PROXIMATE DEFICIENCY OF HOMOGENEOUS DIFFERENTIAL POLYNOMIALS

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We investigate the value distribution of homogeneous differential polynomials by means of proximate deficiency.

1. INTRODUCTION AND DEFINITIONS

Let f be a meromorphic function of finite order ρ_f. A function $\rho_f(r)$ is called a proximate order of f if the following hold :

(i) $\rho_f(r)$ is nonnegative and continuous for $r > r_0$, say,

(ii) $\rho_f(r)$ is differentiable for $r \geqq r_0$ except possibly at isolated points at which $\rho_f'(r+0)$ and $\rho_f'(r-0)$ exist,

(iii) $\lim_{r \to \infty} \rho_f(r) = \rho_f$,

(iv) $\lim_{r \to \infty} r\, \rho_f'(r) \log r = 0$ and

(v) $\limsup_{r \to \infty} \dfrac{T(r,f)}{r^{\rho_f(r)}} = 1$, where $T(r, f)$ is the Nevanlinna characteristic function of f.

In some general setting the existence of such a proximate order is proved in Lahiri[4]. In the paper we use without any explanation the standard notations $T(r, f)$, $m(r, f)$, $N(r, f)$, $m(r, a)$, $N(r, a)$, $S(r, f)$, $\delta(a, f)$, $\Theta(a, f)$, $\Delta(a, f)$ etc. of Nevanlinna theory (cf. Hayman[3]).

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Valiron[7] introduced the quantity $\delta_{\rho_f}(a,f) = 1 - \limsup_{r \to \infty} \frac{N(r,a)}{r^{\rho_f(r)}}$ in the value distribution theory which we may call the proximate deficiency of the value a. Using property (v) of the proximate order one can verify that $\delta(a,f) \leqq \delta_{\rho_f}(a,f)$ for every complex number a. Likewise we can define $\Theta_{\rho_f}(a,f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r,a)}{r^{\rho_f(r)}}$ and call this the proximate deficiency of the value a for distinct a-points. Clearly for every complex number a $\Theta(a,f) \leqq \Theta_{\rho_f}(a,f)$. From the second fundamental theorem it is not difficult to prove that the set $\{a : \Theta_{\rho_f}(a;f) > 0\}$ is countable and $\sum_a \Theta_{\rho_f}(a,f) \leqq 2$.

Definition 1 (cf. Doeringer[1]) — Let $n_{0j}, n_{1j}, ..., n_{kj}$ $(k \geqq 1)$ be nonnegative integers such that for each j, $\sum_{i=0}^{k} n_{ij} \geqq 1$. We call $M_j[f] = A_j(f)^{n_{0j}}(f^{(1)})^{n_{1j}} \ldots (f^{(k)})^{n_{kj}}$, where $T(r,A_j) = S(r,f)$, to be a differential monomial generated by f. The numbers $\gamma_{M_j} = \sum_{i=0}^{k} n_{ij}$ and $\Gamma_{M_j} = \sum_{i=0}^{k} (i+1)n_{ij}$ are called respectively the degree and weight of $M_j[f]$. The expression $P[f] = \sum_{j=1}^{l} M_j[f]$ is called a differential polynomial generated by f. The numbers $\gamma_P = \max_{1 \leqq j \leqq l} \gamma_{M_j}$, $\Gamma_P = \max_{1 \leqq j \leqq l} \Gamma_{M_j}$ are called respectively the degree and the weight of $P[f]$. If $\gamma_{M_j} = \gamma_P$ for $j = 1, 2, ..., l$, then $P[f]$ is called homogeneous.

Yang[8] proved the following theorem.

Theorem A — Let f be transcendental meromorphic with $N(r, f) + N(r, 1/f) = S(r, f)$ and $\gamma_{0j} < \gamma_P$ for $j = 1, 2, ..., l$. Then $\delta(a, P[f]) < 1$ for $a \neq 0, \infty$.

For homogeneous $P[f]$ Gopalakrishna and Bhoosnurmath[2] improved the above theorem and proved the following result.

Theorem B — Let f be a meromorphic function satisfying $\overline{N}(r, f) + \overline{N}(r, 1/f) = S(r, f)$ and $P[f]$ be homogeneous. Then $\Theta(a, P[f]) = 0$ for $a \neq 0, \infty$.

The purpose of the paper is to investigate the proximate deficiency of homogeneous differential polynomials and obtain some results in the direction of Theorem A and Theorem B.

