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ON GLOBAL SOLUTIONS OF NONLINEAR INTEGRAL EQUATIONS IN BANACH SPACES*

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This paper gives some existence theorems of global L^p solutions for nonlinear integral equations in Banach spaces.

1. INTRODUCTION

Let E be a real Banach space, $J = [0, d]$ $(d > 0)$ a compact interval and $L^p(J, E)$ $(p > 1)$ be the space of all strongly measurable functions $x : J \to E$ with $\int_J \|x(t)\|^p\, dt < \infty$, provided with the norm $\|x\|_p = (\int_J \|x(t)\|^p\, dt)^{1/p}$.

Consider the nonlinear integral equation in E :

$$x(t) = x_0(t) + \int_0^t K(t, s) f(s, x(s))\, ds, \qquad \text{... (1)}$$

and formulate the following conditions :

(H_1) p, q are real numbers such that $p, q > 1$ and $p \geq \min\{q, 2\}$; let $q' = q/(q-1)$, $m = \max\{p, q'\}$ and let k be a number such that $1 < k \leq \infty$ and $(1/k) + (1/m) + (1/p) = 1$;

(H_2) $x_0 \in L^p(J, E)$;

(H_3) $(t, x) \to f(t, x)$ is a function from $J \times E$ into a real Banach space E_1 such that $f(t, x)$ is strongly measurable in t and continuous in x, and

$$\|f(t, x)\| \leq c(t) + b\|x\|^{p/q}, \quad \text{for} \quad t \in J \text{ and } x \in E,$$

where $c \in L^q(J, R_+)$ and $b \geq 0$;

(H_4) K is a strongly measurable function from $J \times J$ into the space of continuous linear mappings $E_1 \to E$ and

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$$\iint_{J\times J} \|K(t,s)\|^m \, ds\, dt < \infty.$$

(H_5) there exists a $h \in L^k(J, R_+)$ such that $\alpha_1(f(t,B)) \le h(t)\,\alpha(B)$ for any $t \in J$ and bounded $B \subset E$, where α and α_1 denote the Kuratowski measures of noncompactness in E and E_1 respectively.

Under conditions (H_1)-(H_5), Januszewski[1] (Theorem 2) proved that eqn. (1) has a local L^p solution. In this paper, we use different method to obtain the existence of global L^p solutions of eqn. (1) (see Theorem 1). Moreover, we also discuss the uniqueness of solutions and the convergence of the Tonelli approximate sequence (see Theorem 2 and Theorem 3). Finally, we offer an example for infinite system of scalar nonlinear integral equations.

2. Main Theorems

Consider the integral operator defined by

$$A(x)\ (t) = x_0(t) + \int_0^t K(t,s)\, f(s, x(s))\, ds. \qquad \dots (2)$$

Lemma 1 — Let conditions (H_1)-(H_4) be satisfied. Then A is a bounded and continuous operator from $L^p(J,E)$ into $L^p(J,E)$.

Proof : We have $Ax = x_0 + K_0 F$, where F is the Nemitskii operator $(Fx)\ (t) = f(t, x(t))$ and K_0 is the linear integral operator

$$(K_0 y)\ (t) = \int_0^t K(t,s)\, y(s)\, ds.$$

Using standard argument[1, 2], it can be shown that condition (H_3) implies that F is a bounded and continuous operator from $L^p(J,E)$ into $L^q(J,E)$. On the other hand, (H_1) and (H_4) imply that K_0 is a bounded linear operator from $L^q(J,E)$ into $L^p(J,E)$. Hence, by (H_2), A is a bounded and continuous operator from $L^p(J,E)$ into $L^p(J,E)$. □

Let (H_1)-(H_4) be satisfied and consider the Tonelli approximate sequence $\{x_n^*(t)\}$ of eqn. (1), i.e., $x_n^*(t)$ (n = 1, 2, 3, ...) satisfy

$$x_n^*(t) = \begin{cases} x_0(t), & \text{for } 0 \le t \le d/n; \\ x_0(t) + \displaystyle\int_0^{t-(d/n)} K(t,s)\, f(s, x_n^*(s))\, ds, & \text{for } d/n < t \le d. \end{cases} \qquad \dots (3)$$

$x_n^*(t)$ can be defined by induction as follows : first let $x_n^*(t) = x_0(t)$ for $0 \le t \le d/n$.

Assume that $x_n^*(t)$ has been defined already on $0 \le t \le i\, d/n$. Then, we define $x_n^*(t)$ on $id/n < t \le (i+1)\, d/n$ by

$$x_n^*(t) = x_0(t) + \int_0^{t-(d/n)} K(t,s) f(s, x_n^*(s))\, ds.$$

It is clear that $x_n^*(t) \in L^p(J,E)$, $(n = 1, 2, 3, \ldots)$.

