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ON GENERALIZATIONS OF *H*-CLOSED SPACES

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In this paper, we obtain some generalizations of *H*-closed spaces using preopen sets and their properties. Besides these, we have extended the results of Arya and Bhamini[2] to the *p*-regular and pre-Urysohn spaces.

1. INTRODUCTION

The notion of preopen sets was first introduced by Mashour *et al.*[7]. Subsequently much has been done in the direction of generalizations of separation axioms, covering axioms and mappings using preopen sets. The concept of *H*-closedness was first introduced by Alexandroff and Urysohn[1] and Bourbaki[3] characterized minimal Hausdorff and *H*-closed spaces. With particular reference to *H*-closed spaces. Liu[6] and Porter and Thomas[8] independently proved that in the category of *H*-closed spaces and continuous maps, the projective objects are finite spaces and injective objects are singletons.

Raghavan and Reilly[9], as a continuation of the study of properties of *HP*-closed [Hausdorff *P*-spaces] spaces initiated by Cameron[4] have shown that the class of *HP*-closed spaces has a projective maximum and a projective minimum. Further they have shown that in the category *HP*-closed spaces and continuous maps, the projective objects are discrete spaces.

Thompson[10] generalized *H*-closed spaces using semi-open sets in the name of *S*-closed spaces. Arya and Bhamini[2] have introduced *S*-Urysohn closed and *s*-regular closed spaces using semi-separation axioms and filters using semiopen sets and have given characterizations of such spaces.

With the above works in the background the present paper is an attempt to generalize *H*-closed spaces by using preopen sets. In doing so, we adopt the method employed by Raghavan and Reilly[9]. Analogues of *S*-Urysohn closed and *s*-regular closed spaces using preopen sets are studied along the lines of Arya and Bhamini[2].

Throughout this paper $(X, \mathcal{J})$ represents a topological space without any separation axiom assumed on it. By nbd we mean neighbourhood and cl A represents the closure of a set $A \subset X$ and int A, denotes the interior of a set $A \subset X$.

In section 2, we furnish necessary preliminaries related to preopen sets. Section 3 deals with pre T_2-closed spaces and locally pre T_2-closed spaces and their one point extensions. Sections 4 and 5 are devoted to pre-Urysohn-closed and p-regular closed spaces respectively. As an application of pre T_2-closed spaces, it is shown that in the category of pre T_2-closed spaces and continuous maps, the projective objects are finite spaces.

2. Preliminaries

In this section, some basic definitions and results regarding preopen sets and related concepts are presented.

A subset $A \subset X$ is called preopen if $A \subset$ int cl A and the complement of a preopen set is called preclosed. The family of all preopen sets is denoted by $PO(X)$ and that of preclosed sets by $PF(X)$. For a subfamily $\{B_i \mid i \in I\} \subset PO(X)$, $U B_i \in PO(X)$. For $A \subset Y \subset X$ and $A \in PO(X)$, $A \in PO(Y)$ whenever Y is open in X. For $A \subset Y \subset X$ and $A \in PF(Y)$, $A \in PF(X)$ whenever $Y \in PF(X)$.

If $U \in PO(X)$, $V \in PO(U)$ then $V \in PO(X)$. For any subset $A \subset X$, the conditions that A is preopen, A is the intersection of a regular open set and a dense set and A is the intersection of an open set and a dense set are equivalent. The preinterior of a subset $A \subset X$ is the union of all preopen sets which are contained in A and it is denoted by pint A or A_*. The intersection of all preclosed sets containing a set A is called the preclosure of A and it is denoted by pcl A or A^*. The preinterior of a set A is preopen and its preclosure is preclosed. For $A \subset X_0 \subset X$ and X_0 an open set in X $p\ \mathrm{cl}_{X_0}(A) = p\ \mathrm{cl}\ A \cap X_0$ where $p\ \mathrm{cl}_{X_0} A$ is the preclosure of A in X_0. For a product space $X = X_1 \times X_2$, $A \subset X$ is in $PO(X)$ if and only if $A = A_1 \times A_2$ where $A_1 \in PO(X_1)$ and $A_2 \in PO(X_2)$.

A mapping $f : X \to Y$ is precontinuous if $f^{-1}(V) \in PO(X)$ for every open set V of Y and f is preopen whenever $f(U) \in PO(Y)$ for every open set V of X.

A space X is called pre T_2 if for every pair of distinct points x and y of X, there exist disjoint preopen sets U and V containing x and y respectively. A p-regular space is one in which a point $x \in X$ and a closed set F not containing x are separated by disjoint preopen sets. A space X is called pre Urysohn if for any two distinct points x and y, there exist preopen sets U and V containing x and y respectively such that $p\ \mathrm{cl}\ U \cap p\ \mathrm{cl}\ V = \phi$. An almost p-regular space X is defined as one in which a point $x \in X$ and a regular closed set F not containing x are separated by preopen sets.

We quote the following theorems, without proof, as they are very useful in studying p-regular closed and pre Urysohn-closed spaces.

Theorem A — For a space X the following are equivalent :

(1) X is p-regular

(2) For each $x \in X$ and each open set U of X containing x, there exists $V \in PO(X)$ such that $x \in V \subset p$ cl $V \subset U$.

(3) For each closed set F of X, $\bigcap \{p \text{ cl } V \mid F \subset V \in PO(X)\} = F$.

(4) For each subset A of X and each open set U of X such that $A \bigcap U \neq \phi$, there exists $V \in PO(X)$ such that $A \bigcap V \neq \phi$ and p cl $V \subset U$.

(5) For each nonempty subset A of X and each closed set F of X such that $A \bigcap V \neq \phi$, there exist $V, W \in PO(X)$ such that $A \bigcap V \neq \phi$, $F \subset W$ and $V \bigcap W = \phi$.

Theorem B — Every almost p-regular Hausdorff space is a pre Urysohn space.

Theorem C — For a space X the following are equivalent.

(a) X is almost p-regular

(b) For each point $x \in X$ and a regular open set G containing x, there exists a preopen set U such that $x \in U \subset p$ cl $U \subset G$.

(c) Every regular closed set F is the intersection of all preclosed preneighbourhoods of F.

(d) For every set A and a regular open set B such that $A \bigcap B \neq \phi$, there exists a preopen set U such that

$$A \bigcap U \neq \phi \text{ and } p \text{ cl } U \subset B.$$

(e) For every nonempty set A and a regular closed set B such that $A \bigcap B = \phi$, there exists disjoint preopen sets G and H such that $A \bigcap G \neq \phi$ and $B \subset H$.

3. One Point Extensions

In this section, analogous to H-closed spaces, we define pre T_2-closed spaces and by developing the necessary preliminaries, we prove the existence of a projective maximum and a projective minimum in the class of one point extensions for a locally pre T_2-closed extremally disconnected Hausdorff space.

Definition 1 — Let $(X, \mathcal{J})$ be a topological space. Let $A \subseteq X$. Then (i) A is said to be pre T_2-closed relative to X if and only if every preopen cover $\mathcal{U}$ of A has a finite subfamily $\mathcal{U}' \subset \mathcal{U}$ such that $A \subset \bigcup \{\text{cl } U \mid U \in \mathcal{U}'\}$.

(ii) A is said to be a pre T_2-closed set if and only if $(A, \mathcal{J}/A)$ is pre T_2-closed.

Example 1 — Let $(X, \mathcal{J})$ be a topological space with the indiscrete topology. Every subset of X is preopen and dense in X. So for every preopen cover $\mathcal{U}$ and $U \in \mathcal{U}$, cl $U = X$ and $(X, \mathcal{J})$ is pre T_2-closed.

