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THE TOTAL GLOBAL DOMINATION NUMBER OF A GRAPH

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A total dominating set T of a graph $G = (V, E)$ is a total global dominating set (t.g.d. set) if T is also a total dominating set of $\overline{G}$. The total global domination number $\gamma_{tg}(G)$ of G is the minimum cardinality of a t.g.d. set. In this paper, we characterize t.g.d. sets and bounds are obtained for $\gamma_{tg}(G)$. We exhibit inequalities involving variations on domination numbers and vertex covering number. For graphs with diameter at least five, three of the four domination numbers considered turn out to be identical. Values are given for paths, cycles and complete bipartite graphs. We characterize graphs with $|V|$ vertices in any minimum total global dominating set.

1. INTRODUCTION

The graphs G considered here have order p and size q (i.e. p vertices and q edges) and both G and their complements $\overline{G}$ have no isolates. Any undefined term in this paper may be found in Harary[3].

A set D of vertices in a graph $G = (V, E)$ is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set (see Cockayne and Hedetniemi[2]).

A total dominating set T of G is a dominating set such that the induced subgraph $\langle T \rangle$ has no isolates. The total domination number $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set. This concept was introduced in Cockayne *et al.*[1].

A dominating set D of G is a global dominating set (g.d. set) if D is also a dominating set of $\overline{G}$. The global domination number $\gamma_g(G)$ of G is the minimum cardinality of a g.d. set (Sampathkumar[5]). The purpose of this paper is to study the global aspect of total domination.

A total dominating set T of G is a total global dominating set (t.g.d. set) if T is also a total dominating set of $\overline{G}$. The total global domination number $\gamma_{tg}(G)$ of G is the minimum cardinality of a t.g.d. set.

We note that $\gamma(G)$ and $\gamma_g(G)$ are defined for any G while $\gamma_t(G)$ is only defined for G with $\delta(G) \geq 1$ and $\gamma_{tg}(G)$ is only defined for G with $\delta(G) \geq 1$ and $\delta(\overline{G}) \geq 1$, where $\delta(G)$ is the minimum degree of G.

A γ_t-set is a minimum total dominating set. Similarly a γ_g-set and a γ_{tg}-set are defined.

2. Results

Theorem 1 — A total dominating set T of G is a t.g.d. set if and only if for each vertex $v \in V$ there exists a vertex $u \in T$ such that v is not adjacent to u.

Theorem 2 — Let G be a graph such that neither G nor $\overline{G}$ have an isolated vertex. Then,

(i) $\gamma_{tg}(G) = \gamma_{tg}(\overline{G})$; ... (1)

(ii) $\gamma_t(G) \leq \gamma_{tg}(G)$; ... (2)

(iii) $\gamma_g(G) \leq \gamma_{tg}(G)$; ... (3)

(iv) $(\gamma_t(G) + \gamma_t(\overline{G}))/2 \leq \gamma_{tg}(G) \leq \gamma_t(G) + \gamma_t(\overline{G})$. ... (4)

Next we characterize graphs G which have total global domination number equal to the order p of G.

Theorem 3 — Let G be a graph such that neither G nor $\overline{G}$ have an isolated vertex. Then,

$$\gamma_{tg}(G) = p \qquad \text{... (5)}$$

if and only if $G = P_4$ (a path on 4 vertices) or mK_2 or $\overline{mK_2}$; $m \geq 2$.

Proof : Suppose (5) holds. On the contrary, suppose $G \neq P_4$, mK_2, $\overline{mK_2}$; $m \geq 2$. Then we consider the following cases :

Case 1 : If $\Delta(G)$ and $\Delta(\overline{G})$ are $\leq p-3$, where $\Delta(G)$ is the maximum degree of G, then both G and $\overline{G}$ have no vertices of degree 1 and hence for any vertex $v \in V$, $V - \{v\}$ is a t.g.d. set of G, a contradiction.

Case 2 : If either $\Delta(G)$ or $\Delta(\overline{G}) = p - 2$ say $\Delta(G) = p - 2$ and u is a vertex of degree $p - 2$, then there exists exactly one vertex v such that v is not adjacent to u. If v is also of degree $p - 2$, then, as $G = \overline{mK_2}$; $m \geq 2$, there exists a vertex w such that w is not adjacent to at least two vertices. If some non-neighbour x of w has degree $p - 2$, then $V - x$ is a t.g.d. set. Otherwise each non-neighbour of w has at least two non-neighbours and $V - w$ is a t.g.d. set. Suppose $\deg_G(v) < p-2$. If u has no neighbour of degree 1, then $V - u$ is a t.g.d. set of G, otherwise let u be adjacent to w, $\deg_G(w) = 1$, and let x be adjacent to v (and necessarily adjacent to u and non-adjacent to w) then $\{v, x, u, w\}$ is a t.g.d. set (and $G = P_4$). If $\Delta(\overline{G}) = p - 2$, then we can apply the same argument to $\overline{G}$ (and obtain that $\overline{G} = P_4$, so that $G = \overline{P}_4 \cong P_4$).

This proves the necessity. Sufficiency is obvious.