Lemma 2 — Let conditions (H_1)-(H_4) be satisfied. Assume that

$$\| x_0 \|_p^p + d^{1+(p/q')-2p/m} \| c \|_q^p Q^p < 3^{-p} \| c \|_q^p \int_0^\infty (\| c \|_q^p + b^p x^{p/q})^{-1}\, dx, \qquad \ldots (4)$$

where

$$Q = \left(\int\int_{J \times J} \| K(t,s) \|^m\, ds\, dt \right)^{1/m} < \infty. \qquad \ldots (5)$$

Then the following conclusions hold :

(a) there exists a $g^* \in L^p(J, R_+)$ such that

$$\| x_n^*(t) \| \le g^*(t), \quad \text{for } t \in J \ (n = 1, 2, 3, \ldots);$$

(b) $\| x_n^* - Ax_n^* \|_p \to 0$ as $n \to \infty$;

(c) $\int_J \| x_n^*(t+h) - x_n^*(t) \|^p\, dt \to 0$ as $h \to 0$ uniformly in n $(n = 1, 2, 3, \ldots)$.

PROOF : Consider the scalar integral equation

$$z(t) = \int_0^t 3^p d^{p(1/q')-(1/m)} \left(\| c \|_q^p + b^p (z(s))^{p/q} \right) g_1(s)\, ds, \qquad \ldots (6)$$

where

$$g_1(t) = d^{p(1/m-1/q')} \| c \|_q^{-p} \| x_0(t) \|^p + (g(t))^p, \text{ for } t \in J \qquad \ldots (7)$$

and

$$g(t) = \left(\int_J \| K(t,s) \|^m\, ds \right)^{1/m}, \text{ for } t \in J. \qquad \ldots (8)$$

Using a well known inequality

$$\left(\int_J | y(t) |^{p_1}\, dt \right)^{1/p_1} \le d^{(1/p_1 - 1/p_2)} \left(\int_J | y(t) |^{p_2}\, dt \right)^{1/p_2},$$

$$\text{for} \quad y \in L^{p_2}(J,R),\ p_2 \ge p_1 > 0, \qquad \ldots (9)$$

we see that $g \in L^p(J, R_+)$ and

$$\| g \|_p \leq d^{(1/p - 1/m)} Q, \qquad \dots (10)$$

and so $g_1 \in L(J, R_+)$. We now show that eqn. (6) has exactly one solution $\bar{z}(t)$ in $C(J, R_+)$. In fact, let

$$F(z) = 3^{-p} d^{p((1/m)-(1/q'))} \int_0^z (\| c \|_q^p + b^p x^{p/q})^{-1} dx \qquad \dots (11)$$

and

$$G(t) = \int_0^t g_1(s)\, ds. \qquad \dots (12)$$

Then $F : [0, \infty) \to [0, F(\infty))$ is a strictly increasing and continuously differentiable function and $G : J \to [0, G(d)]$ is a nondecreasing and absolutely continuous function. It is easy to see from (4) and (10)-(12) that $G(d) < F(\infty)$, so, there exists a unique $0 < z_1 < \infty$ such that $G(d) = F(z_1)$. Let

$$\bar{z}(t) = F^{-1}(G(t)), \text{ for } t \in J. \qquad \dots (13)$$

Then $\bar{z} : J \to [0, z_1]$ is a nondecreasing and absolutely continuous function which satisfies

$$F'(\bar{z}(t))\, \bar{z}'(t) = G'(t), \text{ for a.e. } t \in J, \qquad \dots (14)$$

i.e.

$$\bar{z}'(t) = 3^p d^{p((1/q')-(1/m))} (\| c \|_q^p + b^p [\bar{z}(t)]^{p/q})\, g_1(t), \text{ for a.e. } t \in J. \qquad \dots (15)$$

By (13), we have $\bar{z}(0) = 0$, and consequently, (15) implies that function $\bar{z}(t)$ defined by (13) is a solution of eqn. (6) in $C(J, R_+)$. Conversely, if $\bar{z}(t)$ is a solution of eqn. (6) in $C(J, R_+)$, then $\bar{z}(t)$ is nondecreasing and absolutely continuous on J and (15) holds, i.e. (14) is satisfied. This means that

$$\frac{d}{dt} F(\bar{z}(t)) = G'(t), \text{ for a.e. } t \in J$$

which implies by virtue of $F(\bar{z}(0)) = G(0) = 0$ that $F(\bar{z}(t)) = G(t)$ for $t \in J$, i.e. $\bar{z}(t)$ is the function defined by (13).