Example 2 — Consider $(X, \mathcal{J})$ with the cofinite topology. Every infinite set is preopen and dense in X. Hence $(X, \mathcal{J})$ is pre T_2-closed.

Definition 2 — A space $(X, \mathcal{J})$ is said to be locally pre T_2-closed if for each $x \in X$ and an open set U containing x, the closure of U is pre T_2-closed.

Theorem 1 — Let $(X, \mathcal{J})$ be a topological space. Then

(i) if $A \subset X$ is pre T_2-closed relative to X, then A is closed in X, if X is Hausdorff.

(ii) if $A \subset X$ is an open set, A is pre T_2-closed relative to X if and only if A is pre T_2-closed set.

PROOF OF (i) : Let $x \in X - A$. Since X is a Hausdorff space, there exist for each $y \in A$, open neighbourhoods U_y and V_y of x and y respectively such that $U_y \cap V_y = \phi$. Then $\{V_y \mid y \in A\}$ is an open cover hence a preopen cover of A. Since A is pre T_2-closed, there exists a finite subset $B \subset A$ such that $A \subset \bigcup \{\text{cl } V_y \mid y \in B\}$. Let $U = \bigcap \{U_y \mid y \in B\}$. Then U is an open neighbourhood of x such that $A \cap U = \phi$. Thus cl $A = A$ and hence A is closed.

PROOF OF (ii) : Assume A to be a pre T_2-closed set. Then $(A, \mathcal{J}/A)$ is pre T_2-closed. Let $\{U_y \mid y \in A\}$ be a preopen cover of A with $U_y \in PO(X)$ for every $y \in A$. Let $V_y = A \cap U_y$. Since A is open, $V_y \in PO(A)$. So $\{V_y \mid y \in A\}$ is a preopen cover of A in A. As $(A, \mathcal{J}/A)$ is pre T_2-closed, there exists a finite subset $B \subset A$ such that $A \subset \bigcup \{\text{cl}_A V_y \mid y \in B \subset A\}$.

Now $\text{cl}_A V_y = \text{cl } V_y \cap A \subset \text{cl } U_y$. So $A \subset \bigcup \{\text{cl } U_y \mid y \in B \subset A\}$. Thus A is pre T_2-closed relative to X. As A is open, every preopen subset of A is preopen in X and hence the converse part of (ii) is obvious.

Theorem 2 — Let $(X, \mathcal{J})$ be a topological space. Then the following are equivalent.

(i) X is pre T_2-closed.

(ii) The closure of an open subset of X is pre T_2-closed relative to X.

(iii) Every closed-open subset of X is pre T_2-closed set.

PROOF OF (i) $\Rightarrow$ (ii) : Let U be an open subset of X. Let A = cl U. Let $\mathcal{U}$ be a preopen cover of A. Then $\mathcal{D} = \mathcal{U} \cup (X - A)$ is a preopen cover of X so that there exists a finite number of members V_i, $i = 1, 2, \ldots, n$ in $\mathcal{D}$ such that $X = \bigcup \{cl\ V_i \mid i = 1, 2, \ldots, n, V_i \in \mathcal{D}\}$. If no V_i is $(X - A)$ then $A \subset \bigcup \{\text{cl } V_i \mid i = 1, 2, \ldots, n, V_i \in \mathcal{U}\}$. If $V_k = X - A$, since $U \subset \text{int cl } U = \text{int } A = X - \text{cl } (X - A)$ $= (X - \text{cl } V_k) \subset \bigcup \{\text{cl } V_i \mid i = 1, 2, \ldots, k-1, k+1, \ldots, n, V_i \in \mathcal{U}\}$.

$A = \text{cl } U \subset \bigcup \{\text{cl } V_i \mid i = 1, 2, ..., k-1, k+1, ..., n, V_i \in \mathcal{U}\}$ as the right-hand side is closed. Thus $A = \text{cl } U$ is pre T_2-closed relative to X.

PROOF OF (ii) $\Rightarrow$ (iii) : Let A be open and closed. Then $A = \text{cl } A$. So A is pre T_2-closed relative to X. By Theorem (1), A is pre T_2-closed set.

PROOF (iii) $\Rightarrow$ (i) : Immediate.

Remark 1 : If X is extremally disconnected in the above theorem, then the closure of an open set is pre T_2-closed set whenever it is pre T_2-closed relative to X.

Theorem 3 — Let $(X, \mathcal{J})$ be a topological space and $A \subset X$. Then A is pre T_2-closed if and only if every regular open cover has a finite subcover whose closures cover A.

PROOF : Let A be pre T_2-closed. Let $\{V_\alpha / \alpha \in I\}$ be a regular open cover of A with each $V_\alpha \in RO(X)$ where $RO(X)$ represents the set of all regular open sets of X. Then $\{V_\alpha / \alpha \in I\}$ is an open, hence pre open cover of A. As A is pre T_2-closed, $A \subset \bigcup \{\text{cl } V_i \mid i = 1, 2, ..., n\}$. Conversely, let $\{U_\alpha / \alpha \in I\}$ be a preopen cover of A where $U_\alpha \in PO(X)$. Since $U_\alpha \in PO(X)$, $U_\alpha \subset \text{int cl } U_\alpha$ for every α. Now $W_\alpha = \text{int cl } U_\alpha / \alpha \in I\}$ is a regular open cover of A with $W_\alpha \in RO(X)$ and hence for a finite subcollection say $W_{\alpha_1} W_{\alpha_2}, ..., W_{\alpha_n}, A \subset \bigcup \{\text{cl } W_{\alpha_i} \mid i = 1, 2, ..., n\} = \bigcup \{\text{cl (int cl } U_{\alpha_i}) \mid i = 1, 2, ..., n\} = \bigcup \{\text{cl } U_{\alpha_i} \mid i = 1, 2, ..., n\}$.

Thus A is pre T_2-closed.

Corollary 1 — A space $(X, \mathcal{J})$ is pre T_2-closed if and only if every regular open cover has a finite subcover the closures of whose members cover X.

Theorem 4 — If B is open preclosed subset of A which is open pre T_2-closed relative to X, then B is pre T_2-closed relative to X.

PROOF : Let $\mathcal{U}$ $\{U_y \mid y \in B\}$ be a preopen cover of B relative to X. Then $\mathcal{A} = \mathcal{U} \bigcup W$ where $W \bigcap A = (A - B)$, is a preopen cover of A [since A is open and $(A - B)$ is preopen in A, $(A - B) \in PO(X)$; as W is preopen $W \bigcap A \in PO(X)$] for a preopen set $W \in PO(X)$. As B is open in A, $(A - B)$ is closed in A. Thus $(A - B)$ is closed and preopen in A. Since A is pre T_2-closed, $A \subset \bigcup \{\text{cl } U_y \mid U_y \in \mathcal{A}' \subset \mathcal{A}$, $\mathcal{A}'$ is a finite subfamily of $\mathcal{A}\}$. If no U_y is W, $B \subset \bigcup \{\text{cl } U_y \mid U_y \in \mathcal{A}' \subset \mathcal{U}\}$. If some $U_y = W$, then $B \bigcap \text{cl } W = \phi$. For if $x \in B \bigcap \text{cl } W$, B being open $B \bigcap W \neq \phi$. But $(A - B) = A \bigcap W$, a contradiction. So $B \bigcap \text{cl } W = \phi$, $(A - B) \subset \text{cl } W$ and $B \subset \bigcup \{\text{cl } U_y \mid U_y \in \mathcal{A}' - W\}$. Thus B is pre T_2-closed relative to X.

