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THERMAL SHOCK PROBLEM IN THERMOELASTICITY WITHOUT ENERGY DISSIPATION

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The generalized thermoelasticity theory recently developed by Green and Naghdi is used to solve a boundary value problem of an isotropic elastic half-space with its plane boundary either held rigidly fixed or stress-free and subjected to a sudden temperature increase. The approximate small time solutions are obtained by employing the Laplace transform method.

INTRODUCTION

The classical theory of heat conduction predicts that if a material conducting heat is subjected to a thermal disturbance, the effects of the disturbance will be felt instantaneously at the distances infinitely far from its source, which contradicts the physical facts. During the last three decades, nonclassical theories have been developed, which are free from this paradox. Lord and Shulman[1], by incorporating a flux-rate term into the Fourier's law, have formulated a generalized theory which involves hyperbolic type heat transport equation, admitting finite speed for thermal signals (called the thermoelasticity with one relaxation time). Green and Lindsay[2], by including temperature-rate among the constitutive variables, have developed a temperature-rate dependent thermoelasticity, which does not violate the classical Fourier's law of heat conduction when the body under consideration has a center of symmetry, and this theory also predicts a finite speed for heat propagation (called the thermoelasticity with two relaxation times). According to these theories, heat propagation is to be viewed as a wave phenomenon rather than a diffusion phenomenon. A wave-like thermal disturbance is referred as second sound by Chandrasekharaiah[3]. These theories are motivated by experiments exhibiting the actual occurrence of second sound at low temperature and small intervals of time.

Recently, relevant theoretical developments on this subject are due to Green and Naghdi[4-6], which provide sufficient basic modifications in the constitutive equations that permit treatment of a much wider class of heat flow problems. The characterization of material response for the thermal phenomena in Green and Naghdi[4, 5] is based on three types of constitutive response functions. The nature of these three types of constitutive equations is such that when the respective theories are linearized, type I is the same as the classical heat conduction theory (based on Fourier's law), type II predicts a finite speed for heat propagation and involves no energy dissipation and type III permits the propagation of thermal signals at both infinite and finite speeds.

One-dimensional thermal shock problems have been considered by Sherief and Dhaliwal[7], using the generalized theory of thermoelasticity developed by Lord and Shulman[1], the same thermal-shock problems have been considered again by Dhaliwal and Rokne[8], using the generalized theory of thermoelasticity developed by Green and Lindsay[2]. In this paper, the same problems are considered by using the above mentioned type III theory.

Basic Equations

For a homogeneous, isotropic elastic body, the basic equations for the linear generalized theory of thermoelasticity of type III developed in Green and Naghdi[6] are

$$(\lambda + \mu)\, u_{j,ij} + \mu u_{i,jj} - \gamma\theta_{,i} + \rho f_i = \rho \ddot{u}_i \qquad \text{... (1)}$$

$$\rho c \ddot{\theta} + \gamma\theta_0\, \ddot{u}_{i,i} = \rho\, \dot{Q} + k\, \dot{\theta}_{,ii} + k^*\, \theta_{,ii} \qquad \text{... (2)}$$

$$\sigma_{ij} = \lambda u_{i,i}\, \delta_{ij} + \mu(u_{i,j} + u_{j,i}) - \gamma\theta \qquad \text{... (3)}$$

where

λ, μ	=	lamé's constants
γ	=	$\frac{1}{3} E\beta^*/(1-2\nu)$
E	=	Young's modulus
ν	=	Poisson ratio
β^*	=	coefficient of volume expansion
u_i	=	components of the displacement vector
ρ	=	mass density
θ	=	absolute temperature above the reference temperature
θ_0	=	reference temperature

σ_{ij} = components of stress tensor

k = thermal conductivity

k^* = a constant

c = specific heat at constant deformation

Q = heat source per unit volume

f_i = components of the body force per unit volume

a comma followed by a suffix denotes material derivative and a superposed dot denotes the derivative with respect to time.