On account of (3), (9) and (H_1)-(H_4), we have

$$\| x_n^*(t) \| \leq \| x_0(t) \| + d^{(1/q')-(1/m)} g(t) \left\{ \| c \|_q + b \left(\int_0^{t-(d/n)} \| x_n^*(s) \|^p ds \right)^{1/q} \right\},$$

$t \in J$

$$\dots (16)$$

and

$$\| x_n^*(t) - (Ax_n^*)(t) \| = \| x_n^*(t) - x_0(t) - \int_0^t K(t,s) f(s, x_n^*(s))\, ds \|$$

$$\leq d^{(1/q') - (1/m)} g_n(t) \left\{ \| c \|_q + b \left(\int_0^t \| x_n^*(s) \|^p ds \right)^{1/q} \right\}, \quad t \in J, \quad \text{... (17)}$$

where $g(t)$ is defined by (8) and

$$g_n(t) = \begin{cases} g(t), & \text{for } 0 \leq t \leq d/n; \\ \left(\int_{t-(d/n)}^t \| K(t,s) \|^m ds \right)^{1/m}, & \text{for } d/n < t \leq d. \end{cases} \quad \text{... (18)}$$

Let

$$u_n(t) = \int_0^t \| x_n^*(s) \|^p ds, \quad \text{for } t \in J \ (n = 1, 2, 3, \ldots). \quad \text{... (19)}$$

Then $u_n \in C(J, R_+)$ and, by (16),

$$u_n(t) \leq 3^p \int_0^t \| x_0(s) \|^p ds$$

$$+ 3^p \int_0^t d^{p((1/q') - (1/m))} \left(\| c \|_q^p + b^p (u_n(s))^{p/q} \right) (g(s))^p ds$$

$$\leq \int_0^t 3^p d^{p((1/q') - (1/m))} \left(\| c \|_q^p + b^p (u_n(s))^{p/q} \right) g_1(s)\, ds, \quad \text{... (20)}$$

$$\text{for } t \in J \ (n = 1, 2, 3, \ldots).$$

We are now in a position to show that

$$u_n(t) \leq \bar{z}(t), \quad \text{for } t \in J \ (n = 1, 2, 3, \ldots). \quad \text{... (21)}$$

Since

$$\bar{z}(t) = \int_0^t 3^p d^{p((1/q') - (1/m))} \left(\| c \|_q^p + b^p (\bar{z}(s))^{p/q} \right) g_1(s)\, ds, \text{ for } t \in J, \quad \text{... (22)}$$

we have from (3) and (19) that, for $0 \leq t \leq d/n$,

$$u_n(t) = \int_0^t \| x_0(s) \|^p ds \leq 3^p \int_0^t \| x_0(s) \|^p ds \leq \bar{z}(t),$$

i.e. (21) holds for $0 \le t \le d/n$. Assume that (21) holds for $0 \le t \le id/n$. Then, for $id/n < t \le (i+1)\, d/n$, (16) implies

$$\| x_n^*(t) \| \le \| x_0(t) \| + d^{((1/q') - (1/m))}\, g(t)\, (\| c \|_q + b[u_n(t-(d/n))]^{1/q})$$
$$\le \| x_0(t) \| + d^{((1/q') - (1/m))}\, g(t)\, (\| c \|_q + b[\bar{z}(t-(d/n))]^{1/q})$$
$$\le \| x_0(t) \| + d^{((1/q') - (1/m))}\, g(t)\, (\| c \|_q + b[\bar{z}(t)]^{1/q}),$$

and so

$$u_n(t) \le 3^p \int_0^t \| x_0(s) \|^p\, ds + \int_0^t 3^p\, d^{p((1/q') - (1/m))}$$
$$(\| c \|_q^p + b^p\, [\bar{z}(s)]^{p/q})\, (g(s))^p\, ds$$
$$\le \int_0^t 3^p\, d^{p(1/q' - 1/m)}\, (\| c \|_q^p + b^p\, [\bar{z}(s)]^{p/q})\, g_1(s)\, ds = \bar{z}(t).$$

Hence, by induction, (21) holds for all $t \in J$, and consequently,

$$u_n(t) \le z_1 = \bar{z}(d), \text{ for } t \in J \quad (m = 1, 2, 3, ..). \qquad ... (23)$$

It follows from (23) and (16) that

$$\| x_n^*(t) \| \le \| x_0(t) \| + d^{((1/q') - (1/m))}\, g(t)\, (\| c \|_q + b z_1^{1/q}) = g^*(t),$$
$$\text{for } t \in J \ (n = 1, 2, 3, ...) \text{ and } g^* \in L^p(J, R_+).$$

Thus, conclusion (a) is proved. On the other hand, (17), (18) and (23) imply that

$$\| x_n^* - A x_n^* \|_p \le d^{((1/q') - (1/m))}\, \| g_n \|_p\, (\| c \|_q + b z_1^{1/q})$$

and

$$\| g_n \|_p^p \le \int_0^{d/n} [g(t)]^p\, dt$$
$$+ d^{1-(p/m)} \left(\int_{d/n}^{d} dt \int_{t-(d/n)}^{t} \| K(t,s) \|^m\, ds \right)^{p/m} \to 0 \ (n \to \infty),$$

and consequently, conclusion (b) holds. Finally, we prove conclusion (c). Obviously, we may consider $h > 0$ only. By virtue of (H_1)-(H_4), (3), (9) and (23), it is easy to see that