Theorem 5 — Let $B \subset A \subset X$ and A be open in X. Then B is pre T_2-closed relative to X if and only if B is pre T_2-closed relative to A.

PROOF : Let $C = \{U_y \mid y \in B\}$ be a preopen cover of B such that $y \in U_y$. Since

A is open $\{U_y \bigcap A \mid U_y \in C\}$ is a preopen cover of B in the relative topology of A. Since B is pre T_2-closed relative to A, $B \subset \bigcup \{\mathrm{cl}_A (U_y \bigcap A) \mid y \in C, C \subset B$ is a finite subset of $B\}$. $\mathrm{cl}_A (U_y \bigcap A) = \mathrm{cl} \ (U_y \bigcap A) \bigcap A \subset \mathrm{cl}\, U_y$. So $B \subset \bigcup \{\mathrm{cl}\ U_y \mid y \in C\}$ and B is pre T_2-closed relative to X. Conversely, let $\{V_y \mid y \in B\}$ be a preopen cover of B with $V_y \in PO(A)$ for every y. As A is open in X, each $V_y \in PO(X)$. Thus $\{V_y \mid y \in B\}$ is a preopen cover of B relative to X. Since B is pre T_2-closed relative to X, $B \subset \bigcup \{\mathrm{cl} \ V_y \mid y \in D \subset B;\ D$ is finite subset of $B\}$. Now $B = B \bigcap A \subset \bigcup \{\mathrm{cl}\ V_y \mid y \in D\} \bigcap A = \bigcup \{\mathrm{cl}\ U_y \bigcap A \mid y \in D\} = \bigcup \mathrm{cl}_A V_y \mid y \in D\}$.
Hence B is pre T_2-closed relative to A.

Theorem 6 — Let $\{A_i; A_i \subset X;\ i = 1, 2, ..., n\}$ be pre T_2-closed relative to X. Then $\bigcup \{A_i;\ i = 1, 2, ..., n\}$ is pre T_2-closed relative to X.

PROOF : Let $\{U_{ij} : j \in I\}$ be a preopen cover of A_i for $i = 1, 2, ..., n$. Since each A_i is pre T_2-closed, $A_i \subset \bigcup \{\mathrm{cl}\ U_{ij} \mid j = 1, 2, ..., n_i\}$ which shows that

$$A = \bigcup_{i=1}^{n} A_i \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} \mathrm{cl}\, U_{ij}.$$

Thus there exists a finite subfamily $\{U_\alpha;\ \alpha = 1, 2, ..., k\}$ such that $A \subset \bigcup \{\mathrm{cl}\ U_\alpha \mid \alpha = 1, 2, ..., k\}$ for a preopen cover $\{U_\alpha, \alpha = 1, 2, ...\}$ of A.

Theorem 7 — Let X be pre T_2-closed space and Y a compact space. Then the product $X \times Y$ is pre T_2-closed.

PROOF : Let $\mathcal{U}$ be a regular open cover of $X \times Y$. For each $x \in X$, $\{x\} \times Y \cong Y$ and so is compact. As $\mathcal{U}$ is an open cover of $\{x\} \times Y$, a finite number of members, say, U_{ij}^x, $j = 1, 2, ..., n$ of $\mathcal{U}$ cover $\{x\} \times Y$. In fact we can find an open set V_x containing x such that

$$\{x\} \times Y \subset V_x \times Y \subset \bigcup \{U_{ij}^x \mid j = 1, 2, ..., n\} \quad \text{[by the tube lemma]}.$$

Thus $\mathrm{cl}\ (V_x \times Y) = \mathrm{cl}\ V_x \times Y \subset \bigcup \{\mathrm{cl}\ U_{ij}^x \mid j = 1, 2, .., n\}$. This shows that $\mathrm{int}\ (\mathrm{cl}\ V_x \times Y) = \mathrm{int}\ \mathrm{cl}\ V_x \times Y \subset \bigcup \{\mathrm{cl}\ U_{ij}^x \mid j = 1, 2, .., n\}$. Consider $\{W_x = \mathrm{int}\,\mathrm{cl}\, V_x \mid x \in X\}$. This is a regular open cover for X. As X is pre T_2-closed, there exists a finite number of members, say $W_{x_1}, W_{x_2}, ..., W_{x_m}$ of $\{W_x = \mathrm{int}\,\mathrm{cl}\, V_x \mid x \in X\}$ such that $X \subset \bigcup \{\mathrm{cl}\ W_{x_i} \mid i = 1, 2, ..., m\}$. Thus $X \times Y \subset \bigcup \{\mathrm{cl}\ W_{x_i} \times Y \mid i = 1, 2, ..., m\} \subset \bigcup (U\ \mathrm{cl}\ U_{ij}^{x_i} \mid i = 1, 2, ..., n;\ j = 1, 2, ..., m)$. Therefore there exists a finite number of members of $\mathcal{U}$ such that $X \times Y \subset \bigcup \{\mathrm{cl}\ V_k \mid k = 1, 2, ..., m + n,\ V_n \in \mathcal{U}\}$ and $X \times Y$ is pre T_2-closed.

Theorem 8 — Let $f : X \to Y$ be a continuous surjection. If X is pre T_2-closed, so is Y.

PROOF : Let $\{U_\alpha \mid \alpha \in I\}$ be a preopen cover of Y. Since U_α is a preopen set in Y, $U_\alpha \subset \operatorname{int} \operatorname{cl} U_\alpha$ for every $\alpha \in I$. Since f is continuous $\{f^{-1} (\operatorname{int} \operatorname{cl} U_\alpha) \mid \alpha \in I\}$ is an open, hence preopen cover of X. As X is pre T_2-closed, there exists a finite subset $I_0 \subset I$ such that

$$X \subset \bigcup \{\operatorname{cl} (f^{-1} \operatorname{int} \operatorname{cl} U_\alpha) \mid \alpha \in I_0\}.$$

Thus

$$Y = f(X) \subset f(\bigcup \{\operatorname{cl} f^{-1} (\operatorname{int} \operatorname{cl} U_\alpha) \mid \alpha \in I_0\})$$

$$= \bigcup \{f(\operatorname{cl} f^{-1} \operatorname{int} \operatorname{cl} U_\alpha) \mid \alpha \in I_0\}$$

$$\subset \bigcup \{\operatorname{cl} (f(f^{-1} (\operatorname{int} \operatorname{cl} U_\alpha))) \mid \alpha \in I_0 \subset \bigcup \{\operatorname{cl} (\operatorname{int} \operatorname{cl} U_\alpha) \alpha \in I_0\}$$

$$= \bigcup \{\operatorname{cl} U_\alpha \mid \alpha \in I_0\}$$

as f is a surjection. Thus Y is pre T_2-closed.

Proposition 1 — Let $(X, \mathcal{J})$ be an extremally disconnected space. Then the following are equivalent.

(i) X is locally pre T_2-closed

(ii) Each point of X has a neighbourhood which is pre T_2-closed.

(iii) Each point of X has a neighbourhood which is pre T_2-closed set.

PROOF : Follows by the fact that in an extremally disconnected space, closure of an open set is open.

Theorem 9 — Every locally pre T_2-closed extremally disconnected Hausdorff space is regular.

PROOF : Since X is locally pre T_2-closed, every $x \in X$ has a neighbourhood U whose closure is pre T_2-closed. As X is extremally disconnected cl U is open as well. So for a point $x \in X$ and a closed set F not containing x, cl U and X – cl U separate x and F.

