Indian J. pure appl. Math., **27**(6) : 543-549, June 1996

NON-RIEMANNIAN CARTAN GEOMETRY*

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(*Received* 24 *April* 1995; *after revision* 11 *December* 1995;
accepted 29 *December* 1995)

The main aim of the present paper is to introduce and to generalize the concept of strongly non-Riemannian Cartan spaces.

1. INTRODUCTION

A Cartan space[5, 6] is a Hamilton space $H^n = (M, H(x, p))$ in which the fundamental function $H(x, p)$ is $2(p)$-homogeneous in p_i on $T^* M - \{0\}$. We denote it by ζ^n. The fundamental tensor field of ζ^n and its reciprocal are given by

$$g^{ij}(x, p) = \frac{1}{2} \dot{\partial}^i \dot{\partial}^j H,$$

and
$$g_{ij}\, g^{jk} = \delta^k{}_j \; (\dot{\partial}^i := \partial / \partial p_i).$$

The homogeneity of $H(x, p)$ is expressed by

$$(\dot{\partial}^i H) p_i = H \text{ and so } H = g^{ij} p_i p_j.$$

A d-connection on M is $H\Gamma(N) = (H^i{}_{jk}\,,\ C_i{}^{jk})$, where $N = (N_{jk})$ is a non-linear connection, $H = (H^i{}_{jk})$ is a special d-object and $C = (C_i{}^{jk})$ is a d-tensor field of type (2, 1).

For ζ^n we have the canonical metrical d-connection $C\Gamma(N) = (H^i{}_{jk}, C_i{}^{jk})$, with the coefficients given by

$$N_{ij} = \gamma_{ij}{}^0 - \frac{1}{2} \gamma_r{}^0{}_0\, \dot{\partial}^r g_{ij},$$

$$H^i{}_{jk} = \frac{1}{2} g^{ir} (\delta_j g_{rk} + \delta_k g_{jr} - \delta_r g_{jk})$$

$$C_i{}^{jk} = -\frac{1}{2} g_{ir} (\dot{\partial}^j g^{rk} + \dot{\partial}^k g^{jr} - \dot{\partial}^r g^{jk}),$$

*Research supported under Career Award of U.G.C.

where $\delta_i = \partial_i + N_{ri}\dot{\partial}^r \quad (\partial_i := \partial/\partial x^i)$,

$$\gamma_i{}^0{}_j = p_h \gamma^h{}_{ij},\ \gamma_r{}^0{}_0 = \gamma^0{}_{rs}\, g^{sk}{}_{p_k} \quad \text{(Miron}^4\text{)}.$$

Given a d-connection, the h- and v-covariant derivatives are defined for d-tensor field, e.g., K^i_j, by

$$K^i{}_{j|k} = \delta_k K^i{}_j + K^r{}_j H^i{}_{rk} - K^i{}_r H^r{}_{jk}\ ,$$

$$K^i_j\ |^k = \dot{\partial}^k K^i{}_j + K^r{}_j C_r{}^{jk} - K^i{}_r C_j{}^{rk}.$$

In this paper, the notion of Cartan space depends on the definition given by Miron[4-6]. Some authors call it Miron spaces (cf. Miron[6], Singh[7]). Also, the fundamental geometric notions in ζ^n are refered to Miron[5, 6].

2. Strongly Non-riemannian Cartan Spaces

We introduce the following d-tensor fields on T^*M :

$$H^i = 2g^{ij} p_j\ ,\ H^{i_1 i_2 \cdots i_{2s-1}} = \frac{1}{2^{s-1}} \dot{\partial}^{i_1} \dot{\partial}^{i_2} \ldots \dot{\partial}^{i_{2s-1}} H \qquad (s = 2, 3, \ldots).$$

These d-tensor fields are completely symmetric and determine a sequence of covariant d-vector fields which are at each point defined on T^*M, namely

$$\overset{1}{H^i} := H^i,\ \overset{2}{H^i} := H^{i j_1 j_2} g_{j_1 j_2},$$

$$\overset{3}{H^i} := H^{i j_1 j_2 j_3 j_4} g_{j_1 j_2} g_{j_3 j_4},\ \cdots$$

Also, we note that

$$H^{ijk} = -C^{ijk}.$$

We now introduce (Here the reference Matsumoto[3] has been taken as a useful tool) :

Definition 2.1 — A Cartan space ζ^n is called strongly non-Riemannian if the $(n-1)$ vectors $\overset{1}{H^i}, \overset{2}{H^i}, \ldots, \overset{n-1}{H^i}$ are linearly independent :

We now prove :

Theorem 2.1 — In a strongly non-Riemannian Cartan space ζ^n there exists a single d-vector $\overset{n}{H^i}$ such that $\overset{1}{H^i}\overset{2}{H^i}, \ldots, \overset{n-1}{H^i}, \overset{n}{H^i}$ are linearly independent.