$$\| x_n^*(t+h) - x_n^*(t) \| \le w_n(t,h), \quad (h > 0),$$

where

$$w_n(t,h) = \begin{cases} \| x_0(t+h) - x_0(t) \| \, d^{((1/q')-(1/m))} \left(\| c \|_q + b z_1^{1/q} \right) \left(\displaystyle\int_0^h \| K(t+h,s) \|^m ds \right)^{1/m}, \\ \qquad \text{for } 0 \le t \le d/n; \\ \| x_0(t+h) - x_0(t) \| + d^{((1/q')-(1/m))} \left(\| c \|_q + b z_1^{1/q} \right) \cdot \\ \times \left\{ \left(\displaystyle\int_0^d \| K(t+h,s) - K(t,s) \|^m ds \right)^{1/m} + \left(\displaystyle\int_{t-(d/n)}^{t+h-(d/n)} \| K(t+h,s) \|^m ds \right)^{1/m} \right\}, \\ \qquad \text{for } d/n < t \le d, \end{cases}$$

and so

$$\int_J [w_n(t,h)]^p \, dt \le 3^p \int_0^d \| x_0(t+h) - x_0(t) \|^p dt + 3^p d^{1+(p/q')-2(p/m)}$$

$$\times \left(\| c \|_q + b z_1^{1/q} \right)^p \cdot \left\{ \left(\int_0^d dt \int_0^h \| K(t,s) \|^m ds \right)^{p/m} \right.$$

$$+ \left(\int_0^d dt \int_0^d \| K(t+h,s) - K(t,s) \|^m ds \right)^{p/m}$$

$$\left. + \left(\int_{h+(d/n)}^d dt \int_{t-h-(d/n)}^{t-(d/n)} \| K(t,s) \|^m ds \right)^{p/m} \right\}.$$

(notice that, by definition, $x_n^*(t+h) = x_0(t+h) = 0$ and $K(t+h, s) = 0$ if $t+h > d$. As usual, the same symbol 0 denotes the zero number or the zero element of E or the zero element of another Banach space if no confusion is arisen). Observing the area of the integral domain for the double integral

$$\int_{h+(d/n)}^d dt \int_{t-h-(d/n)}^{t-(d/n)} \| K(t,s) \|^m ds$$

is less than $\sqrt{2}\, dh$, we conclude that

$$\lim_{h \to 0^+} \int_J [w_n(t,h)]^p \, dt = 0 \text{ uniformly in } n \ (n = 1, 2, 3, \ldots).$$

Hence, conclusion (c) is proved. □

Remark 1 : Inequality (4) is certainly satisfied if $p \le q$, since in this case

$$\int_0^\infty (\| c \|_q^p + b^p x^{p/q})^{-1} \, dx = \infty.$$

We need also to use the following two known lemmas (see Lemma 1 and Lemma 2 in Januszewski[1]).

Lemma 3 — Let V be a countable set of strongly measurable functions $J \to E$ such that there exists $M \in L(J, R_+)$ such that $\| x(t) \| \le M(t)$ for all $x \in V$ and $t \in J$. Then $\alpha(V(t))$ is integrable on J and

$$\alpha\left(\left\{ \int_J x(t) \, dt : x \in V \right\}\right) \le 2 \int_J \alpha(V(t)) \, dt,$$

where $V(t) = \{x(t) : x \in V\} \subset E$ and α denotes the Kuratowski measure of noncompactness in E.

Lemma 4 — Let V be a countable set of strongly measurable functions $J \to E$ such that

(a) there exists $M \in L^p(J, R_+)$ $(p > 1)$ such that $\| x(t) \| \le M(t)$ for all $x \in V$ and $t \in J$;

(b) $$\lim_{h \to 0} \left(\sup_{x \in V} \int_J \| x(t+h) - x(t) \|^p \, dt \right) = 0.$$

Then $$\alpha_p(V) \le 2 \left(\int_J [\alpha(V(t))]^p \, dt \right)^{1/p},$$

where α_p denotes the Kuratowski measure of noncompactness in space $L^p(J, E)$.

Theorem 1 — Let conditions (H_1)-(H_5) be satisfied. Assume that inequality (4) and the following inequality hold :

$$2d^{(1/p - 1/m)} Q \| h \|_k < 1. \qquad \text{... (24)}$$

Then eqn. (1) has at least one solution in $L^p(J, E)$.