To transform the above equations to nondimensional form, we define the following nondimensional variables.

$$x_i' = x_i/l, \quad t' = ta_0/l, \quad \theta' = \theta/\theta_0, \quad u_i' = u_i/l,$$

$$\sigma_{ij}' = \sigma_{ij}/\mu, \quad \rho' = \rho/\rho_0, \quad Q' = Ql/a_0^3, \quad f_i' = f_i l/a_0^2$$

where

l = a standard length

a_0 = a standard speed

ρ_0 = a standard mass density.

The basic eqns. (1)-(3), dropping primes for convenience, reduce to the following

$$\rho\alpha_1 \ddot{u}_i = \alpha_2 u_{j,ij} + u_{i,jj} - \alpha_3 \theta_{,i} + \rho\alpha_1 f_i \quad \text{... (4)}$$

$$\theta_{,ii} + \alpha_4 \dot{\theta}_{,ii} + \rho\alpha_5 \dot{Q} = \rho\alpha_6 \ddot{\theta} + \alpha_7 \ddot{u}_{i,i} \quad \text{... (5)}$$

$$\sigma_{ij} = \frac{\lambda}{\mu} u_{i,i} \delta_{ij} + (u_{i,j} + u_{j,i}) - \alpha_3 \theta \quad \text{... (6)}$$

where

$$\alpha_1 = \rho_0 a_0^2/\mu, \quad \alpha_2 = (\lambda + \mu)/\mu, \quad \alpha_3 = \gamma\theta_0/\mu,$$

$$\alpha_4 = ka_0/k^* l, \quad \alpha_5 = \rho_0 a_0^4/(k^* \theta_0), \quad \alpha_6 = \rho_0 c a_0^2/k^*,$$

$$\alpha_7 = \gamma a_0^2/k^*.$$

For a one-dimensional problem, all quantities depend only on one space coordinate x and time t, hence, we assume the components of displacement of the form

$$u_x = u(x, t), \quad u_y = 0, \quad u_z = 0.$$

For this case, eqns. (4)-(6), with $Q = 0$, $f_i = 0$, reduce to

$$\rho\alpha_1 \ddot{u} = (\alpha_2 + 1)\, u'' - \alpha_3 \theta' \qquad \text{... (7)}$$

$$\theta'' + \alpha_4 \dot{\theta}'' = \rho\alpha_6 \ddot{\theta} + \alpha_7 \ddot{u}'' \qquad \text{... (8)}$$

$$\sigma = (\alpha_2 + 1)\, u' - \alpha_3 \theta \qquad \text{... (9)}$$

where prime and dot denote derivatives with respect to x and t, respectively, and σ denotes the normal stress.

Introducing the thermoelastic potential function φ defined by

$$u = \frac{\partial \varphi}{\partial x} \,. \qquad \text{... (10)}$$

Equations (7)-(9) reduce to

$$\rho\alpha_1 \ddot{\varphi} = (\alpha_2 + 1)\, \varphi'' - \alpha_3 \theta \qquad \text{... (11)}$$

$$\theta'' + \alpha_4 \dot{\theta}'' = \rho\alpha_6 \ddot{\theta} + \alpha_7 \ddot{\varphi}'' \qquad \text{... (12)}$$

$$\sigma = (\alpha_2 + 1)\, \varphi'' - \alpha_3 \theta = \rho\alpha_1 \ddot{\varphi} \,. \qquad \text{... (13)}$$

Solution in the Laplace Transform Domain

Applying the Laplace transform, defined by

$$\bar{g}(x, p) = \int_0^\infty g(x, t) \exp(-pt)\, dt \qquad \mathrm{Re}(p) > 0$$

to eqns. (10)-(13), we arrive at

$$\bar{u} = \frac{d}{dx} \bar{\varphi} \qquad \text{... (14)}$$

$$\bar{\theta} = \frac{1}{\alpha_3} \left\{ (\alpha_2 + 1) \frac{d^2}{dx^2} - \rho\alpha_1 p^2 \right\} \bar{\varphi} \qquad \text{... (15)}$$

$$\alpha_7 p^2 \frac{d^2}{dx^2} \bar{\varphi} = \left\{ (1 + \alpha_4 p) \frac{d^2}{dx^2} - \rho\alpha_6 p^2 \right\} \bar{\theta} \qquad \text{... (16)}$$

$$\bar{\sigma} = \rho\alpha_1 p^2 \bar{\varphi} \qquad \text{... (17)}$$

where we have assumed the following initial conditions

$$u(x, t) = \dot{u}(x, t) = \theta(x, t) = \dot{\theta}(x, t) = 0 \quad \text{at} \quad t = 0. \qquad \text{... (18)}$$