Theorem 4 — Let G be a graph such that neither G nor $\overline{G}$ have an isolated vertex and T be a γ_t-set of G with each x in T has non-neighbour in T. If there exists a vertex $u \in V-T$ which is adjacent only to vertices in T, then,

$$\gamma_{tg}(G) \leq \gamma_t(G) + 2. \qquad \text{... (6)}$$

PROOF : We consider the following cases :

Case 1 : If $V - T = \{u\}$, then there exists a vertex $v \in T$ such that v is not adjacent to u and hence T is a t.g.d. set. Thus (6) holds.

Case 2 : If $V - T \neq \{u\}$, then there exists a vertex $v \in V-T$ and hence $T \cup \{u, v\}$ is a t.g.d. set. Thus (6) follows.

Now we obtain a lower bound on $\gamma_{tg}(G)$.

Theorem 5 — Let G be a graph such that neither G nor $\overline{G}$ have an isolated vertex. Then,

$$2q - p\,(p - 3) \leq \gamma_{tg}(G). \qquad \text{... (7)}$$

PROOF : Let T be a γ_{tg}-set of G. Then by Theorem 1, each vertex $v \in V$ is not adjacent to at least one vertex in T. This implies

$$q \leq \binom{p}{2} - |V-T| + \frac{|T|}{2}.$$

Thus (7) follows.

In a graph G, a vertex and an edge incident with it are said to cover each other. The vertex covering number $\alpha_0(G)$ equals the minimum number of vertices in a set S which covers every edge of G.

Theorem 6 — Let G be a graph such that neither G nor $\overline{G}$ have an isolated vertex. Then,

$$\gamma_{tg}(G) \leq 2\alpha_0(G). \qquad \text{... (8)}$$

PROOF : Let S be a vertex cover of G with $|S| = \alpha_0(G)$. Let $S = \{u_1, u_2, ..., u_s\}$, $S = \{u_1\}$ is impossible since u_1 has a non-neighbour x, and x has a neighbour which can only be u_1, so $s \geq 2$. Each $u \in S$ has a neighbour, a non-neighbour, or both, in $S - u$. If u has no neighbour in S, then choose $v \in V-S$ a neighbour of u. If u has no non-neighbour in S, then choose in $V - S$ a non-neighbour v of u.

Thus construct $D = \{v_1, v_2, ..., v_{s'}\}$, $s' \leq s$, not all vertices v_i, v_j need be distinct, so $|D| \leq |S|$. If $D = \{v_1\}$ and if v_1 is adjacent to each vertex of S then add a non-neighbour v_1 to D so that $|D| \geq 2$.

$S \cup D$ is a t.g.d. set in G because $x \in V - \{S \cup D\}$ has at least one neighbour which necessarily belongs to S and each vertex of D is a non-neighbour of x. $x \in S$ has by construction both a neighbour and a non-neighbour in $S \cup D$. $x \in D$ has a neighbour in S and as $|D| \geq 2$, x has a non-neighbour in D. Thus $\gamma_{tg}(G) \leq |S \cup D| = |S| + |D| \leq 2|S| = 2\alpha_0(G)$.

For a vertex $v \in V$, the eccentricity is defined as $e(v) = \max \{d(u, v)/u \in V\}$. The maximum of the eccentricities is defined to be the diameter of G, denoted diam (G).

Theorem 7 — Let G be a graph with $\text{diam}(G) \geq 5$. Then $T \subseteq V$ is a total dominating set of G if and only if T is a t.g.d. set.

PROOF : Suppose T is a total dominating set of G. Let $u, v \in V$ such that $d(u, v) \geq 5$. Then $T \cap N(u) \neq \phi$ and $T \cap N(v) \neq \phi$. Let $u_1 \in T \cap N(u)$ and $v_1 \in T \cap N(v)$. Then u_1 and v_1 are not adjacent and further every vertex in $V - \{u_1, v_1\}$ is adjacent to at most one of u_1 and v_1. This implies that $\{u_1, v_1\}$ is a total dominating set of $\overline{G}$ and hence T is a t.g.d. set.

The converse is obvious.

For a set $D \subseteq V$, the minimum degree of the induced subgraph $\langle D \rangle$ is denoted by $\delta(\langle D \rangle)$.

Theorem 8 — Let G be a graph such that neither G nor $\overline{G}$ have an isolated vertex and diam $(G) \geq 5$. A set $D \subseteq V$ with $\delta(\langle D \rangle) \geq 1$, is a g.d. set of G if and only if D is a t.g.d. set.

PROOF : Let D be a g.d. set of G. Suppose there exists a vertex $u \in D$ such that u is adjacent to every vertex in D. Then, $\text{diam}(G) \leq 4$, a contradiction. This implies that D is a total dominating set of $\overline{G}$. Since $\langle D \rangle$ has no isolates, D is a t.g.d. set.

The converse is immediate.