Definition 2 (Yi[9]) — We denote by $n_p(r,a;f)$ the number of zeros of $f - a$ in $|z| \leqq r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\geqq p$ is counted exactly p times; and $N_p(r,a;f)$ is defined in terms of $n_p(r,a;f)$ in the usual way.

Definition 3 — We define $\delta^{p}_{\rho_f}(a,f) = 1 - \limsup_{r\to\infty} \frac{N_p(r,a;f)}{r^{\rho_f(r)}}$. Then

$$\delta_{\rho_f}(a,f) \leqq \delta^{p}_{\rho_f}(a,f) \leqq \delta^{p-1}_{\rho_f}(a,f) \leqq \dots \leqq \delta^{1}_{\rho_f}(a,f) = \Theta_{\rho_f}(a,f) \leqq 1$$

and

$$\sum_{a\neq\infty} \delta^{p}_{\rho_f}(a,f) + \Theta_{\rho_f}(\infty,f) \leqq \sum \Theta_{\rho_f}(\infty,f) \leqq 2.$$

Also for a complex number a we put

$$\Delta_{\rho_f}(a,f) = 1 - \liminf_{r\to\infty} \frac{N(r,a;f)}{r^{\rho_f(r)}}.$$

Definition 4 (Lahiri and Sharma[5]) — For two meromorphic functions f and g we denote by $n(r, b; g \mid f = a, \geqq p)$ the number of b-points of g, counted with proper multiplicities, which are the a-points of f with multiplicities not less than p within $|z| \leqq r$; and $N(r, b; g \mid f = a, \geqq p)$ is defined in the usual manner in terms of $n(r, b; g \mid f = a, \geqq p)$.

Throughout the paper we consider only the homogeneous differential polynomials $P[f]$ generated by a meromorphic function f of finite order so that $P[f]$ is also of finite order. By $P_0[f]$ we denote a homogeneous differential polynomial not containing f i.e., $n_{0j} = 0$ for $j = 1, 2, \dots, l$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 — $T(r, P[f]) \leqq \gamma_P T(r,f) + (\Gamma_P - \gamma_P)\, \overline{N}(r, f) + S(r, f)$.

The proof is omitted.

Lemma 2 — For any $a(\neq\infty)$

$$N(r, 0; P_0[f] \mid f = a, \geqq k) \geqq \gamma_{P_0} N(r, 0; f^{(k)} \mid f = a, \geqq k) + S(r, f).$$

Proof : Since for any finite complex number a $P_0[f] = P_0[f-a]$, without loss of generality we may take $a = 0$.

If $z = z_0$ is a zero of f of multiplicity $p(\geqq k)$ then it is a zero of $M_j[f]$ of multiplicity $= p_j + (p-1)\, l_{1j} + (p-2)\, l_{2j} + \dots + (p-k)\, l_{kj}$, where p_j is the multiplicity of the zero of A_j at $z = z_0$ and we agree to take a pole of multiplicity p_j to be its zero of multiplicity $-\, p_j$

$$= p_j + p\gamma_{P_0} - (l_{1j} + 2l_{2j} + \dots + kl_{kj})$$

$$= p_j + (p-k)\,\gamma_{P_0} + k\,\gamma_{P_0} - (l_{1j} + 2l_{2j} + \dots + kl_{kj})$$

$$\geqq p_j + (p-k)\,\gamma_{P_0}.$$

Hence the multiplicity of the zero of $P_0[f]$ at $z = z_0$ is not less than $\min_{1 \leqq j \leqq l} \left\{ p_j + (p-k)\,\gamma_{P_0} \right\} = \min_{1 \leqq j \leqq l} p_j + (p-k)\,\gamma_{P_0}$. Since $T(r, A_j) = S(r, f)$ $(j = 1, 2, \dots, l)$, considering all the zeros of $P_0[f]$ (which are the zeros of f of multiplicities $\geqq k$) in $|z| \leqq r$ we can deduce that

$$N(r,\ 0;\ P_0[f]\,|\,f = 0,\ \geqq k) \geqq \gamma_{P_0} N(r, 0; f^{(k)}\,|\,f = 0,\ \geqq k) + S(r,\ f).$$

This proves the lemma.

