n(n-1) + 3\Delta_1$$

$$\leq n(n-1) + 3(\Delta-1) - 3\lfloor (2n+2)/5 \rfloor.$$

But $\quad 3(\Delta-1) \leq 3(n-2) = 2n + (n-6) \leq 2n + 4\lfloor (2n+2)/5 \rfloor.$

Thus $\quad 3(\Delta-1) \leq n(n+1) + \lfloor (2n+2)/5 \rfloor - n(n-1) + 3\lfloor (2n+2)/5 \rfloor.$

That is, $\quad n(n+1) + \lfloor (2n+2)/5 \rfloor \geq n(n-1) + 3(\Delta-1) - 3\lfloor (2n+2)/5 \rfloor \geq \sum_{v \in V} d(v).$

Hence $\quad \varepsilon \leq \frac{1}{2}(n(n+1)) + \lfloor (n+1)/5 \rfloor.$ ■

Now we prove the main theorem.

Theorem 1 — The size of a highly irregular graph G of order $2n + 1$, $n \geq 4$, is at most $n(n+1)/2 + \lfloor (n+1)/5 \rfloor$.

PROOF : As G is highly irregular, $v(G) \geq 2\Delta(G)$, that is, $\Delta(G) \leq n$. The proof now follows from Lemma 1 and Lemma 2. ■

The bound presented in Theorem 1 cannot be reduced, i.e., there exists a highly irregular graph of order $2n + 1$ with maximum degree n and having exactly $n(n + 1)/2 + \lfloor (n+1)/5 \rfloor$ edges. The construction of such a graph is given below :

Consider the graph $H_{n,n} \backslash \{u_{n-1} v_m, u_{n-2} v_{m+1}, \ldots, u_{n-(m/2)}, v_{(3m/2)-1}\}$ where $m = \lfloor (2n+2)/5 \rfloor$. Add a new vertex v and new edges $u_{n-1} v, u_{n-2} v, \ldots, u_{n-(m/2)} v, vv_m, vv_{m+1}, \ldots, vv_{(3m/2)-1}$. The resulting graph is the required one. Figure 3 displays the graph with $n = 5$ and $m = 2$.

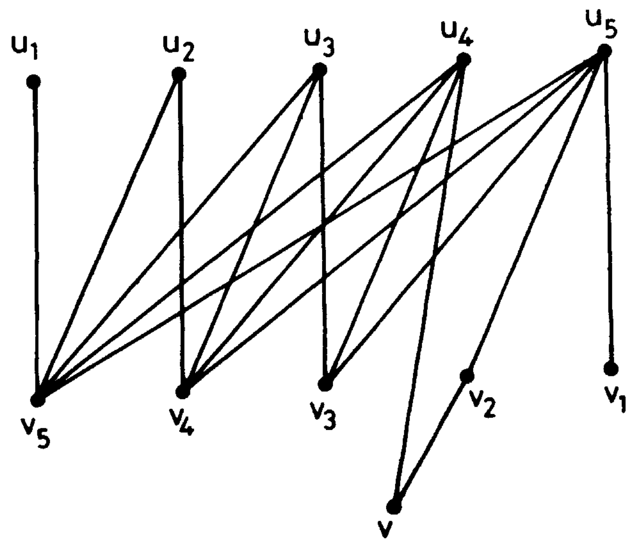

FIG. 3. A highly irregular graph of order 11 and of size 16.

We now ask for which highly irregular graphs G, their complements are also highly irregular.

Theorem 2 — P_1 and P_4 are the unique two highly graphs whose complements are also highly irregular.