PROOF : Let $V = \{x_n^* : n = 1, 2, 3, \ldots\} \subset L^p(J, E)$, where $\{x_n^*(t)\}$ is the Tonelli approximate sequence of eqn. (1) defined by (3). Then $V(t) = \{x_n^* : n = 1, 2, 3, \ldots\} \subset E$ for $t \in J$. By virtue of Lemma 4 and conclusions (a) and (c) of Lemma 2, we have

$$\alpha_p(V) \le 2 \left(\int_J [\alpha(V(t))]^p \, dt \right)^{1/p}. \qquad \text{... (25)}$$

On the other hand, (3) implies that, for $0 < t \le d$,

$$\alpha\,(V(t)) \le \alpha\left(\left\{\int_0^t K(t,s)\,f(s,x_n^*(s))\,ds : n > dr^{-1}\right\}\right)$$

$$+\ \alpha\left(\left\{\int_{t-(d/n)}^{t} K(t,s)\,f(s,x_n^*(s))\,ds : n > dr^{-1}\right\}\right). \quad \ldots (26)$$

Since, by conclusion (a) of Lemma 2,

$$\| K(t,s)\,f(s,x_n^*(s))\| \le \| K(t,s)\|\,(c(s) + b[g^*(s)]^{p/q}) \in L(J,R_+) \quad \ldots (27)$$

as a function of s for a.e. $t \in J$,
we have on account of Lemma 3 and condition (H_5) that, for a.e. $t \in J$,

$$\alpha\left(\left\{\int_0^t K(t,s)\,f(s,x_n^*(s))\,ds : n > dr^{-1}\right\}\right)$$

$$\le 2\int_0^t \alpha\left(\left\{K(t,s)\,f(s,x_n^*(s)) : n > dr^{-1}\right\}\right) ds$$

$$\le 2\int_0^t \| K(t,s)\|\,\alpha_1\,(f(s,V(s))\,ds$$

$$\le 2\int_0^t \| K(t,s)\|\,h(s)\,\alpha(V(s))\,ds$$

$$\le 2g\,(t)\,\| h\|_k\left(\int_J [\alpha(V(s))]^p\,ds\right)^{1/p},$$

where $g(t)$ is defined by (8). Moreover, it is easy to see from (27) that

$$\alpha\left(\left\{\int_{t-(d/n)}^{t} K(t,s)\,f(s,x_n^*(s))\,ds : n > dr^{-1}\right\}\right) = 0, \text{ for a.e. } t \in J.$$

Hence, (26) implies

$$\alpha(V(t)) \le 2g(t)\,\| h\|_k\left(\int_J [\alpha(V(s))]^p\,ds\right)^{1/p}, \text{ for a.e. } t \in J,$$

and so

$$\left(\int_J [\alpha(V(t))]^p\,dt\right)^{1/p} \le 2d^{((1/p)-(1/m))}\,Q\,\| h\|_k\left(\int_J [\alpha(V(s))]^p\,ds\right)^{1/p},$$

which implies by virtue of (24) that

$$\left(\int_J [\alpha(V(t))]^p \, dt \right)^{1/p} = 0. \quad \text{... (28)}$$

It follows from (25) and (28) that $\alpha_p(V) = 0$, i.e. $V = \{x_n^*\}$ is relatively compact in $L^p(J, E)$, so, there is a subsequence $\{x_{ni}^*\}$ which converges in $L^p(J, E)$ to some $x^* \in L^p(J, E)$. Finally, taking limits along n_i in conclusion (b) of Lemma 2 and observing the continuity of A by virtue of Lemma 1, we obtain $x^* = Ax^*$, and the theorem is proved. □

Remark 2 : From the proof of Theorem 2 in Januszewski[1] we see that he used the maximal continuous solution $z_\varepsilon(t)$ of a scalar integral equation of Volterra type (see Januszewski[1], p. 689). According to the theory of differential and integral equations, this maximal solution exists usually on some small interval $J_0 = [0, a]$ $(0 < a < d)$. So, Januszewski's method obtains only a solution of eqn. (1) in $L^p(J_0, E)$, i.e. a local L^p solution. Now, in this paper, we consider a different scalar integral equation (6) and prove that it has a unique nonnegative continuous solution $\bar{z}(t)$ on the entire interval $J = [0, d]$ when inequality (4) is satisfied. Hence, after establishing Lemma 2, in Theorem 1 we obtain a solution of eqn. (1) in $L^p(J, E)$, i.e. a global L^p solution.

Let us list one more condition :

(H_6) there exists a $h \in L^k(J, R_+)$ such that $\| f(t, x) - f(t, y) \| \le h(t) \| x - y \|$ for any $t \in J$ and $x, y \in E$.

Theorem 2 — Let conditions (H_1)-(H_4) and (H_6) be satisfied. Assume that inequalities (4) and (24) hold. Then eqn. (1) has exactly one solution $x^*(t)$ in $L^p(J, E)$. Moreover, the Tonelli approximate sequence $\{x_n^*(t)\}$ defined by (3) converges in norm of $L^p(J, E)$ to $x^*(t)$ as $n \to \infty$.