Lemma 3 — $N\left(r, \dfrac{P[f]}{f^{\gamma_P}}\right) \leqq (\Gamma_P - \gamma_P)\,\{\overline{N}(r,\ f) + \overline{N}(r,\ 1/f)\} + S(r,\ f).$

PROOF : First we note that the poles of $\dfrac{P[f]}{f^{\gamma_P}}$ are the poles and zeros of f because we can ignore the contribution of poles and zeros of the coefficients. Now we consider the following cases :

Case I — If $z = z_0$ is a pole of f of multiplicity p then it is a pole of $(f^{(i)}/f)^{n_{ij}}$ of multiplicity in_{ij} $(i = 0, 1, 2, \dots, k)$ so that it is a pole of $(M_j[f]/f^{\gamma_P})$ of multiplicity $= n_{1j} + 2n_{2j} + \dots + kn_{kj} = \Gamma_{M_j} - \gamma_P$.

Case II — If $z = z_0$ is a zero of f of multiplicity $p\,(\geqq k)$ then $z = z_0$ is a pole of $(f^{(i)}/f)^{n_{ij}}$ of multiplicity in_{ij} $(i = 0, 1, 2, \dots, k)$ so that it is a pole of $(M_j[f]/f^{\gamma_P})$ of multiplicity $= n_{1j} + 2n_{2j} + \dots + kn_{kj} = \Gamma_{M_j} - \gamma_P$.

Case III — If $z = z_0$ is a zero of f of multiplicity p $(< k)$ then $z = z_0$ is a pole of $(f^{(i)}/f)^{n_{ij}}$ of multiplicity in_{ij} $(i = 0, 1, 2, \dots, p)$ and of multiplicity pn_{ij} $(i = p + 1, p + 2, \dots, k)$ so that it is a pole of $(M_j[f]/f^{\gamma_P})$ of multiplicity $= n_{1j} + 2n_{2j} + \dots + pn_{pj} + pn_{p+1,j} + \dots + pn_{kj} \leqq \Gamma_{M_j} - \gamma_P$.

Since if $z = z_0$ is a pole of $(M_j[f]/f^{\gamma_P})$ of multiplicity p_j then it is a pole of $(P[f]/f^{\gamma_P})$ of multiplicity $\max_{1 \leqq j \leqq l} p_j$ almost, considering the above cases we see that

$N\left(r, \dfrac{P[f]}{f^{\gamma_P}}\right) \leqq (\Gamma_P - \gamma_P)\ \{\overline{N}(r,\ f) + \overline{N}(r,\ 1/f)\} + S(r,\ f)$. This proves the lemma.

3. THEOREMS

We first prove a theorem on the growth of $P_0[f]$ which has some applications to the value distribution of $P_0[f]$.

Theorem 1 — If $\sum_{a\neq\infty} \delta^k_{\rho_f}(a,f) > 0$ then $\liminf_{r\to\infty} \frac{T(r,f)}{T(r,P_0[f])} \leqq \frac{1}{\gamma_{P_0} \sum_{a\neq\infty} \delta^k_{\rho_f}(a,f)}$.

PROOF : Setting $F = \sum_{i=1}^{q} \frac{1}{(f-a_i)^{\gamma_{P_0}}}$, where $a_1, a_2, ..., a_q$ are distinct finite complex numbers, and proceeding as in Hayman[3] (pp. 32-33) we get

$$\gamma_{P_0} \sum_{i=1}^{q} m(r,a_i;f) \leqq m(r,F) + O(1) \leqq \sum_{i=1}^{q} m\left(r, \frac{P_0[f]}{(f-a_i)^{\gamma_{P_0}}}\right) + m\left(r, \frac{1}{P_0[f]}\right) + O(1).$$

By the first fundamental theorem we get

$$\gamma_{P_0} \sum_{i=1}^{q} m(r,a_i;f) \leqq T(r,P_0[f]) - N(r,0;P_0[f]) + \sum_{i=1}^{q} m\left(r, \frac{P_0[f-a_i]}{(f-a_i)^{\gamma_{P_0}}}\right) + O(1). \quad ... (1)$$

Since $T(r,f-a_i) = T(r,f) + O(1)$, we get by Milloux theorem (cf. Hayman[3], p.55)

$$\gamma_{P_0} \sum_{i=1}^{q} m(r,a_i,f) \leqq T(r,P_0[f]) - N(r,0;P_0[f]) + S(r,f)$$

$$\leqq T(r,P_0[f]) - \sum_{i=1}^{q} N(r,0;P_0[f] \mid f = a_i, \geqq k) + S(r,f).$$