Theorem 10 — Let X be a locally pre T_2-closed extremally disconnected (Hausdorff) space and Y a compact (Hausdorff) space. The product $X \times Y$ is locally pre T_2-closed.

PROOF : Since X is locally pre T_2-closed, each $x \in X$ has a neighbourhood U_x whose closure is pre T_2-closed. As X is extremally disconnected cl U_x is also open. So cl U_x is a pre T_2-closed set. That is, if $Z = \operatorname{cl} U_x$, then $(Z, \mathcal{J}/Z)$ is a pre T_2-closed space. Thus by Theorem 7, $Z \times Y$ is pre T_2-closed. Now $U_x \times Y$ is an open neighbourhood of $(x, y) \in X \times Y$, $\operatorname{cl} (U_x \times Y) = \operatorname{cl} U_x \times Y = Z \times Y$ is pre T_2-closed set. Thus $X \times Y$ is locally pre T_2-closed [cl U_x is both open and closed in X, and hence cl $U_x \times Y$ is both open and closed in $X \times Y$].

In the following we define one point extensions for topological spaces and extend the same to non-pre T_2-closed spaces.

Definition 3. — A topological space $(Y, \mathcal{U})$ is said to be an extension of $(X, \mathcal{J})$ if $X \subset Y$, $\mathcal{U}/X = \mathcal{J}$, $\text{cl}_Y(X) = Y$ where $\text{cl}_Y(X)$ denotes the closure of X in Y. $(Y, \mathcal{U})$ is a one-point pre T_2-closed extension of $(X, \mathcal{J})$ if $(Y, \mathcal{U})$ is an extension of $(X, \mathcal{J})$; $(Y, \mathcal{U})$ is pre T_2-closed and $(Y - X)$ is singleton.

Theorem 11 — Let $(X, \mathcal{J})$ be a Hausdorff space. If $(Y, \mathcal{U})$ is a one-point Hausdorff pre T_2-closed extension of $(X, \mathcal{J})$ then

(1) X is open in Y.

(2) X is pre T_2-closed if and only if $(Y - X)$ is $\mathcal{U}$-open.

(3) X is locally pre T_2-closed.

PROOF : Let $Y - X = \{\pi\}$. Since Y is Hausdorff, $\{\pi\}$ is closed and X is $\mathcal{U}$ open in Y. Since Y is Hausdorff for each $x \in X$, there exists a $\mathcal{U}$-open neighbourhood U of x such that $\pi \notin \text{cl}_Y U$. Since $\pi \notin U$, and X is $\mathcal{U}$-open, U is open in X. Also $\text{cl}_X U = \text{cl}_Y U \cap X = \text{cl}_Y U$. Since Y is pre T_2-closed $\text{cl}_Y U$ is pre T_2-closed relative to Y by Theorem (2). Since X is open, $\text{cl}_X U = \text{cl}_Y U \subset X \subset Y$, by Theorem (5), $\text{cl}_X U$ is pre T_2-closed relative to X. Thus X is locally pre T_2-closed. If $(Y - X)$ is open relative to Y, X is closed relative to Y. That is, X is closed open subspace of Y. As Y is pre T_2-closed X is pre T_2-closed relative to Y by Theorem (2). Since Y is Hausdorff, X is pre T_2-closed relative to Y implies X is closed in Y and $\{\pi\}$ is open. Thus $(Y - X)$ is open.

Definition 4 — Let $(X, \mathcal{J})$ be a topological space. A filter $\mathcal{F}$ on X is said to be an open filter if and only if (1) $\mathcal{F} \subset \mathcal{J}$; (2) $\phi \notin \mathcal{F}$; (3) if $\{U_i\} \subset \mathcal{F}$. $\bigcap_{i=1}^{n} (U_i) \in \mathcal{F}$; (4) $U \in \mathcal{J}$, $U \supset V \in \mathcal{F}$ then $U \in \mathcal{F}$.

Theorem 12 — Let $(X, \mathcal{J})$ be a locally pre T_2-closed Hausdorff space which is not pre T_2-closed.

Let $X^* = X \cup \{\pi\}$ where $\pi \notin X$. Then

(a) $\mathcal{J}^* = \mathcal{J} \cup \{\{\pi\} \cup V \mid V \in \mathcal{F}\}$ is a Hausdorff topology on X^* where $\mathcal{F}$ is a filter generated by $\{U \mid U \in \mathcal{J},$ such that $(X - U)$ is pre T_2-closed$\}$. (b) $(X^*, \mathcal{J}^*)$ is a one-point pre T_2-closed extension of $(X, \mathcal{J})$. (c) $(X^*, \mathcal{J}^*)$ is the projective minimum in the set of all one point pre T_2-closed extensions of $(X, \mathcal{J})$.

PROOF : Let α $\{U \subset X \mid U \in \mathcal{J}$ and $(X - U)$ is pre T_2-closed$\}$. Since X is not pre T_2-closed, $\phi \notin \alpha$. Let $U_i \in \alpha$ for $i = 1, 2, ..., n$. Since each $(X - U_i)$ is pre

T_2-closed $\bigcup_{i=1}^{n} (X-U_i)$ is pre T_2-closed by Theorem (6). But $\bigcup_{i=1}^{n} (X-U_i)$ = $\left(X-\bigcap_{i=1}^{n} U_i\right)$. So $\bigcap_{i=1}^{n} U_i \in \alpha$. Hence α is a pre open filter base. Then $\mathcal{J}^*$ is a Hausdorff topology on X^*. Let $x, y \in X^*$ with $x \neq y$. If $x, y \in X$, there exist disjoint open sets in X, hence in X^*, containing x and y respectively. If $x \neq \pi$, since X is locally pre T_2-closed, x has an open neighbourhood U whose closure is pre T_2-closed relative to X. Consider U and $\{\pi\} \bigcup (X - \text{cl } U)$. These are the required disjoint $\mathcal{J}^*$-open sets containing x and π respectively. Thus $(X^*, \mathcal{J}^*)$ is a Hausdorff space.

X^* is pre T_2-closed — Consider a $\mathcal{J}^*$ -pre open cover $\mathcal{G}$ of X^*. There exists a $G_0 \in \mathcal{G}$ such that $\pi \in G_0 \subset \text{int}_{X^*} \text{cl}_{X^*} G_0$ where $\text{int}_{X^*} \text{cl}_{X^*} G_0$ is regular open in X^* and hence open in X^*. Since α is a filter base, there exists an open set U such that $\{\pi\} \bigcup U \subset \text{int}_{X^*} \text{cl}_{X^*} G_0 \subset \mathcal{J}^* (G_0)$ and $(X - U)$ is pre T_2-closed relative to X. Further, since X is open, $\{G \bigcap X \mid G \in \mathcal{G}\}$ is a preopen cover of $(X - U)$ and there exist a $G_i \in \mathcal{G}$, $i = 1, 2, 3, ..., n$, such that $(X - U) \subset \bigcup \{\text{cl}_X (G_i \bigcap X) \mid i = 1, 2, ..., n\} \subset \bigcup \{\text{cl}_{X^*} G_i \mid i = 1, 2, ..., n\}$. Thus

$$X^* = \{\{\pi\} \bigcup U\} \; U \; (X - U) \subset \mathcal{J}^*(G_0) \bigcup \{U \mathcal{J}^* (G_i) \mid i = 1, 2, ..., n)$$

$$= \bigcup \{\mathcal{J}^* G_i \mid G_i \in \mathcal{G}, \; i = 0, 1, ..., n\}.$$

Thus X^* is pre T_2-closed.