Proof : We first claim that there exists such a contravariant d-vector $\overset{n}{H^i}$.

Let us put

$$q^i = \{\delta^{ij}, \overset{1}{H^j}, \ldots, \overset{n-1}{H^j}\}.$$

Then under a coordinate transformation $\overline{x}^i = \overline{x}^i(x)$, q^i is transformed as

$$\overline{q}^i = |\partial\overline{x}/\partial x|\, q^k\ (\partial_k \overline{x}^i).$$

Further

$$\overline{g} = |\partial\,\overline{x}/\partial x|^2 g,\ g := |g^{ij}| > 0.$$

Consequently the quantity

$$\overset{n}{H^i} := q^i g^{-1/2},$$

is a d-vector.

We now prove that $\{\overset{1}{H^i}, ..., \overset{n}{H^i}\}$ is linearly independent. We have

$$|\overset{n}{H^i},\ \overset{1}{H^j}, ..., \overset{n-1}{H^j}|$$

$$= \sum_{i=1}^{n} \overset{n}{H^i}(\partial/\partial x^i)\,|x^j, \overset{1}{H^j}, ..., \overset{n-1}{H^j}|$$

$$= \sum_{i=1}^{n} \overset{n}{H^i} q^i = \sum_{i=1}^{n} g^{-1/2}(q^i)^2 > 0,$$

where $\qquad x \in T_p^*(T^* M)$.

Thus the strongly non-Riemannian ζ^n admits n-linearly independent d-vector $\overset{r}{H^i}$, r = 1, 2, ..., n in unique way. q.e.d.

Let us now assume that g^{ij} in ζ^n is positive definite. Then we can, apply the Gramm-Schmidt method to orthonormalization of the d-vector fields of the system $\{\overset{1}{H^i}, ..., \overset{n}{H^i}\}$ by means of g^{ij}.

Theorem 2.2 — If a strongly non-Riemannian ζ^n has a positive definite metric g^{ij}, there exist n at each point orthonormal d-vector fields $\overset{1}{e_i}, \overset{2}{e_i}, ..., \overset{n}{e_i}$, where $\overset{1}{e_i} = p_i/\sqrt{H}$.

And hence

$$g^{ij} \overset{\alpha}{e_i} \overset{\beta}{e_j} = \delta^{\alpha\beta}. \qquad \text{... (2.1)}$$

Remark : If we denote by $\mathcal{R} = \{(x,p); \overset{1}{e_i}, ..., \overset{n}{e_i}\}$, such that for α = 2, 3, ..., n.

$$\overset{\alpha}{e}_i := \frac{1}{\sqrt{E^{\alpha-1}E^{\alpha}}} \begin{vmatrix} \langle 1,1\rangle \ldots \langle 1,\alpha-1\rangle & \overset{1}{H}_i \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \langle \alpha,1\rangle \ldots \langle \alpha,\alpha-1\rangle & \overset{\alpha}{H}_i \end{vmatrix},$$

where

$$E^{\alpha} = \begin{vmatrix} \langle 1,1\rangle \ldots \langle 1,\alpha\rangle \\ \\ \langle \alpha,1\rangle \ldots \langle \alpha,\alpha\rangle \end{vmatrix}, \quad (\alpha = 1, 2, \ldots, n),$$

$$\overset{\alpha}{H}_i := g_{ij}\overset{\alpha}{H^j}, \quad \langle \alpha, \beta \rangle = g^{ij}\overset{\alpha}{H}_i \overset{\beta}{H}_j,$$

then it is easy to verify (2.1). Moreover, the orthonormal frame $\mathcal{R}$ is Berwald type, Moor type and Miron type according as $n = 2$, $n = 3$ or $n \geq 4$.

Note : The n- d-vectors $\overset{\alpha}{e}_i$ are all homogeneous of degree zero in a generalized Hamilton space. Here we point out an important result : a generalized Hamilton space is a Cartan space if and only if

$$S_i^{\ jk} = -\frac{1}{2} g_{il}\,(\dot{\partial}^k g^{lj} - \dot{\partial}^j g^{lk}),$$

holds good. The proof is easy if we remind the (0) p-homogeneity of g^{ij} in p_i and the d-tensor $\tilde{g}^{ij} := \frac{1}{2}\dot{\partial}^i \dot{\partial}^j\,(g^{kl}p_k p_l)$, (identical with the weakly regular metric in Miron[6]) in the generalized Hamilton space is non-degenerate. For Cartan space, $\tilde{g}^{ij} = g^{ij}$, $H(x,p) = g^{ij}p_i p_j$.