PROOF : Let G be a highly irregular graph of order $n \geq 2$ whose complement is also highly irregular. Since G is highly irregular, $\delta(G) = 1$ and hence $\Delta(G^c) = n - 2$. But as G^c is also highly irregular, the order of G^c is at least $2\Delta(G^c)$, i.e., $n \geq 2(n-2)$ which implies that $n \leq 4$. Since K_2 and P_4 are the only highly irregular graphs of order at most 4 and since K_2^c is not highly irregular, we conclude that G is isomorphic to P_4 and hence the proof follows. ■

3. Highly Irregular Bipartite Graphs

Theorem 3 — For any positive integer $n \neq 3$, 5 or 7, there is a highly irregular bipartite graph of order n.

PROOF : For $n = 1$ and $n = 2m$, $m \geq 1$, K_1 and $H_{m,m}$ are the respective required graphs. So it is enough to prove the result for odd $n \geq 9$. Now for $n = 2m + 1$, $m \geq 4$, consider $H_{m-1,m-1} \backslash \{u_{m-2} v_2\}$. Add new vertices u_m, v_m, v_{m+1} and new edges $u_{m-2} v_m, v_2 u_m, u_m v_{m+1}$ which gives the required highly irregular bipartite graph of order $2m + 1$. Figure 4 displays a highly irregular bipartite graph of order 13. ■

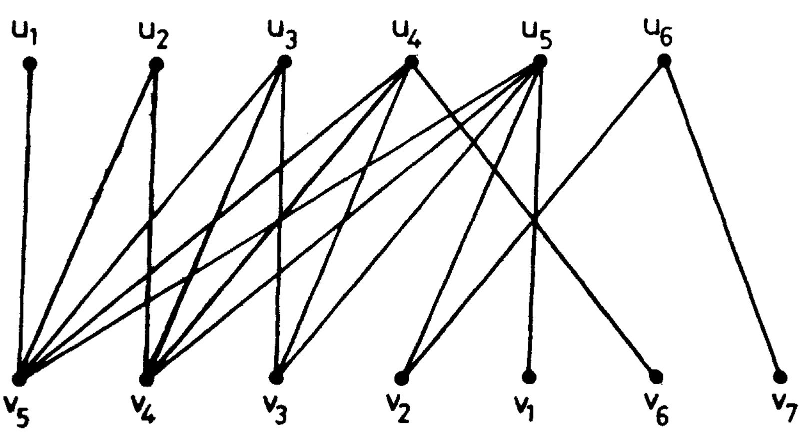

FIG. 4. A highly irregular bipartite graph of order 13.

Theorem 4 — For any two integers n and m, $n \geq m \geq 3$, there is a highly irregular bipartite graph with maximum degree m and with bipartite sizes (n, n).

PROOF : We prove this result in two cases.

Case (i) — Let $m = 3$ and $n = 3a + b$ where $a \geq 1$ and $0 \leq b \leq 2$.

If $b = 0$, then $n = 3a$. When $a = 1$, $H_{3,3}$ is the required graph. When $a \geq 2$, consider a-copies of $H_{3,3} \backslash \{u_2 v_2\}$. Now add new edges $u_2^1 v_2^2$, $u_2^2 v_2^3$, ..., $u_2^{a-1} v_2^a$, $u_2^a v_2^1$ where u_i^j (or v_i^j) denotes the vertex u_i (or v_i) of the jth copy. The resulting graph G is highly irregular and has maximum degree 3. Figure 5 displays a graph with $m = 3$ and $n = 9$.

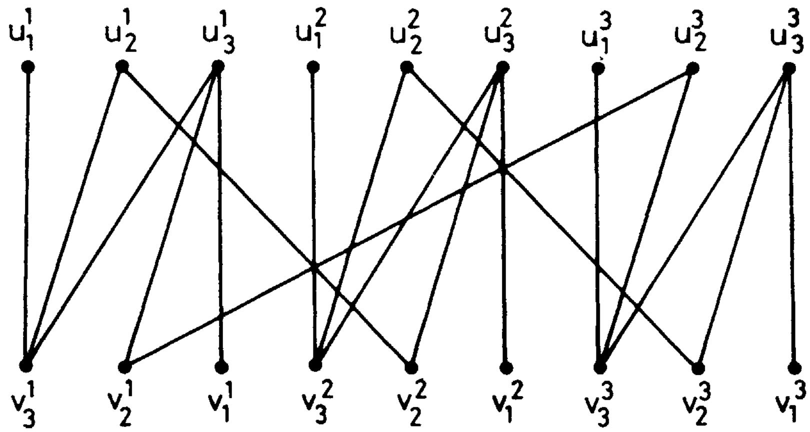

FIG. 5.