PROOF : It is clear that (H_6) implies (H_5), so, by Theorem 1, eqn. (1) has a solution $x^*(t)$ in $L^p(J, E)$. Let $y^*(t)$ be any solution of eqn. (1) in $L^p(J, E)$, and let $t^* = \sup \{t \in J : x^*(s) = y^*(s) \text{ for a.e. } 0 \le s \le t\}$. We now show that $t^* = d$. In fact, if $t^* < d$, then we can choose t' with $t^* < t' < d$ such that

$$d^{(1/p - 1/m)} \| h \|_k \left(\int\int_{J^* \times J^*} \| K(t, s) \|^m \, ds \, dt \right)^{1/m} < 1, \quad \text{... (29)}$$

where $J^* = [t^*, t']$. By virtue of (H_6), we have, for $t \in J^*$,

$$\| x^*(t) - y^*(t) \| = \left\| \int_{t^*}^{t} K(t, s) [f(s, x^*(s)) - f(s, y^*(s))] \, ds \right\|$$

$$\le \int_{t^*}^{t} \| K(t, s) \| \, h(s) \, \| x^*(s) - y^*(s) \| \, ds$$

$$\leq \| h \|_k \left(\int_{J^*} \| x^*(s) - y^*(s) \|^p ds \right)^{1/p} \left(\int_{J^*} \| K(t,s) \|^m ds \right)^{1/m},$$

and therefore

$$\left(\int_{J^*} \| x^*(t) - y^*(t) \|^p dt \right)^{1/p}$$

$$\leq \| h \|_k \left(\int_{J^*} \| x^*(s) - y^*(s) \|^p ds \right)^{1/p}.$$

$$\times \, d^{((1/p)-(1/m))} \left(\int\int_{J^* \times J^*} \| K(t,s) \|^m ds\, dt \right)^{1/m},$$

which implies on account of (29) that

$$\left(\int_{J^*} \| x^*(t) - y^*(t) \|^p dt \right)^{1/p} = 0.$$

Consequently, $x^*(t) = y^*(t)$ for a.e. $t \in J^*$, and this contradicts the definition of t^*. Hence $t^* = d$ and the uniqueness of solutions in $L^p(J,E)$ of eqn. (1) is proved. Finally, we show that

$$\| x_n^* - x^* \|_p \to 0 \qquad (n \to \infty). \qquad \text{... (30)}$$

If (30) is not true, then there exists an $\varepsilon > 0$ and a subsequence $\{x_{ni}^*\}$ such that

$$\| x_{ni}^* - x^* \|_p \geq \varepsilon \quad (i = 1, 2, 3, \ldots). \qquad \text{... (31)}$$

By the proof of Theorem 1, $\{x_n^*\}$ is relatively compact in $L^p(J,E)$, so, $\{x_{ni}^*\}$ contains a subsequence which converges in $L^p(J,E)$ to some $\bar{x} \in L^p(J,E)$. No loss of generality, we may assume that $\{x_{ni}^*\}$ itself converges to $\bar{x}$, i.e.

$$\| x_{ni}^* - \bar{x} \|_p \to 0 \qquad (i \to \infty). \qquad \text{... (32)}$$

By conclusion (b) of Lemma 2, Lemma 1 and (32), we have $\bar{x} = A\bar{x}$, i,.e. $\bar{x}$ is a solution in $L^p(J,E)$ of eqn. (1). Hence, the uniqueness just proved implies that $\bar{x} = x^*$, which contradicts (31) and (32). The proof is complete. □

In the following, we use some partial ordering in E to replace the condition about the Kuratowski measure of noncompactness. Let E be partially ordered by a cone of E, i.e. $x \leq y$ iff $y - x \leq P$. P is said to be normal if there exists a positive constant N such that $0 \leq x \leq y$ implies $\| x \| \leq N \| y \|$, and P is said to be fully regular if every nondecreasing and bounded in norm sequence has a limit in E. It is well known, the full regularity implies the normality and, conversely, the normality implies the full regularity when E is reflexive (see Guo and Lakshmikantham[3], Section 1.2). Now, we formulate a condition :

(H_7) P is fully regular, $x_0(t) \geq 0$ for $t \in J$, $f(t,x) \geq 0$ and $f(t,x)$ is nondecreasing in x for $t \in J$ and $x \geq 0$, and $K(t,s) \geq 0$ for $t, s \in J$ with $t \geq s$ (i.e. $K(t,s)x \geq 0$ for $x \geq 0$ and $t, s \in J$ with $t \geq s$).

Theorem 3 — Let conditions (H_1)-(H_4) and (H_7) be satisfied. Assume that inequality (4) holds. Then, the Tonelli approximate sequence $\{x_n^*(t)\}$ defined by (3) is nondecreasing and converges in norm of $L^p(J,E)$ to some $x^*(t)$, and $x^*(t)$ is the minimal non-negative solution of eqn. (1) in $L^p(J,E)$.

