ON SEMINORMS, SPECTRAL RADIUS AND PTAK'S SPECTRAL FUNCTION IN BANACH ALGEBRAS

S. J. BHATT AND H. V. DEDANIA*

Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar 388 120 *(Gujarat)*

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A seminorm satisfying the square inequality (for some positive constants k and c, $kp(x)^2 \leq p(x^2) \leq cp(x)^2$ for all x) (respectively C^*-inequality) on any algebra (respectively *-algebra) is weakly submultiplicative (for some M, $p(xy) \leq Mp(x)p(y)$ for all x, y); and is equivalent to a submultiplicative uniform (respectively C^*-) seminorm. This is proved, and used to show that the spectral radius r is the only spectral seminorm satisfying the square (in) equality on a Banach algebra commutative modulo the radical; whereas Ptak's spectral function s ($s(x) = r(x^* x)^{1/2}$) is the only spectral seminorm satisfying the C^*-(in) equality on a hermitian Banach *-algebra.

§1. By an algebra A is meant a linear associative algebra, not necessarily unital. A seminorm on A is a function $p : A \to [0, \infty)$ such that $p(x + y) \leq p(x) + p(y)$ and $p(\lambda x) = |\lambda| p(x)$ for all $x, y \in A$ and for all scalars λ. p satisfies the square property (resp. square inequality) if for all $x \in A$, $p(x^2) = p(x)^2$ (resp. if there exist $k > 0$, $c > 0$ such that $kp(x)^2 \leq p(x^2) \leq cp(x)^2$). p satisfies the power inequality if there exists $k > 0$, $c > 0$ such that $kp(x)^n \leq p(x^n) \leq cp(x)^n$ for all x. It is submultiplicative (resp. weakly submultiplicative) if $p(xy) \leq p(x)p(y)$ (resp. $p(xy) \leq Mp(x)p(y)$ for some $M > 0$) for all x, y in A. p is spectral if the set A^{qr} of all quasiregular elements in A is open in (A, p). A seminorm p on a *-algebra A satisfies the C^*-property (resp. C^*-inequality) if $p(x^*x) = p(x)^2$ [resp. $kp(x)^2 \leq p(x^* x) \leq cp(x)^2$ for some $k > 0$, $c > 0$] for all x.

Theorem — Let p be a spectral (linear) seminorm on a Banach algebra A. Let $N_p = \{x \in A \mid p(x) = 0\}$.

(a) If p satisfies the square inequality, then p is equivalent to the spectral radius r, N_p = rad A (radical of A) and A/N_p is commutative. If p satisfies the square property, then $p = r$.

(b) Let A be a *-algebra. If p satisfies the C^*-inequality, then p is equivalent to Ptak's spectral function $s(x) = r(x^* x)^{1/2}$, N_p = srad A (star radical of A) and A is hermitian. If p satisfies the C^*-property, then $p = s$.

**Current address* : Department of Mathematics, University of Leeds, LS2 9JT, Leeds, U.K.

Lemma — Let p be a seminorm on an algebra A.

(a) Let there exists $c > 0$ be such that $p(x^2) \leq cp(x)^2 \, (x \in A)$. Then $p(x^n) \leq 3^{n-2} c^{n-1} p(x)^n$ $(x \in A,\ n \geq 2)$. If A is commutative, then $p(xy) \leq 3cp(x)\, p(y)$ $(x, y \in A)$.

(b) If p satisfies either square property or the power inequality, then $p(x^n) = p(x)^n$ for all $x \in A$ and for all $n \in \mathbb{N}$.

(c) Let p satisfy the square inequality. Then p is weakly submultiplicative, p is equivalent to a submultiplicative seminorm with the square property and A/N_p is commutative and semisimple.

(d) Let A be a *-algebra. Let p satisfy the C^*-inequality. Then p is weakly submultiplicative, the involution is p-continuous and p is equivalent to a submultiplicative seminorm with C^*-property.