Let (Y, T) be any one point pre T_2-closed extension of $(X, \mathcal{J})$. Define $f : Y \to X^*$ such that $f(x) = x$ for all $x \in X$ and $f(\xi) = \pi$ where $\xi = Y - X$. Since X is open in X^* and Y, f is the identity map on X and the spaces involved are Hausdorff, it is enough to verify the continuity of f at ξ. Indeed if G is a basic open neighbourhood of π, then $G = \{\pi\} \bigcup U$ where $(X - U)$ is pre T_2-closed relative to X. As X is open, $(X - U)$ is pre T_2-closed relative to Y by Theorem (5). As Y is Hausdorff $(X - U)$ is closed in Y. So $Y - (X - U) = \{\pi\} \bigcup U$ is open in Y. Since $f^{-1}(G) = \{\xi\} \bigcup U$, f is continuous at ξ. Thus f is continuous and X^* is the projective minimum.

Theorem 13 — Let $(X, \mathcal{J})$ be a locally pre T_2-closed extremally disconnected Hausdorff space which is not pre T_2-closed. Let $X^* = X \bigcup \{\pi\}$ where $\pi \notin X$. Then

(a) $\mathcal{J}^\# = \mathcal{J} \bigcup \{\{\pi\} \bigcup U \mid U \in \mathcal{J}, (X - \text{int cl } U)$ is pre T_2-closed relative to $X\}$ is a Hausdorff topology on X^*.

(b) $(X^*, \mathcal{J}^\#)$ is a one point pre T_2-closed extension of $(X, \mathcal{J})$

(c) $(X^*, \mathcal{J}^\#)$ is the projective maximum in the set of all one point pre T_2-closed extensions of $(X, \mathcal{J})$.

PROOF : Let $\beta = \{U \mid U \in \mathcal{J}, (X - \text{int cl } U) \text{ is pre } T_2\text{-closed relative to } X\}$. Then as $(X - \text{int cl } U) = \text{cl }(X - \text{cl } U)$, $\text{cl }(X - \text{cl } U)$ is closed-open subset of X as X is extremally disconnected. Now $\phi \notin \beta$, $\bigcap_{i=1}^{n} U_i \in \beta$ for U_i, $(i = 1, 2, ..., n) \in \beta$. Further if $U \in \beta$ and $U \subset V$ where $V \in \mathcal{J}$ then $(X - \text{int cl } V) \subset (X\text{-int cl } U)$. As $(X - \text{int cl } U)$ is pre T_2-closed relative to X and $(X - \text{int cl } U)$ is an open set of X, $(X - \text{int cl } U)$ is pre T_2-closed by Theorem (1). Now $(X - \text{int cl } V)$ is pre T_2-closed by Theorem (2). Thus $V \in \beta$ and β is an open filter on X.

$\mathcal{J}^*$ is a Hausdorff topology — Let $x, y \in X^*$. If $x \neq y$ and $x, y \in X$, then there exist disjoint open sets in X, hence X^*, containing them. If $x \neq \pi$, there exists a neighbourhood U of x such that cl U is pre T_2-closed. As X is extremally disconnected cl U is open and int cl U = cl U. Now, the required neighbourhoods of x and π are U and $\{\pi\} \bigcup (X - \text{int cl } U)$ respectively.

$(X^*, \mathcal{J}^\#)$ is pre T_2-closed — Consider a $\mathcal{J}^\#$-preopen cover $\mathcal{G}$ of X^*. There exists a $G_0 \in \mathcal{G}$ such that $\pi \in G_0 \subset \text{int}_{X^*} \text{cl}_{X^*} G_0$. Now $\text{int}_{X^*} \text{cl}_{X^*} G_0$ is a $\mathcal{J}^\#$-open set containing π and therefore, we can find $U \in \mathcal{J}$ such that $\{\pi\} \bigcup U \subset \text{int}_{X^*} \text{cl}_{X^*} G_0 = U'$ (say) with $(X - \text{int}_X \text{cl}_X U)$ pre T_2-closed relative to X. Now $\{G \bigcap X \mid G \in \mathcal{G}\}$ is a $\mathcal{J}$-preopen cover of $(X - U)$ as X is open. Since $U \subset \text{int}_X \text{cl}_X U$, $(X - \text{int}_X \text{cl}_X U) \subset (X - U)$, $\{G \bigcap X \mid G \in \mathcal{G}\}$ is also a cover of $(X - \text{int}_X \text{cl}_X U)$. Since $(X - \text{int}_X \text{cl}_X U)$ is pre T_2-closed.

$$(X - \text{int}_X \text{cl}_X U) \subset \bigcup \{\text{cl}_X (G_i \bigcap X) \mid i = 1, 2, ..., n\}, G_i \in \mathcal{G}\}.$$

Now $X^* = \{\{\pi\} \bigcup \text{int}_X \text{cl}_X U\} \bigcup (X - \text{int}_X \text{cl}_X U)$. As $\{\pi\} \bigcup U \subset \{\pi\} \bigcup \text{int}_X \text{cl}_X U \subset \{\pi\} \bigcup \text{cl}_X U \subset \text{cl}_{X^*} U' = \text{cl}_{X^*} (\text{int}_{X^*} \text{cl}_{X^*} G_0) = \text{cl}_{X^*} G_0$. So

$$X^* \subset \text{cl}_{X^*} G_0 \bigcup \{\bigcup \{\text{cl}_{X^*} G_i \mid i = 1, 2, ..., n, G_i\}\}$$

$$= \bigcup \{\text{cl } G_i \mid i = 0, 1, 2, ..., n, G_i \in \mathcal{G}\}.$$

Thus $(X^*, \mathcal{J}^\#)$ is pre T_2-closed.

$(X^*, \mathcal{J}^\#)$ is the projective maximum — Let (Y, T) be any other one point pre T_2-closed extension of $(X, \mathcal{J})$. Define $f : X^* \to Y$ by $f(x) = x$ for all $x \in X$ and $f(\pi) = \eta$ where $\eta = Y - X$. Since X is Hausdorff $X \in \mathcal{J}^\#$, $X \in T$, and f is the identity map on X, it is enough to verify continuity of f at π. Let U be a T-open neighbourhood of η. $f^{-1}(U) = (U \bigcap X) \bigcup \{\pi\}$. Since Y is pre T_2-closed,

$\mathrm{cl}_Y(Y-\mathrm{cl}_Y U)$ is pre T_2-closed. Now $\mathrm{cl}_Y(Y-\mathrm{cl}_Y U) \subset X \subset Y$ where $X \in \mathcal{T}$. So $\mathrm{cl}_Y(Y-\mathrm{cl}_Y U)$ is pre T_2-closed relative to X. Now $\mathrm{int}_X \mathrm{cl}_X (U \cap X) = \mathrm{int}_Y \mathrm{cl}_Y U \cap X$ so that $Y - \mathrm{int}_Y \mathrm{cl}_Y U = X - \mathrm{int}_X \mathrm{cl}_X (U \cap X)$. This implies $\{\pi\} \cup (U \cap X) \in \mathcal{J}^{\#}$. Thus f is continuous.

4. Pre-Urysohn Closed Spaces

This section deals with the definition and characterization of a pre-Urysohn closed space using filters.

Definition 5 — A pre-Urysohn space X is said to be pre-Urysohn closed if it is closed in every pre-Urysohn space in which it can be embedded.

