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STRONG A-SUMMABILITY AND A-STATISTICAL CONVERGENCE

K. DEMIRCI

Department of Mathematics, Ankara University, Tandoǧan, 06100, Ankara, Turkey

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In this paper, we extend the definition of strong A-summability to a definition of strong A-summability with respect to an Orlicz function when A is a nonnegative regular matrix summability method. Via the ideal in l_∞, we show that strong A-summability with respect to an Orlicz function which satisfies Δ_2-condition and strong A-summability and A-statistical convergence are equivalent on bounded sequences.

1. INTRODUCTION

Throughout the following we let e denote the sequence which is identically 1 and let

$$s = \{\text{all complex or real valued sequences}\}$$

$$l_\infty = \left\{ x \in s : \| x \| = \sup_n |x_n| < \infty \right\}.$$

If $x, y \in s$, we let xy denote the sequence $(x_k y_k)$.

An Orlicz function is a function $F : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $F(0) = 0$, $F(x) > 0$ for $x > 0$ and $F(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz F is replaced by

$$F(x + y) \leq F(x) + F(y)$$

then this function is called modulus function (Maddox[7], Ruckle[9]).

Let $A = (a_{nk})$ be a nonnegative regular matrix summability method. Following Connor[1] we define the following sequences spaces

$$W_0(A) = \left\{ x \in s : \lim_n \sum_{k=1}^{\infty} a_{nk} |x_k| = 0 \right\}$$

and

$$W(A) = \{x \in s : \text{ for some } L, x - Le \in W_0(A)\}.$$

If $x \in W(A)$, we say that x is strongly A-summable to L.

If K is a subset of positive integers and given $\varepsilon > 0$, we let $\chi_{K(x;\,\varepsilon)}$ denote the characteristic function of the set $K(x;\,\varepsilon) = \{k \in \mathbf{N} : |x_k| \geq \varepsilon\}$.

Recall that x is said to be A-statistically convergent to L if $\chi_{K(x-Le;\,\varepsilon)}$ is contained in $W_0(A)$ for every $\varepsilon > 0$. In this case we write $x \to L(AS)$ and $(AS) := \{x \in s :$ for some $L,\ x \to L(AS)\}$. (see Connor[1], Kolk[5]).

Note that if we take $A = C_1$ (the Cesàro matrix of order 1), then we get the original definition of statistical convergence (Fast[2], Fridy[3] and Salat[10]).

We now give the definition of strong A-summability with respect to an Orlicz function F by using an Orlicz function in the same fashion as Connor[1] and Kolk[5] define strong A-summability with respect to a modulus function.

Let F be an Orlicz function and $A = (a_{nk})$ is a nonnegative regular summability method. We define the following sequences spaces

$$W_0(A, F) = \left\{ x \in s : \lim_n \sum_{k=1}^{\infty} a_{nk} F(|x_k|) = 0 \right\}$$

and

$$W(A, F) = \{x \in s : \text{ for some } L,\ x - Le \in W_0(A, F)\}.$$

If $x \in W(A, F)$, we say that x is strongly A-summable to L with respect to an Orlicz function F.

An Orlicz function F is said to satisfy Δ_2-condition on for all values of u, if there exists a constant $H > 0$, such that

$$F(2u) \leq HF(u), \quad (u \geq 0).$$

The Δ_2-condition is equivalent to the satisfaction of inequality

$$F(tu) \leq HtF(u)$$

for all values of u and for $t > 1$ (Krasnoselkii and Rutitsky[6]).

2. The Main Results

In this section we establish some relations among $W(A)$, $W(A, F)$ and (AS) and show that all three of these notions are equivalent for bounded sequences.

Using the same technique as in Theorem 7 of Prashar and Choudhary[8], we immediately conclude that $W_0(A) \subseteq W_0(A, F)$ and $W(A) \subseteq W(A, F)$ for an Orlicz function F which satisfies Δ_2-condition.

Now we have the first result.

Lemma 1 — If F is an Orlicz function which satisfies Δ_2-condition, then $W_0(A, F) \bigcap l_\infty$ is an ideal in l_∞.

PROOF : Let $x \in W_0(A, F)$ and $y \in l_\infty$. We show that $xy \in W_0(A, F) \bigcap l_\infty$. Since $y \in l_\infty$, there exist $H_1 > 1$ such that $\| y \| < H_1$. In this case $|x_k y_k| < H_1 |x_k|$ for all k. Since F is nondecreasing and satisfies Δ_2-condition, we have

$$F(|x_k y_k|) < F(H_1 |x_k|) \leq H H_1 F(|x_k|), \quad (H > 0).$$

Since

$$\lim_n \sum_{k=1}^{\infty} a_{nk} F(|x_k|) = 0,$$

we get

$$\lim_n \sum_{k=1}^{\infty} a_{nk} F(|x_k y_k|) = 0.$$

Thus $xy \in W_0(A, F) \bigcap l_\infty$.