Corollary 8.1 — Let G be a graph such that neither G nor $\overline{G}$ have an isolated vertex and D be a γ_g-set of G with $\delta(\langle D \rangle) \geq 1$. If diam $(G) \geq 5$, then,

$$\gamma_t(G) = \gamma_{tg}(G) \quad \text{... (9)}$$

$$\gamma_g(G) = \gamma_{tg}(G). \quad \text{... (10)}$$

Proposition A (Kulli and Patwari[4]) — (i) For any complete bipartite graph $K_{m,n}$ with $2 \leq m \leq n$, $\gamma_t(K_{m,n}) = 2$.

(ii) For any cycle C_p with $p \geq 4$ vertices,

$$\gamma_t(C_p) = (p/2) + 1 \text{ if } p \equiv 2 \pmod 4;$$
$$= \lceil p/2 \rceil \text{ otherwise;}$$

where $\lceil x \rceil$ is a least integer not less than x.

(iii) For any path P_p with $p \geq 4$ vertices,

$$\gamma_t(P_p) = (p/2) + 1 \text{ if } p \equiv 2 \pmod 4;$$
$$= \lceil p/2 \rceil \text{ otherwise.}$$

Now we list the exact values of $\gamma_{tg}(G)$ for some standard graphs.

Proposition 9 — (i) For any complete bipartite graph $K_{m,n}$ with $2 \leq m \leq n$,

$$\gamma_{tg}(K_{m,n}) = 4. \quad \text{... (11)}$$

(ii) For any cycle C_p with $p \geq 4$ vertices,

$$\gamma_{tg}(C_p) = (p/2) + 1 \text{ if } p \equiv 2 \pmod 4; \quad \text{... (12)}$$

$$= \lceil p/2 \rceil \text{ otherwise.} \quad \text{... (13)}$$

(iii) For any path P_p with $p \geq 4$ vertices,

$$\gamma_{tg}(P_p) = (p/2) + 1 \text{ if } p \equiv 2 \pmod 4; \quad \text{... (14)}$$

$$= \lceil p/2 \rceil. \quad \text{... (15)}$$

Theorem 10 — Let diam $(G) = k$.

(i) If $k = 4$, then $\gamma_{tg}(G) \leq \gamma_t(G) + 1$. ... (16)

(ii) If $k = 3$, then, (6) holds.

PROOF : Let T be a γ_t-set of G. Suppose $k = 4$ and u, v be two vertices with $d(u, v) = 4$. Then $T \cap N[u] \neq \phi$. Let $u_1 \in T \cap N[u]$. Then no vertex in G is adjacent to both u_1 and v and hence $\{u_1, v\}$ is a total dominating set of $\overline{G}$. Thus $T \cup \{v\}$ is a t.g.d. set. Hence (16) holds.

If $k = 3$ and u, v be two vertices with $d(u, v) = 3$, then no vertex in G is adjacent to both u and v and hence $\{u, v\}$ is a total dominating set of $\overline{G}$. Thus $T \cup \{u, v\}$ is a t.g.d. set. Hence (6) holds.

Similarly, we can prove

Theorem 11 — Let D be a γ_g-set of G such that $\langle D \rangle$ has no isolates and diam $(G) = k$.

(i) If $k = 4$, then, $\gamma_{tg}(G) \leq \gamma_g(G) + 1$. ... (17)

(ii) If $k = 3$, then, $\gamma_{tg}(G) \leq \gamma_g(G) + 2$. ... (18)

When diam$(G) = 2$, then the difference between $\gamma_{tg}(G)$ and $\gamma_t(G)$ as well as between $\gamma_{tg}(G)$ and $\gamma_g(G)$ may be very large. For example, let $G = \overline{C}_p$ with $p \geq 19$ vertices. Then any two vertices u and v at a distance at least four in C_p form a γ_t-set for G and hence $\gamma_t(G) = 2$. Also, by (1), (12) and (13), $\gamma_{tg}(G) = \gamma_{tg}(C_p) \geq \lceil p/2 \rceil$. Further, $\gamma_g(G) = \gamma_g(C_p) = \lceil p/3 \rceil$.

Theorem 12 — Let diam (G) and diam $(\overline{G}) \geq 3$.

If G is connected, then

$$\gamma_{tg}(G) \leq \min \{p - \Delta(G) + 2, \ \delta(G) + 4\}. \quad \text{... (19)}$$

If G is disconnected, then

$$\gamma_{tg}(G) \leq \min \{p - \Delta(G) + 3, \ \delta(G) + 3\}. \quad \text{... (20)}$$

PROOF : By (9) and Theorem 10,

$$\gamma_{tg}(G) \leq \gamma_t(G) + 2.$$

$$\gamma_{tg}(\overline{G}) \leq \gamma_t(\overline{G}) + 2.$$

Since $\gamma_{tg}(G) = \gamma_{tg}(\overline{G})$,

$$\gamma_{tg}(G) \leq \min\ \{\gamma_t(G) + 2,\ \gamma_t(\overline{G}) + 2\}.$$

Suppose G is connected. Since $\Delta(G) < p - 1$, by Cockayne *et al*[1].

$$\gamma_t(G) \leq p - \Delta(G).$$

$$\gamma_t(\overline{G}) \leq p - \Delta(\overline{G}) + 1 = 2 + \delta(G).$$

Thus (19) holds.

Similarly, we can prove (20), when G is disconnected.

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