3. On Generalization of Strongly Non-Riemannian ζ^n

First we introduce :

Definition 3.1 — A Cartan space ζ^n is said to be parallelizable if it admits n linearly independent at each point d-vector fields.

We now prove :

Proposition 3.1 — If a Cartan space ζ^n admits $(n-1)$ linearly independent at each point d-vector fields, then ζ^n is parallelizable.

PROOF : We have the well-known ε-tensor.

$$\varepsilon_{ij\ldots k} = \sqrt{g}\,\delta^{12\ldots n}_{ij\ldots k}$$

where $g = |\det(g_{ij})|$ and $\delta^{12,\ldots n}_{ij\ldots k}$

is the generalized Kronecker delta.

Let $(n-1)$ linearly independent at each point d-vector fields be $\overset{1}{v}_i, ..., \overset{n-1}{v}_i$. We now define a d-vector field $\overset{n}{v}_i$ such that

$$\overset{n}{v}_i = \varepsilon_{hk...ji}\, \overset{1}{v}_r g^{rh} \overset{2}{v}_s g^{sk} \dots \overset{n-1}{v}_t g^{tj}.$$

So the d-vector $\overset{n}{v}_i$ is orthogonal to $\overset{\alpha}{v}_i$ $(\alpha = 1, ..., n-1)$ and defined at each point on ζ^n. q.e.d.

Important particular cases are mentioned as :

Proposition 3.2 — A 2-dimensional Cartan space is parallelizable. A 3-dimensional Cartan space with $C_i \neq 0$ is parallelizable.

Proposition 3.3 — A strongly non-Riemannian Cartan space is parallelizable.

Next, for a parallelizable Cartan space ζ^n, we can apply the Gramm-Schmidt method to orthonormalization of the system $\{\overset{1}{v}_i, ..., \overset{n}{v}_i\}$ by means of g^{ij}. Then we obtain

Theorem 3.1 — For a parallelizable Cartan space ζ^n, there exist n orthonormal at each point d-vector fields $\overset{1}{z}_i, ..., \overset{n}{z}_i$ where $\overset{1}{z}_i = p_i/\sqrt{H}$.

Now it follows that

$$g^{ij}\, \overset{\alpha}{z}_i \overset{\beta}{z}_i = \delta^{\alpha\beta}. \qquad ... (3.1)$$

We denote the inverse matrix of the matrix $(\overset{\alpha}{z}_i)$ by $(\underset{\beta}{z}_i)$, so

$$\overset{\alpha}{z}_i \underset{\alpha}{z}^j = \delta^j_i, \quad \overset{\alpha}{z}_i \underset{\beta}{z}^j = \delta^{\alpha}_{\beta}. \qquad ... (3.2)$$

Here after, the object in reference to $C\Gamma(N)$ will be recognized by the letter c placed on them.

Let us now consider canonical metrical connection $C\Gamma(\overset{c}{N}) = (\overset{c}{H^i}_{jk}, \overset{c}{C_i}^{jk})$ as a fixed connection. And put

$$X^i{}_{jk} = \underset{\alpha}{z}^i \overset{\alpha}{z}_{c\,j|k}, \quad Y_j{}^{ik} = \underset{\alpha}{z}^i \overset{\alpha}{z}_j \overset{c}{|}{}^k.$$

Lemma — Two d-tensor fields $X^i{}_{jk}$ and $Y^i{}_{jk}$ satisfy

$$g^{ih} X^i{}_{hk} + g^{jh} X^i{}_{hk} = 0,$$

$$g^{ih} Y_h{}^{jk} + g^{jh} Y_h{}^{ik} = 0. \qquad ... (3.3)$$

PROOF : From (3.1) and (3.2), we get

$$g^{ij} \overset{\alpha}{z_i} = \delta^{\alpha\beta} \underset{\beta}{z^j}, \quad g^{ij} = \delta^{\alpha\beta} \overset{i}{z_\alpha} \overset{j}{z_\beta},$$

and hence

$$0 = g^{ij}{}_{|k} = -X^i{}_{hk} g^{hj} - X^j{}_{hk} g^{hi}.$$

Analogously, we have the second equation of (3.3). q.e.d.