By Sebestyen[9], a seminorm with the C^*-property on any involutive algebra is submultiplicative. Also, by Dedania[6], a seminorm with the square property on any algebra is submultiplicative. Parts (d) and (c) of above lemma contain inequality analogues of these results. Our proof of (c) is based on essentially the same arguments as in Dedania[6]. Further, it is shown in Theorem 3 of Bhatt *et al.*[4] that if $| \ |$ is a norm on a C^*-algebra $(A, \| \ \|)$ such that the involution is $| \ |$ - continuous and $| \ |$ satisfies the C^*-inequality, then $| \ |$ is equivalent to $\| \ \|$. It was asked therein whether $| \ |$ continuity of the involution can be omitted or not. Incidently, part (d) of above lemma affirmatively answers this. This theorem gives extrinsic characterizations of spectral radius r and Ptak's spectral function s. In a Banach algebra A, r has the square property. It is a seminorm if(f) $A/\text{rad } A$ is commutative (Aupetit[1], Theorem 2, p. 48), in which case, r is a spectral seminorm (Bonsall and Duncan[5], Theorem 2.9, p. 12; Palmer[8], Theorem 3.1). On the other hand, if A is a Banach *-algebra, then s has the C^*-property $s(a)^2 = s(a^* a)$ $(a \in A)$. It is a seminorm if(f) A is hermitian (Bonsall and Duncan[5], §41), in which case, s [= m, the Gelfand Naimark seminorm (Bonsall and Duncan[5], § 36)] is a spectral seminorm (Palmer[8], p. 295). The theorem shows that these properties uniquely determine r and s. As a whole, this theorem supports a Meta Theorem (suggested by the similarity between the square property and the C^*-property of the respective norms) envisaged in Bhatt[3] that there is a structural analogy between (certain aspects of) uniform Banach algebras and C^*-algebras; at a more general level, between Banach algebras commutative modulo the radical and hermitian Banach *-algebras.

§2. *Proof of Lemma* — (a) Let A be commutative. Then the assumption $p(x^2) \leq cp(x)^2$ applied to the identity $4xy = (x+y)^2 - (x-y)^2$ implies that $4p(xy) \leq 2c(p(x) + p(y))^2$ for all x, y. Thus $p(xy) \leq 2c$ if $p(x) \leq 1$, $p(y) \leq 1$. Hence $p(xy) \leq 2cp(x)\, p(y)$ for all x, y. The rest is easily proved by passing to the commutative subalgebra of A generated by x and using induction.

(b) If p satisfies the power inequality, then for all $x \in A$, $p(x) \leq \liminf p(x^n)^{1/n} \leq \limsup p(x^n)^{1/n} \leq p(x)$. Thus $p(x) = \lim\limits_{n \to \infty} p(x^n)^{1/n}$ exists, which gives the square property. Now assume p has the square property. We can assume A to be commutative by considering the subalgebra generated by x. By Dedania[6], p is submultiplicative. Now p coincides with the spectral radius in an appropriate uniform

Banach algebra. This gives the desired conclusion.

(c) The weak submultiplicativity follows from the following steps :

Step (1) : $p(xy+yx) \le 6cp(x)\,p(y) \quad (x, y \in A)$

Using the identity $(x+y)^2 = x^2 + y^2 + xy + yx$,

$$p(xy+yx) \le c[p(x+y)^2 + p(x)^2 + p(y)^2]$$
$$\le 2c[p(x)^2 + p(y)^2 + p(x)\,p(y)].$$

If $p(x) \le 1$, $p(y) \le 1$, then $p(xy+yx) \le 6c$. Hence (1) follows.

Step (2) : $p(xyx) \le 42c^2 p(x)^2 p(y) \quad (x, y \in A)$

The identity $(x+y)^3 = x^3 + y^3 + xyx + yxy + x^2y + yx^2 + xy^2 + y^2x$ gives $xyx + yxy = (x+y)^3 - x^3 - y^3 - (x^2y + yx^2) - (xy^2 + y^2x)$.

Applying part (a) and Step (1),

$$p(xyx + yxy) \le 3c^2 p\,(x+y)^3 + 3c^2\, p(x)^3 + 3c^2\, p(y)^3$$
$$+ p(x^2y + yx^2) + p(y^2x + xy^2)$$
$$\le 3c^2\,[p(x+y)^3 + p(x)^3 + p(y)^3] + 6cp(x^2)\,p(y) + 6cp(x)\,p(y^2)$$
$$\le 3c^2\,[(p(x) + p(y))^3 + p(x)^3 + p(y)^3]$$
$$+ 6c^2\,[p(x)^2\,p(y) + p(x)\,p(y)^2].$$

Let $p(x) \le 1$, $p(y) \le 1$. Then $p(xyx + yxy) \le 42c^2$; and replacing x by $-x, p(xyx - yxy) \le 42c^2$. Hence $2p(xyx) = p(2(xyx)) \le p(xyx + yxy) + p(xyx - yxy)$ implies $p(xyx) \le 42c^2$. Hence $p(xyx) \le 42c^2 p(x)^2 p(y)$ for all x, y in A.