In the sequel we need the following lemmas :

Lemma 2 — Let M be an ideal in l_∞ and let $x \in l_\infty$. Then x is in the closure of M in l_∞ if and only if $\chi_{K(x;\,\varepsilon)} \in M$ for all $\varepsilon > 0$ (Connor[1]).

Lemma 3 — If A is a nonnegative regular matrix summability method, then $W_0(A) \bigcap l_\infty$ is a closed ideal of l_∞ (Freedman and Sember[4]).

Now we have the following :

Theorem 4 — Let x be a bounded sequence, F be an Orlicz function which satisfies Δ_2-condition and A be a nonnegative regular matrix summability method. Then $W(A, F) \bigcap l_\infty = W(A) \bigcap l_\infty$.

PROOF : It is enough to show that $W_0(A, F) \bigcap l_\infty = W_0(A) \bigcap l_\infty$. Since $W_0(A) \subseteq W_0(A, F)$, therefore $W_0(A) \bigcap l_\infty \subseteq W_0(A, F) \bigcap l_\infty$. We show that $W_0(A, F) \bigcap l_\infty \subseteq W_0(A) \bigcap l_\infty$. We note that

$$\sum_{k=1}^{\infty} a_{nk} F(\chi_{K(x;\,\varepsilon)}(k)) = F(1) \sum_{k=1}^{\infty} a_{nk} \chi_{K(x;\,\varepsilon)}(k) \qquad \ldots (1)$$

for all $n \in \mathbf{N}$. Let $x \in W_0(A, F) \bigcap l_\infty$ and $\varepsilon > 0$ be given. Now define the sequence $y \in l_\infty$ by

$$y_k := \begin{cases} \dfrac{1}{x_k}, & \text{if } |x_k| \geq \varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Since $xy = \chi_{K(x;\,\varepsilon)}$, $\chi_{K(x;\,\varepsilon)} \in W_0(A, F) \bigcap l_\infty$ hence

$$\lim_n \sum_{k=1}^{\infty} a_{nk} F(\chi_{K(x;\,\varepsilon)}(k)) = 0.$$

So, we have

$$\lim_n \sum_{k=1}^{\infty} a_{nk}\, \chi_{K(x;\,\varepsilon)}(k) = 0$$

from (1). Lemma 3 asserts that $x \in W_0(A) \bigcap l_\infty$. Thus $W_0(A, F) \bigcap l_\infty \subseteq W_0(A) \bigcap l_\infty$.

Finally we establish a relationship between strong A-summability with respect to an Orlicz function F which satisfies Δ_2-condition and A-statistical convergence when A is a nonnegative regular matrix summability method.

Theorem 5 — Let be F an Orlicz function which satisfies Δ_2-condition and A be a nonnegative regular matrix summability method. Then

(i) $W(A, F) \subset (AS)$

(ii) $(AS) \bigcap l_\infty \subset W(A, F)$

(iii) $(AS) \bigcap l_\infty = W(A, F) \bigcap l_\infty$.

PROOF : (i) Let $x \in W_0(A, F)$ and $y \in l_\infty$. Since $y \in l_\infty$, there exists $H_1 > 1$ such that $\| y \| < H_1$. Hence $|x_k y_k| < H_1 |x_k|$ for all k. Since F is nondecreasing and satisfies Δ_2-condition, we have

$$F(|x_k y_k|) < H H_1 F(|x_k|), \qquad (H > 0).$$

Hence

$$\sum_{k=1}^{\infty} a_{nk} F(|x_k y_k|) \leq H H_1 \sum_{k=1}^{\infty} a_{nk} F(|x_k|) \to 0, \ \ (n \to \infty),$$

which yields that

$$\lim_n \sum_{k=1}^{\infty} a_{nk} F(|x_k y_k|) = 0.$$

So, $xy \in W_0(A, F)$ and $\varepsilon > 0$ be given. Now define the sequence by y

$$y_k := \begin{cases} \dfrac{1}{x_k}, & \text{if } |x_k| \geq \varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4 asserts that $xy = \chi_{K(x,\varepsilon)} \in W_0(A, F) \bigcap l_\infty = W_0(A) \bigcap l_\infty$. This yields that x is A-statistically convergent to zero.

(ii) Let $x \in (AS) \bigcap l_\infty$. It follows from the definition that $\chi_{K(x-Le;\varepsilon)} \in W_0(A) \bigcap l_\infty$ for every $\varepsilon > 0$ where $K(x - Le; \varepsilon) = \{k \in \mathbf{N} : |x_k - L| \geq \varepsilon\}$. Now Lemmas 2 and 3 imply that $x - Le$ is strongly A-summable to 0 and hence by Theorem 4, $x \in W(A, F)$.

(iii) This is an immediate consequence of (i) and (ii).

Kolk[5] proved that if a sequence is A-statistically convergent to L, then it must have a subsequence which is convergent to L.

We now have the following :

Corollary 6 — If x is strongly A-summable to L with respect to an Orlicz function which satisfies Δ_2-condition, then x has a subsequence which is convergent to L.

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