By the first fundamental theorem and Lemma 2 we get

$$\gamma_{P_0} qT(r,f) \leqq T(r,P_0[f]) + \gamma_{P_0} \sum_{i=1}^{q} N(r,a_i,f) - \sum_{i=1}^{q} N(r,0;P_0[f] \mid f = a_i, \geqq k) + S(r,f)$$

$$= T(r,P_0[f]) + \gamma_{P_0} \sum_{i=1}^{q} N_k(r,a_i,f) + S(r,f) \quad ... (2)$$

i.e., $$\gamma_{P_0} q \frac{T(r,f)}{r^{\rho_f(r)}} \leqq \frac{T(r,P_0[f])}{r^{\rho_f(r)}} + \gamma_{P_0} \sum_{i=1}^{q} \frac{N_k(r,a_i;f)}{r^{\rho_f(r)}} + o(1) \frac{T(r,f)}{r^{\rho_f(r)}}$$

which gives

$$\gamma_{P_0} q \leqq \limsup_{r\to\infty} \frac{T(r,P_0[f])}{r^{\rho_f(r)}} + \gamma_{P_0} \sum_{i=1}^{q} \limsup_{r\to\infty} \frac{N_k(r,a_i;f)}{r^{\rho_f(r)}}$$

i.e., $$\gamma_{P_0} \sum_{i=1}^{q} \delta^k_{\rho_f}(a_i,f) \leqq \limsup_{r\to\infty} \frac{T(r,P_0[f])}{r^{\rho_f(r)}} .$$

Since q is arbitrary, it follows that

$$\gamma_{P_0} \sum_{a\neq\infty} \delta^k_{\rho_f}(a,f) \leqq \limsup_{r\to\infty} \frac{T(r,P_0[f])}{r^{\rho_f(r)}} . \qquad \ldots (3)$$

This gives

$$\gamma_{P_0} \sum_{a\neq\infty} \delta^k_{\rho_f}(a,f) \leqq \limsup_{r\to\infty} \frac{T(r,P_0[f])}{T(r,f)} \cdot \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f(r)}}$$

$$= \limsup_{r\to\infty} \frac{T(r,P_0[f]}{T(r,f)}$$

from which the theorem follows. This proves the theorem.

Remark 1 : Singh and Kulkarni[6] proved that if f is a meromorphic function of finite order then $\liminf_{r\to\infty} \frac{T(r,f)}{T(r,f')} < \infty$. We generalize this result to the homogeneous differential polynomials with a definite bound on the right hand side

Corollary 1 — If $\sum_{a\neq\infty} \delta^k_{\rho_f}(a,f) > 0$ then

$$1 - \frac{\Gamma_{P_0} - \gamma_{P_0}\delta_{\rho_f}(\infty,f) - (\Gamma_{P_0} - \gamma_{P_0})\,\Theta_{\rho_f}(\infty,f)}{\gamma_{P_0}\Sigma\,\delta^k_{\rho_f}(a,f)} \leqq \Delta(\infty, P_0[f]).$$

PROOF : Since $\frac{N(r,P_0[f])}{T(r,P_0[f])} \leqq \left\{ \gamma_{P_0} \frac{N(r,f)}{r^{\rho_f(r)}} + (\Gamma_{P_0} - \gamma_{P_0}) \frac{\overline{N}(r,f)}{r^{\rho_f(r)}} + o(1) \right\} \frac{r^{\rho_f(r)}}{T(r,P_0[f])}$ it follows by (3) that

$$1 - \Delta(\infty, P_0[f]) \leqq \{\gamma_{P_0}(1-\delta_{\rho_f}(\infty,f)) + (\Gamma_{P_0} - \gamma_{P_0})(1 - \Theta_{\rho_f}(\infty,f))\}$$

$$\times \frac{1}{\gamma_{P_0} \sum_{a\neq\infty} \delta^k_{\rho_f}(a,f)}$$

and the corollary is proved.

Corollary 2 — If $\delta_{\rho_f}(\infty, f) = 1$ and $\sum_{a \neq \infty} \delta^k_{\rho_f}(a, f) > 0$ then $\Delta(\infty, P_0[f]) = 1$.

This corollary follows from Corollary 1.

Corollary 3 — $\gamma_{P_0} \sum_{a \neq \infty} \delta^k_{\rho_f}(a, f)(1 - \Delta(\infty, P_0[f])) \leqq 1 - \Theta_{\rho_f}(\infty, f)$.