Definition 6 — A filter base $\mathcal{F}$ is said to be a pre-Urysohn filter base, if whenever x is not an adherent point of $\mathcal{F}$, there exists a preopen set U containing x such that p cl $U \cap \mathrm{cl}\, F = \phi$ for some $F \in \mathcal{F}$.

Definition 7 — An open cover $\mathcal{U}$ of X is said to be a pre-Urysohn cover if there exists a preopen cover $\mathcal{V}$ of X such that for each $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ such that p cl $V \subset U$.

Theorem 14 — Let X be an almost p-regular Hausdorff space. Then the following are equivalent :

(a) X is pre-Urysohn closed

(b) Every pre-Urysohn cover $\mathcal{U}$ of X has a finite subfamily $\mathcal{U}^*$ such that the closures of whose members cover X.

(c) Every open pre-Urysohn filter base has non-empty adherence.

Proof (a) $\Rightarrow$ (b) : Let $(X, \mathcal{J})$ be pre-Urysohn closed and $\mathcal{U}$ a pre-Urysohn cover of X. Suppose that for no finite subfamily of $\mathcal{U}$, the closures of the members of the subfamily cover X. Let $p \notin X$ and $Y = X \cup \{p\}$. Let $\mathcal{J}^* = \mathcal{J} \cup \{\{p\} \cup \left(X - \bigcup_{i=1}^{n} \mathrm{cl}\, U_i\right)\}$. Then $\mathcal{J}^*$ is a topology on Y which is pre-Urysohn. For, if $x, y \in Y$ with $x \neq y$, $\neq p$ there exist disjoint $\mathcal{J}$-preopen sets, hence $\mathcal{J}^*$-preopen sets H and K containing x and y respectively such that

$$p\ \mathrm{cl}_X H \cap p\ \mathrm{cl}_X K = \phi \ [\text{as } X \in \mathcal{J}^*,\ p\ \mathrm{cl}_X H = p\ \mathrm{cl}_{X^*} H \cap X,$$

$$p\ \mathrm{cl}_X K = p\ \mathrm{cl}_{X^*} K \cap X]$$

so
$$p\ \mathrm{cl}_{X^*} H \cap p\ \mathrm{cl}_{X^*} K = \phi.$$

Suppose $y = p$. Since $\mathcal{U}$ is a pre-Urysohn cover of X, there exists a preopen cover $\mathcal{V}$ of X such that for each $V \in \mathcal{V}$, there exists a $U_v \in \mathcal{U}$ such that p cl $V \subset U_v$. Let $x \in V \in \mathcal{V}$. Then $(X - p\ \mathrm{cl}\ U_v)$ is a preopen set containing $y = p$.

Since $U_v \subset p$ cl U_v, $(X - p \text{ cl } U_v) \subset (X - U_v)$ which implies p cl $(X - p \text{ cl } U_v) \subset (X - U_v)$ as $(X - U_v)$ is closed. As p cl $V \subset U_v$, p cl $V \cap (X - U_v) = \phi$ and p cl $V \cap p$ cl $(X - p \text{ cl } U_v) = \phi$. Thus V and $(X - p \text{ cl } U_v)$ are the disjoint preopen sets containing x and p such that p cl $V \cap p \text{ cl } (X - p \text{ cl } U_v) = \phi$. So $(Y, \mathcal{J})$ is pre-Urysohn. But X is not a closed set of Y, since $p \in \text{cl}_Y X$. This is a contradiction.

(b) $\Rightarrow$ (c) : Let $\mathcal{F}$ be an open pre-Urysohn filter base without any adherent point. Then $\mathcal{U} = \{X - \text{cl } F;\ F \in \mathcal{F}\}$ is an open cover of X. Indeed $\mathcal{U}$ is a pre-Urysohn cover of X. Let $x \in X$. Since x is not an adherent point of $\mathcal{F}$, there exists a preopen set V_x such that p cl $V_x \cap$ cl $F_x = \phi$ for some $F_x \in \mathcal{F}$. Therefore there exists an $(X - \text{cl } F_x) \in \mathcal{U}$ such that p cl $F_x \subset (X - \text{cl } F_x)$. Hence $\mathcal{V} = \{V_x \mid x \in X,\ p \text{ cl } V_x \subset (X - \text{cl } F_x)\}$ is a preopen cover of X such that for each $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that p cl $V \subset U$. So by (b),

$$X = \bigcup \{\text{cl } (X - \text{cl } F_{x_i}) \mid i = 1, 2, \ldots, n\}$$

$$\subset \bigcup \{\text{cl } (X - F_{x_i}) \mid i = 1, 2, \ldots, n\}$$

$$= \bigcup \{(X - F_{x_i}) \mid i = 1, 2, \ldots\}$$

since each F_{x_i} is open. Thus $X = X - \bigcap \{F_{x_i} \mid i = 1, 2, \ldots, n\}$ which means that $\bigcap_{i=1}^{n} F_{x_i} = \phi$ which is a contradiction to the fact that $\mathcal{F}$ is a filter base. Thus (c) is proved.

(c) $\Rightarrow$ (a) : Let X be an almost p-regular Hausdorff space which is not pre-Urysohn closed. Let Y be a pre-Urysohn space in which X is embedded. If possible suppose that X is not a closed subset of Y. Let $p \in (\text{cl } X - X)$ and let $\mathcal{U} = \{U \cap X \mid U$ is an open neighbourhood of p in $Y\}$. Since $(U \cap X)$ is open it is preopen in X. So there exists a regular open set G such $(U \cap X) \subset G$ and cl $(U \cap X) =$ cl G. Let $\mathcal{U}' = \{G \subset X \mid G \supset U \cap X$, G is regular open in X, U is a neighbourhood of p in $Y\}$. Then $\mathcal{U}'$ is a regular open filter base in X. Since X is almost p-regular and Hausdorff it is pre-Urysohn and $\mathcal{U}'$ is a pre-Urysohn filter base in X. Now $\mathcal{U}'$ has no adherent point in X. For if $x_0 \in X$ is an adherent point of $\mathcal{U}'$, for every preopen set U_{x_0} containing x_0, p cl $U_{x_0} \cap$ cl $G \neq \phi$ for every $G \in \mathcal{U}$. This implies that cl $U_{x_0} \cap$ cl $G \neq \phi$ which is a contradiction to the hypothesis that X is almost p-regular Hausdorff. So $\mathcal{U}'$ has no adherent point, which is a contradiction. Thus the proof is complete.

Theorem 15 — Every closed-open subset of an almost p-regular Hausdorff pre-Urysohn closed space is pre-Urysohn closed.

PROOF : Let X be a pre-Urysohn closed space which is almost p-regular Hausdorff and let $Y \subset X$ be closed-open subset. Let $\mathcal{F}$ be a pre-Urysohn filter base in Y with empty adherence. Since Y is closed-open it is regular open and hence is almost p-regular. As X is Hausdorff, Y is Hausdorff. So Y is almost p-regular Hausdorff and hence pre-Urysohn. Also $\mathcal{F}$ is an open filter base in X. Now we claim that $\mathcal{F}$ is a pre-Urysohn filter base in X. If every point of X is an adherent point of $\mathcal{F}$ in X, then $\mathcal{F}$ is vacuously pre-Urysohn in X. Let x be a point of X which is not an adherent point of $\mathcal{F}$ in X. Let every preopen subset U of X containing x have nonempty intersection with every $F \in \mathcal{F}$. That is, every preopen set containing x has nonempty intersection with Y. So $x \in p\,\text{cl}\,(Y) = Y$ as Y is preclosed. Now $U \cap Y$ is preopen in Y as Y is open in X and $x \in U \cap Y$. Thus $U \cap Y$ is a preopen subset of Y containing x, having non-empty intersection with every $F \in \mathcal{F}$. Also, Y being open, every preopen subset of Y is of the form $U \cap Y$, where $U \in PO(X)$. Hence every preopen set of Y containing x intersects every member $F \in \mathcal{F}$ which is a contradiction to the fact that $\mathcal{F}$ is a pre-Urysohn filter base in Y. Y being closed, $\mathcal{F}$ cannot have an adherent point in X, since it has empty adherence in Y. This is a contradiction. Hence every pre-Urysohn filter base in Y has non-empty adherence in Y and Y, therefore is pre-Urysohn closed.