If $b = 1$, then $n = 3a + 1$. Now consider the graph $G - u_2^a v_2^1$. Introduce new vertices u and v and new edges uv_2^1 and $u_2^a v$. The resulting graph H is the required graph.

If $b = 2$, then $n = 3a + 2$. In this case, add new vertices x and y and new edges xv and uy to the graph H which gives the required graph.

Case (ii) — Let $m \geq 4$ and $n = ma + b$ where $a \geq 1$ and $0 \leq b \leq m-1$.

Subcase (i) : Let $a = 1$. Then $n = m + b$ when $b = 0$, $G_0 \cong H_{m,m}$ is the required graph. When $1 \leq b \leq m-2$, consider the graph $H_{m,m} \backslash \{v_2 u_{m-1}, v_3 u_{m-2}, ..., v_{b+1} u_{m-b}\}$. Add new vertices $u_1', u_2', ..., u_b', v_1', v_2', ..., v_b'$ and new edges $v_2 u_1', v_3 u_2', ..., v_{b+1} u_b', u_{m-1} v_1', u_{m-2} v_2', ..., u_{m-b} v_b'$. The resulting graph G_b is the required graph. Figure 6 displays a graph with $m = 5$ and $n = 7$.

Finally when $b = m - 1$, G_{m-1}, the graph obtained from the graph G_{m-2} together with the vertices u_{m-1}', v_{m-1}' and the edges $u_{m-2}' v_{m-1}'$, $v_{m-2}' u_{m-1}'$ is the required one.

Subcase (ii) : Let $a > 1$. Now $n = am + b$. Consider the graph G_b along with $a - 1$ copies of $H_{m,m}$. Delete the edges $u_{m-1}^1 v_{m-1}^1, u_{m-1}^2 v_{m-1}^2, ..., u_{m-1}^{a-1} v_{m-1}^{a-1}$ where u_i^j (or v_i^j) denotes the vertex u_i (or v_i) of the jth copy of $H_{m,m}$ and delete the edge

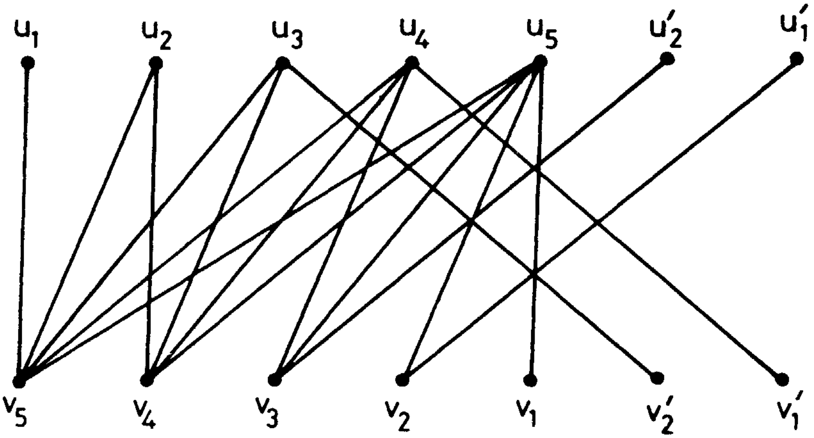

FIG. 6.