PROOF : Since $\overline{N}(r, P_0[f]) \leqq \overline{N}(r, f) + S(r, f)$, from (3) we get

$$\gamma_{P_0} \sum_{a \neq \infty} \delta^k_{\rho_f}(a, f) \leqq \limsup_{r \to \infty} \frac{T(r, P_0[f])}{r^{\rho_f(r)}} \leqq \frac{\limsup\limits_{r \to \infty} \dfrac{\overline{N}(r, f)}{r^{\rho_f(r)}}}{\limsup\limits_{r \to \infty} \dfrac{\overline{N}(r, P_0[f])}{T(r, P_0[f])}}$$

$$= \frac{1 - \Theta_{\rho_f}(\infty, f)}{1 - \overline{\Delta}(\infty, P_0[f])}$$ and this proves the corollary.

Corollary 4 — If $\Theta_{\rho_f}(\infty, f) = 1$ and $\sum_{a \neq \infty} \delta^k_{\rho_f}(a, f) > 0$ then $\overline{\Delta}(\infty, P_0[f]) = 1$. This corollary follows from Corollary 3.

Theorem 2 — If $\overline{N}(r, f) = S(r, f)$ then for non-zero finite distinct b, c

$$\Theta_{\rho_{P_0}}(b, P_0[f]) + \Theta_{\rho_{P_0}}(c, P_0[f]) - \Delta_{\rho_{P_0}}(0, P_0[f]) \leqq 1 - 2 \sum_{a \neq \infty} \delta(a, f).$$

PROOF : From the second fundamental theorem we get for two nonzero finite distinct numbers b, c

$$2T(r, P_0[f]) \leqq \overline{N}(r, b; P_0[f]) + \overline{N}(r, c; P_0[f]) + \overline{N}(r, 0; P_0[f])$$

$$+ \overline{N}(r, P_0[f]) + S(r, P_0[f]).$$

By Lemma 1 we obtain

$$2T(r, P_0[f]) \leqq \overline{N}(r, b; P_0[f]) + \overline{N}(r, c; P_0[f]) + \overline{N}(r, 0; P_0[f])$$

$$+ \overline{N}(r, P_0[f]) + S(r, f). \qquad \text{... (4)}$$

Since $\overline{N}(r, P_0[f]) = \overline{N}(r, f) + S(r, f)$, $P_0[f] = P_0[f - a_i]$ from (1) and (4) we get by Milloux theorem (Hayman[3], p. 55)

$$2\gamma_{P_0} \sum_{i=1}^{q} m(r, a_i; f) \leqq \overline{N}(r, b; P_0[f]) + \overline{N}(r, c; P_0[f]) + \overline{N}(r, 0; P_0[f])$$

$$- 2N(r, 0; P_0[f]) + S(r, f) \leqq \overline{N}(r, b; P_0[f])$$

$$+ \overline{N}(r, c; P_0[f]) - N(r, 0; P_0[f]) + S(r, f).$$

i.e.,

$$2\gamma_{P_0}\sum_{i=1}^{q}\liminf_{r\to\infty}\frac{m(r,a_i;f)}{T(r,f)}$$

$$\leqq\left\{\limsup_{r\to\infty}\frac{\overline{N}(r,b;P_0[f])}{r^{\rho_{P_0}(r)}}+\limsup_{r\to\infty}\frac{\overline{N}(r,c;P_0[f])}{r^{\rho_{P_0}(r)}}-\liminf_{r\to\infty}\frac{N(r,0;P_0[f])}{r^{\rho_{P_0}(r)}}\right\}$$

$$\times\liminf_{r\to\infty}\frac{r^{\rho_{P_0}(r)}}{T(r,f)}. \qquad \ldots (5)$$

Now by Lemma 1 we get

$$\liminf_{r\to\infty}\frac{r^{\rho_{P_0}(r)}}{T(r,f)}\leqq\frac{1}{\limsup\limits_{r\to\infty}\dfrac{T(r,P_0[f])}{r^{\rho_{P_0}(r)}}},\ \limsup_{r\to\infty}\frac{\gamma_{P_0}T(r,f)+S(r,f)}{T(r,f)}=\gamma_{P_0}.$$

Because q is arbitrary, this gives from (5) that

$$\Theta_{\rho_{P_0}}(b,P_0[f])+\Theta_{\rho_{P_0}}(c,P_0[f])-\Delta_{\rho_{P_0}}(0,P_0[f])$$
$$\leqq 1-2\sum_{a\neq\infty}\delta(a,f).$$

This proves the theorem.