Step (3) : $p(xy - yx) \le mp(x)\,p(y) \ (x, y \in A)$ for some $m > 0$

By (1) and (2), the identity

$$(xy - yx)^2 + (xy + yx)^2 = 2[(xyx)y + y(xyx)]$$

gives

$$kp(xy - yx)^2 \le 2p[(xyx)y + y(xyx)] + p((xy + yx)^2)$$
$$\le 12cp\,(xyx)\,p(y) + cp(xy + yx)^2$$
$$\le 504c^3\, p(x)^2\, p(y)^2 + 36c^3\, p(x)^2\, p(y)^2.$$

Hence $p(xy - yx) \le mp(x)\,p(y)$ for some $m > 0$ and for all x, y.

Step (4) : p is weakly submultiplicative

Taking $m_1 = \max\,\{m, 6c\}$ and using the inequality

$$2p(xy) = p(2(xy)) \le p(xy + yx) + p(xy - yx),$$

we get $p(xy) \le m_1\, p(x)\, p(y)$ for all x, y.

Now without loss of generality, we may assume p to be a norm and $p(xy) \le mp(x)\,p(y) \quad (x, y \in A)$ for some $m > 1$. Then $q(x) = mp(x)$ defines a submultiplicative norm with square inequality $\frac{k}{m}\, q(x)^2 \le q(x^2) \le \frac{c}{m}\, q(x)^2 \ (x \in A)$. By the remark following the proof of Corollary 8 of Bonsall and Duncan[5] (p. 77), A is commutative. Define $q_1(x) = \lim_{n \to \infty} q(x^n)^{1/n}$. Then, by Corollary 3 of Bonsall and Duncan[5] (p. 19), q_1 is a submultiplicative seminorm with square property. By iteration, for any $n \in \mathbb{N}$ and any $x \in A$,

$$\left(\frac{k}{m}\right)^{2^n - 1} q(x)^{2^n} \leq q(x^{2^n}) \leq \left(\frac{c}{m}\right)^{2^n - 1} q(x)^{2^n};$$

hence $\frac{k}{m} q(x) \leq q_1(x) \leq \frac{c}{m} q(x)$. This shows that q_1 is a norm equivalent to q, hence to p.

(d) Let $k > 0$, $c > 0$ be such that $kp(x)^2 \leq p(x^* x) \leq cp(x)^2$ for all x in A. Let x, $y \in A$; let g, d be any two complex numbers. The well known identity

$$4gdxy = \sum_{n=0}^{3} i^n (\bar{d} y^* + (-i)^n gx)(dy + i^n \bar{g} x^*)$$

$$= \Sigma\, i^n (dy + i^n \bar{g} x^*)^* (dy + i^n \bar{g} x^*)$$

implies that $4|g||d|p(xy) \leq c\, \Sigma\, [|d|p(y) + |g|p(x^*)]^2$. For $\varepsilon > 0$, choosing $g = 1/(p(x^*) + \varepsilon)$, $d = 1/(p(y) + \varepsilon)$, one gets $p(xy) \leq 4cp(x^*)p(y)$ for all x, y. This implies that

$$kp(xy)^2 \leq p((xy)^* xy) = p(y^* x^* xy) \leq 4cp(y)\, p(x^* xy) \leq 16c^2 p(y)\, p(x)\, p(xy).$$

Hence $p(xy) \leq mp(x)\, p(y)$ for all x, y in A and $m > 0$. Hence for any x, $kp(x^*)^2 \leq p(xx^*) \leq mp(x)\, p(x^*)$ implies that involution is p-continuous.

Next we show that p is equivalent to a C^*-seminorm. Without loss of generality we may assume $m > 1$ and p to be a norm. On A, let $q_1(x) = mp(x)$, $q(x) = \max \{q_1(x), q_1(x^*)\}$. Then q is a submultiplicative norm with C^*-inequality, $k_1 q(x)^2 \leq q(x^* x) \leq c_1 q(x)^2$ $(x \in A)$, $k_1 = k/m$ and $c_1 = c/m$. Let B be the Banach $*$-algebra obtained by completing (A, q). Then $k_1 q(z)^2 \leq q(z^* z) \leq c_1 q(z)^2$ for all z in B. We can assume that B is unital, otherwise, on the unitification B_e of B, the operator norm $|z + \lambda 1| = \sup \{q(zu + \lambda u) : q(u) \leq 1,\ u \in B\}$ extends q and its satisfies C^*-inequality. Now, for any $h = h^*$ in B, $k_1 q(e^{ih})^2 \leq q(e^{-ih} e^{ih}) = q(1) = w$ (say). By Corollary 2 of Aupetit[1] (p. 123), B is locally C^*-equivalent; hence by a theorem of Cuntz (Aupetit[1], Theorem 4, p. 126), there exists a norm $\|\cdot\|$ on B, equivalent to q, such that $(B, \|\cdot\|)$ is a C^*-algebra. Hence $\|\cdot\|$ is equivalent to p on A.