5. p-REGULAR CLOSED SPACES

In this section along with the characterization of p-regular closed spaces, a few properties of such spaces are studied.

Definition 8 — A p-regular space $(X, \mathcal{J})$ is said to be p-regular closed if it is closed in every p-regular space in which it can be embedded.

Definition 9 — A cover $\mathcal{U}$ is said to be p-regular cover if there exists a preopen cover $\mathcal{V}$ such that preclosures of whose members refine $\mathcal{U}$.

Definition 10 — A filter base is said to be a p-regular filter base if it is equivalent to a preclosed filter base.

Theorem 14 — The following are equivalent for a p-regular space X.

(1) X is p-regular closed.

(2) Every open p-regular cover has a finite subcover.

(3) Every open p-regular filter base has non-empty adherence.

PROOF (1) $\Rightarrow$ (2) : Let $\mathcal{U}$ be an open p-regular cover of X and suppose that $\mathcal{U}$ does not have a finite subcover. Let $Y = X \cup \{p\}$ where $p \notin X$. Let

$$\mathcal{J}^* = \mathcal{J} \cup \left\{ \{p\} \cup \left(X - \bigcup_{i=1}^{n} U_i \right) : U_i \in \mathcal{U} \right\}.$$

Then $\mathcal{J}^*$ is a topology on Y and $(Y, \mathcal{J}^*)$ is p-regular.

Case (i) — Let $x \in Y$ where $x \neq p$ and B be a $\mathcal{J}^*$-closed set containing neither x nor p. Then $(Y - B)$ is a $\mathcal{J}^*$-open set containing both x and p. Since $\mathcal{U}$ is a p-regular cover, there exists a preopen cover $\mathcal{V}$ of X such that $\{p \text{ cl } V;\ V \in \mathcal{V}\}$ is a refinement of $\mathcal{U}$. Since $x \in X$, $x \in V$ for some $V \in \mathcal{V}$. Now $x \in Y - B$ where $(Y - B)$ is an open set containing p. So $(Y - B) = \{p\} \bigcup \left\{\left(X - \bigcup_{i=1}^{n} U_i\right) \mid U_i \in \mathcal{U}\right\}$.

Therefore $x \in V \bigcap \left\{\{p\} U \left(X - \bigcup_{i=1}^{n} U_i\right)\right\}$ which is a preopen set in $\mathcal{J}^*$, being the intersection of an open set and a preopen set. Also $(Y - B) = \{p\}$ $\bigcup \left(X - \bigcup_{i=1}^{n} U_i\right) = Y - \left(\bigcup_{i=1}^{n} U_i\right)$. Hence $B = \bigcup_{i=1}^{n} U_i$. Thus $V \bigcap \{p\}$ $\bigcup \left(X - \bigcup_{i=1}^{n} U_i\right)$ and $\bigcup_{i=1}^{n} U_i$ are disjoint $\mathcal{J}^*$-preopen sets containing x and B respectively.

Case (ii) — Suppose that $x = p$ and B is a $\mathcal{J}^*$-closed set not containing p. Then $(Y - B) = \{p\} \bigcup \left(X - \bigcup_{i=1}^{n} U_i\right)$ is a $\mathcal{J}^*$-open and hence a preopen set containing p and $\bigcup_{i=1}^{n} U_i$ is a $\mathcal{J}^*$-open set containing x.

Case (iii) — Suppose $x \neq p$ and let B be a closed set continuing p. $(Y - B)$ is open, not containing p and therefore a $\mathcal{J}$-open set containing x. As X is p-regular, there exists a preopen set V containing x such that $x \in V \subset p \text{ cl } V \subset (Y - B)$. As X is open, V is preopen in Y. Thus $(Y, \mathcal{J}^*)$ is p-regular.

Since every open set containing p intersects X, X is not closed in Y. This is a contradiction. So $\mathcal{U}$ has a finite subcover.

(2) $\Rightarrow$ *(3)* : Let $\mathcal{F}$ be an open p-regular filter base without an adherent point in X. Then $\{X - \text{cl } F \mid F \in \mathcal{F}\}$ is an open cover of X. If $x \in X$ and x is not an adherent point of $\mathcal{F}$, then $x \notin \text{cl } F_x$ for some $F_x \in \mathcal{F}$. Since X is p-regular there exists a preopen set V_x containing x such that $x \in V_x \subset p \text{ cl } V_x \subset (X - \text{cl } F_x)$. Thus $\{V_x;\ x \in X \text{ and } p \text{ cl } \ V_x \subset X - \text{cl } F_x\} = \mathcal{V}$ is a preopen cover of X such that $\{p \text{ cl } V_x; x \in X\}$ is a refinement of $\{X - \text{cl } F_x \mid F_x \in \mathcal{F}\}$. Hence $\{X - \text{cl } F_x \mid F_x \in \mathcal{F}\}$ is a p-regular cover of X. Therefore there exist finitely many members $F_1, F_2, \ldots, F_n$ of

$\mathcal{F}$ such that $X \bigcup_{i=1}^{n} (X - \mathrm{cl}\, F_i) \subset \bigcup_{i=1}^{n} (X - F_i) = X - \bigcap_{i=1}^{n} F_i$ so that $\bigcap_{i=1}^{n} F_i = \phi$, a contradiction to the fact that $\mathcal{F}$ is a filter base.

Theorem 17 — Let X be a p-regular closed space. Let $Y \subset X$ be a closed-open subset. Then Y is p-regular closed.

PROOF : Every semi-open subset of a p-regular space is p-regular. Since Y is closed-open, it is semi-open and therefore p-regular. We need only show that $(Y, \mathcal{J}/Y)$ possesses that every open p-regular cover has a finite subcover. Let $\mathcal{U}$ be an open p-regular cover of Y. Let $\mathbf{H} = \{H \subset X,\ U = H \bigcap Y,\ U \in \mathcal{U}\} \bigcup (X - Y)$. Then $\mathbf{H}$ is an open p-regular cover of X is seen by the following argument. Since $\mathcal{U}$ is a p-regular cover of Y, there exists a preopen cover $\mathcal{V}'$ of Y such that $\{p \text{ cl } V \mid V \in \mathcal{V}'\}$ refines $\mathcal{U}$. Y being open, each $V \in \mathcal{V}'$ is a preopen subset of X. Now $(X - Y)$ is both open and closed in X. Hence $\mathcal{V} = \mathcal{V}' \bigcup (X - Y)$ is a preopen cover of X such that preclosures of members of $\mathcal{V}$ refine $\mathbf{H}$. Thus $\mathbf{H}$ is a p-regular cover of X. As Y is open, $\mathbf{H}$ is an open p-regular cover of X. Hence $X = \bigcup_{i=1}^{n} H_i \bigcup (X - Y)$ for some finite subfamily $H_1, H_2, ..., H_n$ of $\mathbf{H}$. Then

$$Y = \bigcup_{i=1}^{n} (H_i \bigcap Y) = \bigcup_{i=1}^{n} U_i .$$

Theorem 18 — A continuous image of a p-regular closed space onto a regular space is regular-closed.