Corollary 5 — If $\overline{N}(r,f)=S(r,f)$ and $\sum_{a\neq\infty}\delta(a;f)=1$ then $\Delta_{\rho_{P_0}}(0,P_0[f])=1$ and $\Theta_{\rho_{P_0}}(b,P_0[f])=0$ for every b $(\neq 0,\infty)$.

Corollary 6 — If $\overline{N}(r,f)=S(r,f)$ and $\sum_{a\neq\infty}\delta(a;f)>1/2$ then for every distinct nonzero finite numbers b, c $\Theta_{\rho_{P_0}}(b,P_0[f])+\Theta_{\rho_{P_0}}(c,P_0[f])<1$.

In the next theorem we see that under somewhat stronger hypothesis the conclusion of Corollary 6 holds for $P[f]$ instead of $P_0[f]$.

Theorem 3 — If $\overline{N}(r,f)=S(r,f)$ and $\delta(0,f)>\frac{1}{2}$ then $\Theta_{\rho_P}(b,P[f])+\Theta_{\rho_P}(c,P[f])<1$ for every nonzero finite distinct b, c.

PROOF : By the first fundamental theorem and Milloux theorem (Hayman[3], p. 55) we get

$$\gamma_P\,m(r,0;f)\leqq m\left(r,\frac{P[f]}{f^{\gamma_P}}\right)+m(r,0;P[f])$$

$$=T(r,P[f])-N(r,0;P[f])+S(r,f). \qquad \ldots (6)$$

Again by the second fundamental theorem and Lemma 1 we obtain for two distinct nonzero finite numbers b, c

$$2T(r, P[f]) \leqq \overline{N}(r, b; P[f]) + \overline{N}(r, c; P[f])$$

$$+ \overline{N}(r, 0; P[f]) + \overline{N}(r, P[f]) + S(r, f).$$

Because $\overline{N}(r, P[f]) \leqq \overline{N}(r, f) + S(r, f) = S(r, f)$, we get from above

$$2T(r, P[f]) \leqq \overline{N}(r, b; P[f]) + \overline{N}(r, c; P[f])$$

$$+ \overline{N}(r, 0; P[f]) + S(r, f). \quad \text{... (7)}$$

Combining (6) and (7) we obtain

$$2\,\gamma_P\, m(r, 0; f) \leqq \overline{N}(r, b; P[f]) + \overline{N}(r, c; P[f])$$

$$+ \overline{N}(r, 0; P[f]) - 2N(r, 0; P[f]) + S(r, f)$$

$$\leqq \overline{N}(r, b; P[f]) + \overline{N}(r, c; P[f])$$

$$- N(r, 0; P[f]) + S(r, f)$$

so that

$$2\,\gamma_P \liminf_{r \to \infty} \frac{m(r, 0; f)}{T(r, f)}$$

$$\leqq \left\{ \limsup_{r \to \infty} \frac{\overline{N}(r, b; P[f])}{r^{\rho_P(r)}} + \limsup_{r \to \infty} \frac{\overline{N}(r, c; P[f])}{r^{\rho_P(r)}} \right.$$

$$\left. - \liminf_{r \to \infty} \frac{N(r, 0; P[f])}{r^{\rho_P(r)}} \right\} \liminf_{r \to \infty} \frac{r^{\rho_P(r)}}{T(r, f)}. \quad \text{... (8)}$$

Since $\displaystyle \liminf_{r \to \infty} \frac{r^{\rho_P(r)}}{T(r, f)} \leqq \frac{1}{\displaystyle \limsup_{r \to \infty} \frac{T(r, P[f])}{r^{\rho_P(r)}}} \cdot \limsup_{r \to \infty} \frac{\gamma_P\, T(r, f) + S(r, f)}{T(r, f)} = \gamma_P$

from (8) it follows that

$$\Theta_{\rho_P}(b; P[f]) + \Theta_{\rho_P}(c; P[f]) \leqq 1 + \Delta_{\rho_P}(0; P[f]) - 2\delta(0, f)$$

from which the theorem follows. This proves the theorem.

In the following theorem we see that for the functions of finite order a better result than Theorem B is achieved.

Theorem 4 — If $\overline{N}(r, f) + \overline{N}(r, 1/f) = S(r, f)$ then for every $a \neq 0, \infty$ · $\Theta_{\rho_P}(a, P[f]) = 0$.