PROOF : By Lemma 2, there is a $g^* \in L^p(J,R_+)$ such that

$$\|x_n^*(t)\| \le g^*(t), \text{ for } t \in J \quad (n = 1, 2, 3, ...). \qquad ... (33)$$

We now show that $\{x_n^*(t)\}$ is nondecreasing, i.e.

$$x_1^*(t) \le x_2^*(t) \le \ldots \le x_n^*(t) \le \ldots, \text{ for } t \in J. \qquad ... (34)$$

In fact, condition (H_7) and the definition of $x_n^*(t)$ (see (3)) imply

$$x_n^*(t) \ge x_0(t) \ge 0, \quad \text{for } t \in J \quad (n = 1, 2, 3, ..). \qquad ... (35)$$

Consequently, for $0 \le t \le d/(n+1)$, we have $x_n^*(t) = x_0(t) = x_{n+1}^*(t)$, and, for $d/(n+1) < t \le d/n$, we have $x_n^*(t) = x_0(t) \le x_{n+1}^*(t)$. Hence, $x_n^*(t) \le x_{n+1}^*(t)$ for $0 \le t \le d/n$. Assume that $x_n^*(t) \le x_{n+1}^*(t)$ for $0 \le t \le id/n$. Then, since $f(t, x)$ is nondecreasing in x, we have, for $id/n < t \le (i+1)d/n$,

$$x_n^*(t) = x_0(t) + \int_0^{t-(d/n)} K(t,s) f(s, x_n^*(s))\, ds$$

$$\le x_0(t) + \int_0^{t-(d/n)} K(t,s) f(s, x_{n+1}^*(s))\, ds$$

$$\le x_0(t) + \int_0^{t-(d/(n+1))} K(t,s) f(s, x_{n+1}^*(s))\, ds = x_{n+1}^*(t).$$

Thus, by induction, $x_n^*(t) \le x_{n+1}^*(t)$ for all $t \in J$, i.e. (34) holds. It follows from (33), (34) and the full regularity of P that the following limit exists :

$$\lim_{n \to \infty} x_n^*(t) \doteq x^*(t), \quad \text{for a.e. } t \in J, \qquad ... (36)$$

and $\|x^*(t)\| \le g^*(t)$ for a.e. $t \in J$, hence $x^* \in L^p(J,E)$ and, by (33),

$$\|x_n^*(t) - x^*(t)\|^p \le 2^p [g^*(t)]^p, \text{ for a.e. } t \in J. \qquad ... (37)$$

Now, (36), (37) and the dominated convergence theorem imply

$$\lim_{n \to \infty} \|x_n^* - x^*\|_p^p = \lim_{n \to \infty} \int_J \|x_n^*(t) - x^*(t)\|^p\, dt = 0. \qquad ... (38)$$

It follows from conclusion (b) of Lemma 2, (38) and the continuity of A that x^* =

Ax^*, i.e. $x^*(t)$ is a solution of eqn. (1) and $x^*(t)$ is non-negative on account of (35) and (36). Let $\bar{x} \in L^p(J,E)$ be any solution of eqn. (1) satisfying $\bar{x}(t) \geq 0$ for $t \in J$. Then, $\bar{x}(t) \geq x_0(t)$ for $t \in J$, and so $x_n(t) = x_0(t) \leq \bar{x}(t)$ for $0 \leq t \leq d/n$. Assume that $x_n(t) \leq \bar{x}(t)$ for $0 \leq t \leq id/n$. Then, for $id/n < t \leq (i+1)d/n$, we have

$$x_n^*(t) = x_0(t) + \int_0^{t-(d/n)} K(t,s) f(s, x_n^*(s))\, ds \leq x_0(t) + \int_0^{t-(d/n)} K(t,s) f(s, \bar{x}(s))\, ds$$

$$\leq x_0(t) + \int_0^t K(t,s) f(s, \bar{x}(s))\, ds = \bar{x}(t).$$

Hence, by induction, $x_n(t) \leq \bar{x}(t)$ for all $t \in J$ $(n = 1, 2, 3, \ldots)$. Taking limit as $n \to \infty$, we get $x^*(t) \leq \bar{x}(t)$, and the theorem is proved. □

Remark 3 : In the same way, we can discuss the solutions in $L_r^p(J,E)$ of the nonlinear integral equation

$$x(t) = x_0(t) + \int_0^t r(s) K(t,s) f(s, x(s))\, ds,$$

where $r : J \to R_+$ is a bounded and integrable function such that mes $\{t \in J : r(t) = 0\} = 0$ and $L_r^p(J,E)$ denotes the Banach space of all strongly measurable functions $x : J \to E$ with $\int_J r(t) \|x(t)\|^p dt < \infty$, provided with the norm

$$\|x\|_{p,r} = \left(\int_J r(t) \|x(t)\|^p dt \right)^{1/p}.$$

3. An Example

Consider the infinite system of nonlinear integral equations :

$$x_n(t) = n^{-1/2}(t-2)t^{-1/5} + \int_0^t e^{-(2+ts)}(t-s)^{-1/3} \sin(t-2s)\, [n^{-1} s^{-1/4}$$

$$+ s x_{n+1}(s) - s^2 \cos(ns - x_{2n}(s))]\, ds, \quad 0 \leq t \leq 1 \quad (n = 1, 2, 3, \ldots). \qquad \ldots (39)$$