$u_{m-1} v_{m-1}$ from G_b. Then add edges $v^1_{m-1} u^2_{m-1}, v^2_{m-1} u^3_{m-1}, \ldots, v^{a-2}_{m-1} u^{a-1}_{m-1}$ and $v_{m-1} u^1_{m-1}$, $u_{m-1} v^{a-1}_{m-1}$. The resulting graph is the required one.

Figure 7 displays a graph with $m = 4$ and $n = 10$.

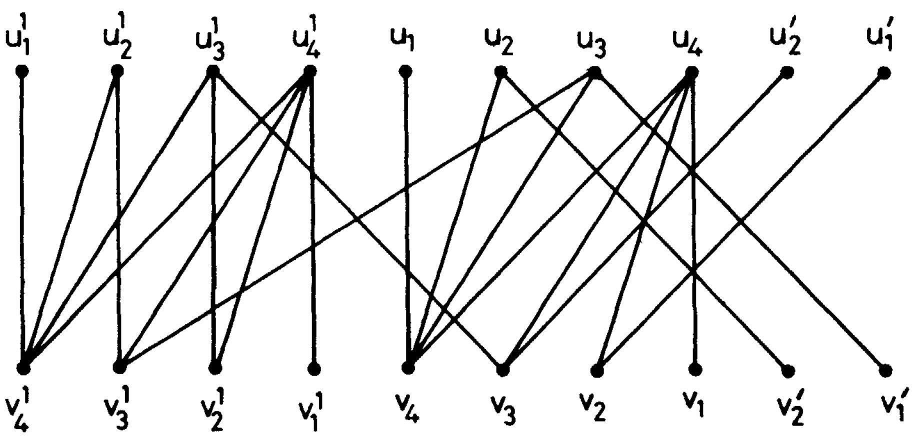

FIG. 7.

Theorem 5 — The maximum degree of a highly irregular bipartite graph with bipartite sizes (n, m), $n < m$ is at most $n - 1$.

PROOF : Let G be a highly irregular bipartite graph with bipartite sets (X, Y) such that $|X| = n$, $|Y| = m$ and $n < m$. Let the maximum degree of G be Δ. As G is highly irregular, Y has a vertex v of degree Δ and hence $\Delta \leq n$. We claim that $\Delta \neq n$. Otherwise X has a vertex u_n and Y has a vertex v_n both of degree n. Let $N(u_n) = \{v_1, v_2, \ldots, v_n\}$ and $N(v_n) = \{u_1, u_2, \ldots, u_n\}$ so that $d(u_i) = d(v_i) = i$, $1 \leq i \leq n$. Label the remaining $m - n$ vertices in Y by $v_{n+1}, v_{n+2}, \ldots, v_m$. Now for

$1 \le i \le n$, $N(v_i) = \{u_{n-i+1}, u_{n-i+2}, ..., u_n\}$. As G is highly irregular, there can be no edge $u_i v_j$, $1 \le i \le n$ and $n < j \le m$. Hence the vertices $v_{n+1}, v_{n+2}, ..., v_m$ are isolated, a contradiction. Thus $\Delta \le n-1$. ■

Theorem 6 — For any n and m, $3 \le n \le m \le n + (n/2) - 1$, there is a highly irregular bipartite graph with bipartite sizes (n, m).

PROOF : If $n = 3$ or $n = m$, then $H_{n,n}$ is the required graph. So we assume that $4 \le n < m$. Now consider $H_{n-1, n-1} \backslash \{v_2 u_{n-2}, v_3 u_{n-3}, ..., v_{m-n+1} u_{2n-m-1}\}$. Introduce new vertices $v_n, v_{n+1}, ..., v_m, u_n$ and new edges $u_n v_n, u_{2n-m-1} v_{n+1}, u_{2n-m} v_{n+2}, ..., u_{n-3} v_{m-1}, u_{n-2} v_m, v_2 u_n, v_3 u_n, ..., v_{m-n+1} u_n$. The resulting graph is highly irregular with bipartite sizes (n, m). This is because $m - n + 1 \le n/2$ by hypothesis. (If $j = m - n + 1 > n/2$ then v_j will be adjacent to the vertices u_j and u_n both of degree j). Figure 8 displays a highly irregular bipartite graph with bipartite sizes (8, 11).