PROOF : Let $f : X \rightarrow Y$ be a continuous mapping from a p-regular closed space X to a regular space Y. Let $\mathcal{U}$ be a regular cover of Y. Then $\mathcal{U}$ is open. Let $\mathcal{V}$ be a closed refinement of $\mathcal{U}$. Since f is continuous $\{f^{-1}(U),\ U \in \mathcal{U}\}$ is an open cover of X. Also f being continuous, $\{f^{-1}(V),\ V \in \mathcal{V}\}$ is a closed, hence preclosed refinement of X. It is also a preclosed cover of X. Thus $f^{-1}(\mathcal{U})$ is an open regular cover of X. Hence $X = \bigcup_{i=1}^{n} f^{-1}(U_i)$ for a subfamily $U_1, U_2, ..., U_n$ of $\mathcal{U}$ and $Y = f(X) = \bigcup_{i=1}^{n} U_i$. So Y is regular closed.

6. As an application of pre T_2-closed spaces, we prove that in the category of pre T_2-closed extremally disconnected Hausdorff spaces and continuous maps, the projective objects are finite spaces.

If $(X, \mathcal{J})$ is locally pre T_2-closed extremally disconnected Hausdorff space and $(X^*, \mathcal{J}^*)$ is any one point pre T_2-closed extension, then $\mathcal{J}^* \subset \mathcal{J}^\#$ if $\mathcal{J}^\#$ is the projective maximum topology. If $(X^*, \mathcal{J}^*)$ is extremally disconnected Hausdorff space, then $(X^*, \mathcal{J}^*)$ is pre T_2-closed.

Theorem 19 — In the category of pre T_2-closed extremally disconnected Hausdorff spaces and continuous maps, the projective objects are finite spaces.

PROOF : That X is projective when X is finite is obvious. Conversely, if X is projective, let us show that X is discrete so that it is finite.

Suppose X is not discrete. Then there exists a point $a \in X$ such that a is not isolated. Let $Y = X - \{a\}$. Let π be a point not in $Y \times N^*$ where $N^* = N \bigcup \{w\}$ is the one point compactification of N which is discrete. Let $A = (Y \times N^*) \bigcup \{\pi\}$. Let us introduce a topology $\mathcal{U}$ on A as follows.

(i) Let $Z = (x, n) \in Y \times N$. Basic neighbourhoods of Z in A are of the form $G \times \{n\}$ where G is an open neighbourhood of x in Y. For $(x, w) \in Y \times \{w\}$, basic neighbourhoods of it in A are of the form $G \times [n, w]$ where G is an open neighbourhood of x in Y and $[n, w]$ is such that $N^* - [n, w]$ is finite.

(ii) If $Z = \pi$, basic open neighbourhoods of it in A are of the form $(Q \times N) \bigcup \{\pi\}$ where Q is a deleted open neighbourhoods of a, (that is, $a \notin Q$, and $Q \bigcup \{a\}$ is open in X).

The space $(A, \mathcal{U})$ is clearly a Hausdorff space.

$(A, \mathcal{U})$ is pre T_2-closed — Since $\{a\}$ is closed in X, Y is open in X. As X is pre T_2-closed, Y is locally pre T_2-closed by Theorem (11). Since N^* is a compact space, $Y \times N^*$ is locally pre T_2-closed by Theorem (10). If we write, $B = Y \times N^*$ and if the topology of $Y \times N^*$ is denoted by τ, then (B, τ) is locally pre T_2-closed Hausdorff space. Now for an open set U and a point $Z \in U, U \in \mathcal{U}_Z$, where $\mathcal{U}_Z$ denotes the neighbourhoods system at Z. Also cl $(Q \times \{n\})$ = cl $Q \times \{n\}$ where cl Q is open in Y. [Since Y is an open subspace of X, Y is extremally disconnected] and cl $\{G \times [n, w]\}$ = cl $Q \times [n, w]$ since $w \in [n, w]$ and cl $(Q \times N \bigcup \{\pi\})$ = (cl $Q \times N^*) \bigcup \{\pi\}$ we find that closure of an open set in A is open in A. Since $\mathcal{U}$ is a Hausdorff topology, $B = A - \{\pi\}$ is open in A and hence B is extremally disconnected. So (B, τ) satisfies the conditions of the Theorem 13. Also we see that $\text{cl}_A(B) = A$ and so $(A, \mathcal{U})$ is a one point extension of (B, τ).

To prove $(A, \mathcal{U})$ is pre T_2-closed, let us start with an open neighbourhood of a in X, say, $Q \bigcup \{a\}$. Since $(X, \mathcal{J})$ is a one point pre T_2-closed extension of $(Y, \mathcal{J}/Y)$ $Q \bigcup \{a\}$ is an open neighbourhood of a in $X = Y^* = Y \bigcup \{a\}$ endowed with the projective maximum topology $(\mathcal{J}/Y)^\#$. Hence $Y - \text{int}_Y \text{cl}_Y Q$ is pre T_2-closed relative to Y. Thus $B - \text{int}_B \text{cl}_B (Q \times N) = \text{cl}_B (B - \text{cl}_B Q \times N) = \text{cl}_B (B - \text{cl}_Y Q \times N^*) = (Y - \text{int}_Y \text{cl}_Y Q) \times N^*$ which is pre T_2-closed relative to B. Thus $(Q \times N) \bigcup \{\pi\}$ is open in A endowed with the projective maximum topology on A. Thus $\mathcal{U} \subset \mathcal{J}^\#$. Since $(A, \mathcal{J}^\#)$ is pre T_2-closed, $(A, \mathcal{U})$ is pre T_2-closed.

Now let us define a topology $\mathcal{U}'$ on $A = Y \times N^* \cup \{\pi\}$ as follows. Let neighbourhoods of $Z \in Y \times N^*$ be as in $(A, \mathcal{U})$ and a basic $\mathcal{U}'$-open neighbourhood of π be of the form $(Q \times N^*) \cup \{\pi\}$ where Q is an open deleted neighbourhood of a.

By argument similar to those in the previous paragraph, it follows $(A, \mathcal{U}')$ is pre T_2-closed as $(A, \mathcal{U})$ is pre T_2-closed $[i : (A, \mathcal{U}) \to (A, \mathcal{U}')$ is continuous because $i^{-1}(Q \times N^*) \cup \{\pi\} = (Q \times N) \cup \{\pi\} \cup (Q \times \{w\})$ is open in $(A, \mathcal{U})]$, by Theorem 8.

Define $f : X \to (A, \mathcal{U})$ such that $f(a) = \pi$, $f(y) = (y, w)$ for each $y \in Y$. Then $f^{-1}(Q \times N^*) \cup \{\pi\} = Q \cup \{a\}$ is open. Again $f^{-1}(Q(y) \times N^* - L) = Q(y)$ where $Q(y)$ denotes an open neighbourhood of y and L is a finite subset of N. Thus $f^{-1}(Q(y) \times N^* - L)$ is open and f is continuous. Since X is projective, there is a continuous mapping g such that $i \circ g = f$. Clearly $g(a) = \pi$, otherwise, $f(a) \neq \pi$. Similarly $g(y) = (y, w)$ for each $y \in Y$. Then $g^{-1}(Q \times N) \cup \{\pi\} = \{a\}$ which is not open and this violates the continuity of g.

Thus X is discrete. Since X is pre T_2-closed X is finite.

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