Next we have :

Theorem 3.2 — The set of all d-connections $H\Gamma(N) = (H^i{}_{jk}, C_i{}^{jk})$, in a parallelizable Cartan space ζ^n having the properties

(i) $g^{ij}{}_{|k} = 0, \; g^{ij} \big|^k = 0;$

(ii) $\overset{\alpha}{z}_{j|k} = 0, \; \overset{\alpha}{z}_i \big|^k = 0$

is given by

$$N_{ij} = \overset{c}{N}_{ij} + A_{ij},$$

$$H^i{}_{jk} = \overset{c}{H^i}{}_{jk} + A_{rk} (\overset{c}{C}_j{}^{ir} - Y_i{}^{jr}) + X^j{}_{ik},$$

$$C_i{}^{jk} = \overset{c}{C}_i{}^{jk} + Y_i{}^{jk} \qquad \text{... (3.4)}$$

where A_{ij} is an arbitrary d-tensor field of the type (0, 2).

PROOF : The proof is a straight forward after noticing the transformations of d-connections

$$N_{ij} = \overset{c}{N}_{ij} + A_{ij},$$

$$H^i{}_{jk} = \overset{c}{H^i}{}_{jk} + \overset{c}{C}_j{}^{il} A_{lk} + B^i{}_{jk},$$

$$C_i{}^{jk} = \overset{c}{C}_i{}^{jk} + D_i{}^{jk},$$

where A_{ij}, $B^i{}_{jk}$ and $D_i{}^{jk}$ are arbitrary d-tensor fields of the type (0, 2), (1, 2) and (2, 1) respectively, and $\overset{\alpha}{z}_{i|k} = 0, \; \overset{\alpha}{z}_i \big|^k = 0$. Further by means of Lemma it is easy to verify the metrical property (i). q.e.d.

An important particular case is obtained when we take $A_{ij} = 0$, namely

Theorem 3.3 — The connection $H\Gamma(\overset{c}{N}) = (H^i{}_{jk}, C_i{}^{jk})$ in a parallelizable Cartan space ζ^n is given by

$$H^i{}_{jk} = \overset{c}{H^i}{}_{jk} + X^j{}_{ik},$$

$$C_i{}^{jk} = \overset{c}{C}_i{}^{jk} + Y_i{}^{jk}, \quad \dots (3.5)$$

and has the properties (i) and (ii) in Theorem (3.2).

Theorem 3.4 — All curvature tensors of the connection $H\Gamma(N)$ in (3.4) or $H\Gamma(\overset{c}{N})$ in (3.5) vanish.

PROOF : The h- and v- covariant derivatives, with respect to the d-connection $H\Gamma(N)$, verify the Ricci formulas :

$$\overset{\alpha}{z}_{i|k|l} - \overset{\alpha}{z}_{i|l|k}$$

$$= -\overset{\alpha}{z}_r R_i{}^r{}_{kl} - T^r{}_{kl} \overset{\alpha}{z}_{i|r} - R_{rkl} \overset{\alpha}{z}_i \Big|^r,$$

$$\overset{\alpha}{z}_{i|k} \Big|^l - \overset{\alpha}{z}_i \Big|^l{}_{|k}$$

$$= -\overset{\alpha}{z}_r P_i{}^r{}_k{}^l - C_k{}^{rl} \overset{\alpha}{z}_{i|r} - P_{rk}{}^l \overset{\alpha}{z}_i \Big|^r,$$

$$\overset{\alpha}{z}_i \Big|^k \Big|^l - \overset{\alpha}{z}_i \Big|^l \Big|^k = -\overset{\alpha}{z}_r S_i{}^{rkl} - S_r{}^{kl} \overset{\alpha}{z}_i \Big|^r.$$

Thus we have $R_j{}^i{}_{kl} = 0$, $P_j{}^i{}_k{}^l = 0$, $S_j{}^{ikl} = 0$ by virtue of $\overset{\alpha}{z}_{i|k} = 0$ and $\overset{\alpha}{z}_i \Big|^k = 0$.

q.e.d.

For the connection $H\Gamma(\overset{c}{N})$ in (3.5), we have $H_{|k} = 0 = H_{c}\big|_{k}$, and h-deflection tensor $D_{jk} = 0$, because we can always take $\overset{\alpha}{z}_i = \frac{P_i}{\sqrt{H}}$.

Remark 1 : It is a conjecture that a Cartan space $\zeta^n\,(n > 4)$ with (α, β)-metric is not strongly non-Riemannian.

Remark 2 : The Lie derivatives of the connection coefficients in (3.4) or in (3.5) may be found directly by the formulae (1.25), (3.5) and (3.6) in Igarashi[1].

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