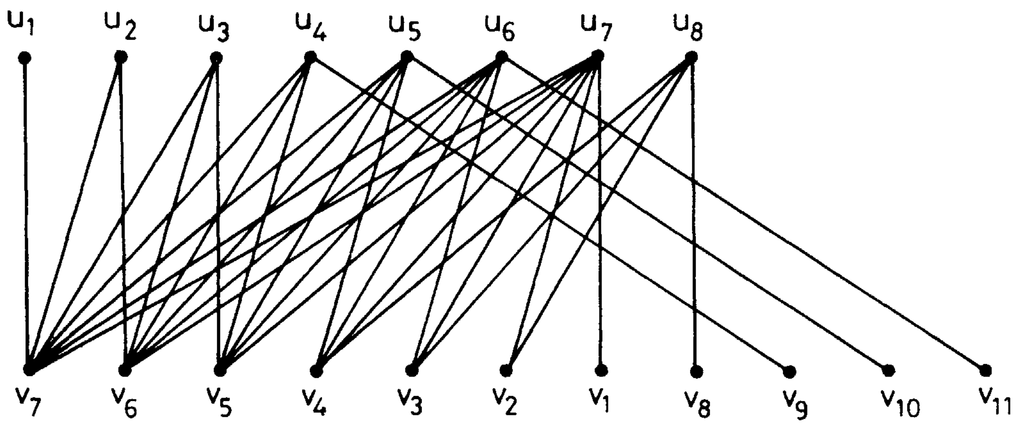

FIG. 8.

Theorem 7 — Every bipartite graph with bipartite sizes (n, m), $n \ge m \ge 3$ is an induced subgraph of a highly irregular bipartite graph with bipartite sizes $(2(n+m) - 2, 2(n+m) - 2)$.

PROOF : Let G be a bipartite graph with bipartite sets (X, Y) such that $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_m\}$ where $n \ge m \ge 3$. Consider another copy $G(U, V)$ of G where $U = \{u_1, u_2, ..., u_n\}$ and $V = \{v_1, v_2, ..., v_m\}$. For $1 \le i \le n$ and $1 \le j \le m$, join x_i with v_j and u_i with y_j if $x_i y_j$ is not an edge in G. Denote the resultant graph by H. Consider the graph $H \cup \{x_{n+1}, x_{n+2}, ..., x_{n+m-1}; y_{m+1}, y_{m+2}, ..., y_{m+n-1}; v_{m+1}, v_{m+2}, ..., v_{m+n-1}; u_{n+1}, u_{n+2}, ..., u_{n+m-1}\}$. Introduce new edges $u_i v_{m+j}$ and $x_i y_{m+j}$ for $1 \le i \le j \le n-1$ and edges $v_i u_{n+j}$ and $x_i y_{n+j}$ for $1 \le i \le j \le m-1$. If G is not a complete bipartite graph, then the resulting graph H_1 is a highly irregular bipartite graph and it contains G as an induced subgraph. If G is a complete bipartite graph, then H_1 is disconnected. To make it connected, take $H \setminus \{x_{n-1}y_{m+n-1}, u_{n-1}v_{m+n-1}\}$ and add new edges $x_{n-1}v_{m+n-1}$ and $u_{n-1}y_{m+n-1}$. The resulting graph H_2 is then the required one.

ACKNOWLEDGEMENT

The author thanks Professor R. Balakrishnan for his valuable guidance, the Council of Scientific and Industrial Research, New Delhi for its financial support (Grant No. 9/3(19)92/EMR-1) and the anonymous referees for their useful remarks.

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