We omit the proof because it is similar to that of Theorem B.

Theorem 5 — If $\overline{N}(r, f) + \overline{N}(r, 1/f) = S(r, f)$ then

(i) $\delta(\infty, f) \leqq \delta_{\rho_P}(\infty, P[f]) \leqq \Delta_{\rho_P}(\infty, P[f]) \leqq \Delta(\infty, f)$

and

(ii) $\delta_{\rho_P}(0, P[f]) \leqq \Delta(0, f)$.

PROOF : By Lemma 3 we get

$$N(r, P[f]) \leqq N(r, P[f]/r^{\gamma_P}) + \gamma_P N(r, f)$$

$$\leqq (\Gamma_P - \gamma_P) \{\overline{N}(r, f) + \overline{N}(r, 1/f)\} + \gamma_P N(r, f) + S(r, f)$$

$$= \gamma_P N(r, f) + S(r, f) \qquad \dots (9)$$

and so

$$\limsup_{r \to \infty} \frac{N(r, P[f])}{r^{\rho_P(r)}} \leqq \gamma_P \limsup_{r \to \infty} \frac{N(r, f)}{T(r, f)} \cdot \limsup_{r \to \infty} \frac{T(r, f)}{T(r, P[f])}. \qquad \dots (10)$$

From (9) and (10) we get

$$1 - \delta_{\rho_P}(\infty, P[f]) \leqq \gamma_P (1 - \delta(\infty, f) \cdot (1/\gamma_P)$$

i.e.,
$$\delta(\infty, f) \leqq \delta_{\rho_P}(\infty, P[f]). \qquad \dots (11)$$

Also by Milloux theorem (Hayman[3], p.55) we see that

$$T(r, P[f]) = m(r, P[f]) + N(r, P[f])$$

$$\leqq \gamma_P m(r, f) + m\left(r, \frac{P[f]}{f^{\gamma_P}}\right) + N(r, P[f])$$

$$= \gamma_P m(r, f) + N(r, P[f]) + S(r, f) \qquad \dots (12)$$

i.e.,
$$\frac{T(r, P[f])}{r^{\rho_P(r)}} - \frac{N(r, P[f])}{r^{\rho_P(r)}} \leqq \gamma_P \left\{ \frac{m(r, f)}{T(r, f)} + o(1) \right\} \frac{T(r, f)}{r^{\rho_P(r)}}$$

which gives because $\limsup_{r \to \infty} \frac{T(r, P[f])}{r^{\rho_P(r)}} = 1$

$$1 - \liminf_{r \to \infty} \frac{N(r, P[f])}{r^{\rho_P(r)}} \leqq \gamma_P \limsup_{r \to \infty} \frac{m(r, f)}{T(r, f)} \limsup_{r \to \infty} \frac{T(r, f)}{T(r, P[f])}. \qquad \dots (13)$$

From (9) and (13) we get

$$\Delta_{\rho_P}(\infty, P[f]) \leqq \Delta(\infty, f). \qquad \dots (14)$$

Now (i) follows from (11) and (14).

Again by the first fundamental theorem and Lemma 3 we get

$$\gamma_P T(r,f) = T(r, 1/f^{\gamma_P}) + O(1)$$

$$\leqq \gamma_P m(r,0;f) + (\Gamma_P - \gamma_P)\{\overline{N}(r,f) + \overline{N}(r,1/f)\}$$

$$+ N(r,0;P[f]) + O(1)$$

$$= \gamma_P m(r,0;f) + N(r,0;P[f]) + S(r,f),$$

which gives $\gamma_P \left\{ 1 + o(1) - \dfrac{m(r,0;f)}{T(r,f)} \right\} \dfrac{T(r,f)}{T(r,P[f])} \cdot \dfrac{T(r,P[f])}{r^{\rho_P(r)}} \leqq \dfrac{N(r,0;P[f])}{r^{\rho_P(r)}}$.

By Lemma 1 we get because $\overline{N}(r,f) = S(r,f)$

$$\gamma_P \left\{ 1 - \limsup_{r\to\infty} \frac{m(r,0;f)}{T(r,f)} \right\} \liminf_{r\to\infty} \frac{T(r,f)}{T(r,P[f])} \limsup_{r\to\infty} \frac{T(r,P[f])}{r^{\rho_P(r)}}$$

$$\leqq \limsup_{r\to\infty} \frac{N(r,0;P[f])}{r^{\rho_P(r)}}$$

i.e.,
$$1 - \limsup_{r\to\infty} \frac{m(r,0;f)}{T(r,f)} \leqq \limsup_{r\to\infty} \frac{N(r,0;P[f])}{r^{\rho_P(r)}}$$

i.e.,
$$\delta_{\rho_P}(0,P[f]) \leqq \Delta(0,f).$$

This proves the theorem.