Conclusion — System (39) has exactly one solution $\{\bar{x}_n(t)\}$ satisfying

$$\int_0^1 (\sup_n |\bar{x}_n(t)|)^2 dt < \infty.$$

Proof : Let $E = l^\infty = \{x = (x_1, \ldots, x_n, \ldots) : \sup_n |x_n| < \infty\}$ with norm $\|x\| = \sup_n |x_n|$. Then, system (39) can be regarded as an equation of the form (1) in E, where $x = (x_1, \ldots, x_n, \ldots)$, $J = [0, 1]$,

$$x_0(t) = ((t-2)\,t^{-1/5}, \ldots, n^{-1/2}(t-2)\,t^{-1/5}, \ldots), \qquad \ldots (40)$$

$$f(t,x) = (f_1(t,x), \ldots, f_n(t,x), \ldots) \text{ with}$$

$$f_n(t,x) = n^{-1}t^{-1/4} + tx_{n+1} - t^2\cos(nt - x_{2n}) \qquad \ldots (41)$$

and

$$K(t,\ s) = \begin{cases} e^{-(2+ts)}(t-s)^{-1/3}\sin(t-2s), & \text{for } t \geq s; \\ 0, & \text{for } t < s. \end{cases} \qquad \ldots (42)$$

Let $p = q = 2$. Then $m = 2$, $k = \infty$ and (H_1) is satisfied. From (40) we see that

$$\int_0^1 \| x_0(t) \|^2\, dt = \int_0^1 (t-2)^2 t^{-2/5}\, dt < \infty,$$

i.e. $x_0 \in L^2(J, E)$ and (H_2) is satisfied. It is clear that (41) implies

$$\| f(t,x) \| \leq (t^2 + t^{-1/4}) + \| x \|, \quad \text{for } t \in J \text{ and } x \in E,$$

and so (H_3) is satisfied for $E_1 = E = l^\infty$, $c(t) = t^2 + t^{-1/4} \in L^2(J, R_+)$ and $b = 1$. (H_4) is also satisfied, since, by (42),

$$\iint\limits_{J\times J} \| K(t,s) \|^2\, ds\, dt = \int_0^1 dt \int_0^t e^{-2(2+ts)}(t-s)^{-2/3}\sin^2(t-2s)\, ds$$

$$\leq e^{-4}\int_0^1 dt \int_0^t (t-s)^{-2/3}\, ds = (9/4)\, e^{-4} < \infty. \qquad \ldots (43)$$

On account of $p = q = 2$ and Remark 1, we see that (4) is satisfied. Using mean value theorem, (41) implies

$$| f_n(t,x) - f_n(t,y) | \leq t\, | x_{n+1} - y_{n+1} | + t^2\, | \cos(nt - x_{2n}) - \cos(nt - y_{2n}) |$$

$$\leq | x_{n+1} - y_{n+1} | + | x_{2n} - y_{2n} | \leq 2\, \| x - y \|,$$

and so

$$\| f(t,x) - f(t,y) \| \leq 2\, \| x - y \|, \text{ for } \ t \in J \text{ and } x,\ y \in E.$$

Consequently, (H_6) is satisfied for $h(t) \equiv 2$. It is clear that (24) is also satisfied since $d = 1$, $p = m = 2$, $\| h \|_k = 2$ and, by (43), $Q \leq (3/2)\, e^{-2}$. Hence, our conclusion follows from Theorem 2. □

Remark 4 : Let the Tonelli approximate sequence for system (39) be $\{x^*_{n,m}(t)\}$, i.e.

$$x^*_{n,m}(t) = \begin{cases} n^{-1/2}(t-2)\,t^{-1/5}, & \text{for } 0 \le t \le 1/m; \\ n^{-1/2}(t-2)\,t^{-1/5} + \int_0^{t-(1/m)} e^{-(2+ts)}(t-s)^{-1/3}\sin(t-2s) \\ \quad \times [n^{-1}s^{-1/4} + s x^*_{n+1,m}(s) - s^2\cos(ns - x^*_{2n,m}(s))]\,ds, \\ \quad \text{for } 1/m < t \le 1 \quad (n, m = 1, 2, 3, \ldots). \end{cases}$$

Then, Theorem 2 implies that

$$\int_0^1 (\sup_n |x^*_{n,m}(t) - \overline{x}_n(t)|)^2\,dt \to 0 \quad \text{as} \quad m \to \infty,$$

and consequentlty, there is a subsequence $\{m_i\} \subset \{m\}$ such that

$$x^*_{n,m_i}(t) \to \overline{x}_n(t) \quad \text{as } i \to \infty \text{ uniformly in } n \text{ for a.e. } t \in [0, 1].$$

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