Remark : In part (d) of the lemma, A/N_p need not be hermitian. Take A to be the disc algebra with involution $f^*(z) = \overline{f(\bar{z})}$ and the C^*-norm $p(f) = \sup \{|f(z)| : z \text{ is real}, -1 \leq z \leq 1\}$.

Proof of the Theorem — By Lemma, p is weakly submultiplicative and is equivalent to a submultiplicative seminorm q with the square property. Since p is spectral, q is spectral. By Theorem 3.1 of Palmer[8] and by the above Lemma, $q(x) = \lim q(x^n)^{1/n} = r(x)$ $(x \in A)$. Thus p is equivalent to r. Now Theorem 2 of Aupetit[1] (p. 48) implies that $A/\text{rad } A$ is commutative. Further, by Theorem 2 of Aupetit[1] (p. 23), $N_p = N_r$ = rad A. If p has square property, then p is submultiplicative, hence $p = q = r$.

(b). (i) We can assume that A is unital. Let p have C^*-property. By Sebestyen[9], p is submultiplicative. Hence $p \leq m$, which is the greatest C^*-seminorm (Bonsall and Duncan[5], § 36). Since p is spectral, m is spectral. Hence for $x \in A$, $m(x)^2 = m(x^*x) = \lim m((x^*x)^n)^{1/n} = r(x^*x) = \lim p((x^*x)^n)^{1/n} = p(x^*x) = p(x)^2$, as well as, $s(x)^2 = s(x^*x) = r(x^*x) = p(x)^2$. Thus A is hermitian (Bonsall and Duncan[5], Theorem 11, p. 227) and $N_p = N_m = s \text{ rad} A$.

(ii) Assume that p has C^*-inequality. By part (c) of the Lemma, p is equivalent to a C^*-seminorm p_1. Hence by above (i), p is equivalent to $p_1 = s$, A is hermitian and $N_p = N_{p_1}$ = srad A (Bonsall and Duncan[5], Theorem 4, p. 223).

§3. *Remarks* : (1) A Frechet algebra is a complete metrizable locally m-convex algebra. An important unsolved problem about Frechet algebras is the Michael problem (see Michael[7], p. 53). Is every character (i.e. multiplicative linear functional) on a commutative Frechet algebra continuous ? The above Lemma and the closed graph theorem immediately imply that given a commutative Frechet algebra A, every character on A is continuous iff every seminorm with square inequality on A is continuous.

(2) For a seminorm on a Banach algebra, the square inequality is not sufficient to imply submultiplicativity; e.g. on a Banach algebra $(A, \|\cdot\|)$ with $\|x^2\| = \|x\|^2$ for all x, take $|x| = \tau\|x\|$, $0 < \tau < 1$. Also, in the presence of submultiplicativity, the square inequality is not sufficient to give the square property. On the supnorm Banach algebra $(C(X), \|\cdot\|_\infty)$ of continuous functions on a compact Hausdorff space X, this is exhibited by the well known norm

$$|f| = \sup\left\{\frac{1}{2}(|f(x)+f(y)| + |f(x)-f(y)|) : x, y \in X\right\}.$$

This also shows that the square inequality does not imply the power inequality

$$kp(x)^n \leq p(x^n) \leq cp(x)^n.$$

(3) Let there be $c > 0$ such that $p(x^2) \leq cp(x)^2$ for all x. Is p weakly submultiplicative, at least in Banach algebras ? Related questions are : Let A be an algebra which is a Banach space in which square is continuous. Is multiplication continuous ? Is any Banach space norm on $C(X)$ equivalent to the supnorm ? The numerical radius on a Banach algebra is known to be weakly submultiplicative; not necessarily submultiplicative; and on a uniform Banach algebra $(A, \|\cdot\|)$, $r = \|\cdot\|$ = the numerical radius. Characterize those Banach algebras in which numerical radius is submultiplicative.

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