Corollary 7 — If $\overline{N}(r,f) + \overline{N}(r,1/f) = S(r,f)$ then $\Delta_{\rho_f}(\infty,f) = \Delta(\infty,f)$.

Since $\Delta(\infty,f) \leqq \Delta_{\rho_f}(\infty,f)$, the corollary follows from (i) of Theorem 5 by putting $P[f] = f$.

Theorem 6 — $\delta(\infty, P_0[f]) \sum_{a\neq\infty} \delta^k_{\rho_f}(a,f) \leqq \Delta^*_{\rho_f}(\infty,f)$, where

$$\Delta^*_{\rho_f}(\infty,f) = \limsup_{r\to\infty} \frac{m(r,f)}{r^{\rho_f(r)}}.$$

PROOF : From (12) we get replacing $P[f]$ by $P_0[f]$

$$\limsup_{r\to\infty} \frac{T(r,P_0[f])}{r^{\rho_f(r)}} \leqq \gamma_{P_0} \limsup_{r\to\infty} \frac{m(r,f)}{r^{\rho_f(r)}}$$

$$+ \limsup_{r\to\infty} \frac{N(r,P_0[f])}{T(r,P_0[f])} \limsup_{r\to\infty} \frac{T(r,P_0[f])}{r^{\rho_f(r)}}$$

i.e., $$\delta(\infty, P_0[f]) \limsup_{r\to\infty} \frac{T(r, P_0[f])}{r^{\rho_f(r)}} \leqq \gamma_{P_0} \Delta^*_{\rho_f}(\infty, f). \quad \ldots (15)$$

Now the theorem follows from (3) and (15). This proves the theorem.

Theorem 7 — $\delta_{\rho_{P_0}}(\infty, P_0[f]) \sum_{a\neq\infty} \delta_k(a, f) \leqq \Delta(\infty, f)$.

PROOF : Replacing $P[f]$ by $P_0[f]$ in (12) we get because

$$\limsup_{r\to\infty} \frac{T(r, P_0[f])}{r^{\rho_{P_0}(r)}} = 1$$

$$1 \leqq \gamma_{P_0} \limsup_{r\to\infty} \frac{m(r,f)}{T(r,f)} \limsup_{r\to\infty} \frac{T(r,f)}{T(r, P_0[f])} + \limsup_{r\to\infty} \frac{N(r, P_0[f])}{r^{\rho_{P_0}}}.$$

i.e., $$\delta_{\rho_{P_0}}(\infty, P_0[f]) \leqq \gamma_{P_0} \Delta(\infty, f) \limsup_{r\to\infty} \frac{T(r,f)}{T(r, P_0[f])}$$

i.e., $$\delta_{\rho_{P_0}}(\infty, P_0[f]) \liminf_{r\to\infty} \frac{T(r, P_0[f])}{T(r,f)} \leqq \gamma_{P_0} \Delta(\infty, f). \quad \ldots (16)$$

Also from (2) we get

$$\gamma_{P_0} q \leqq \frac{T(r, P_0[f])}{T(r,f)} + \gamma_{P_0} \sum_{i=1}^{q} \frac{N_k(r, a_i; f)}{T(r,f)} + o(1)$$

which gives

$$\gamma_{P_0} \sum_{a\neq\infty} \delta_k(a, f) \leqq \liminf_{r\to\infty} \frac{T(r, P_0[f])}{T(r,f)}. \quad \ldots (17)$$

The theorem follows from (16) and (17). This proves the theorem.

Concluding remark — Let f be a meromorphic function of finite lower order λ and $\lambda_f(r)$ be a lower proximate order of f. Now we may define the lower proximate deficiency $\delta_{\lambda_f}(a, f)$ as follows : $\delta_{\lambda_f}(a, f) = \liminf_{r\to\infty} \frac{m(r, a; f)}{r^{\lambda_f(r)}}$. Then clearly $\delta(a, f) \leqq \delta_{\lambda_f}(a, f)$ and the scope of investigation of the value distribution of a differential polynomial by means of lower proximate deficiency remains open.

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