Now elimination of $\bar{\theta}$ between eqns. (15) and (16) leads to the following fourth order differential equation in $\bar{\varphi}$:

$$\left\{(1+\alpha_4 p)\frac{d^4}{dx^4}-(b_1p+b_2)p^2\frac{d^2}{dx^2}+b_3p^4\right\}\bar{\varphi}=0 \qquad \dots (19)$$

where

$$b_1=\rho\alpha_1\alpha_4/(\alpha_2+1)$$

$$b_2=\rho\alpha_6+(\rho\alpha_1+\alpha_3\alpha_7)/(\alpha_2+1)$$

$$b_3=\rho^2\alpha_1\alpha_6/(\alpha_2+1).$$

Solution for $\bar{\varphi}$ from eqn. (19), with the regularity condition taken as

$$\bar{\varphi}\to 0 \quad \text{as} \quad x\to\infty \qquad \dots (20)$$

is given by

$$\bar{\varphi}=A_1\exp(-\lambda_1 x)+A_2\exp(-\lambda_2 x) \qquad \dots (21)$$

where A_1, A_2 are functions of p, and λ_1 and λ_2 are the positive roots of the equation

$$(1+\alpha_4 p)\lambda^4-(b_1p+b_2)p^2\lambda^2+b_3p^4=0 \qquad \dots (22)$$

given by

$$\lambda_i=p\left\{\frac{(b_1p+b_2)+(-1)^{i+1}\sqrt{(b_1p+b_2)^2-4b_3(1+\alpha_4 p)}}{2(1+\alpha_4 p)}\right\}^{\frac{1}{2}}, \quad i=1,2. \qquad \dots (23)$$

By simple substitution from eqn. (21) into eqns. (14), (15) and (17), we obtain

$$\bar{u}=-\lambda_1A_1\exp(-\lambda_1 x)-\lambda_2A_2\exp(-\lambda_2 x) \qquad \dots (24)$$

$$\bar{\theta}=B_1A_1\exp(-\lambda_1 x)+B_2A_2\exp(-\lambda_2 x) \qquad \dots (25)$$

$$\bar{\sigma}=\rho\alpha_1p^2\{A_1\exp(-\lambda_1 x)+A_2\exp(-\lambda_2 x)\} \qquad \dots (26)$$

in terms of the two unknown functions A_1 and A_2, where

$$B_i=C_1\lambda_i^2-C_2p^2, \quad i=1,2; \quad C_1=\frac{\alpha_2+1}{\alpha_3}, \quad C_2=\rho\frac{\alpha_1}{\alpha_3}. \qquad \dots (27)$$

Statement and Solution of Problem A

We consider the elastic half-space $x \geq 0$ at a uniform reference temperature, with its plane boundary held rigidly-fixed and subjected to sudden heating such that the boundary conditions are

$$u(x,t)=0 \qquad \text{at} \quad x=0 \qquad \dots (28)$$

$$\theta(x, t) = T_0 H(t) \qquad \text{at} \quad x = 0 \qquad \text{... (29)}$$

and that

$$\{u(x, t), \theta(x, t)\} \to 0 \quad \text{as} \quad x \to \infty \quad t > 0 \qquad \text{... (30)}$$

where T_0 is constant and $H(t)$ is the Heaviside unit function. The half-space $x \geq 0$ is assumed to be at rest initially and has reference temperature θ_0 and zero velocity such that the initial conditions are given by eqn. (18). The boundary conditions (28)-(29) and the regularity condition (30) may be transformed to

$$\bar{u}(x, p) = 0 \qquad \text{at} \quad x = 0 \qquad \text{... (31)}$$

$$\bar{\theta}(x, p) = \frac{T_0}{p} \qquad \text{at} \quad x = 0 \qquad \text{... (32)}$$

$$\{\bar{u}(x, p), \bar{\theta}(x, p)\} \to 0 \quad \text{as} \quad x \to \infty. \qquad \text{... (33)}$$

Using the expressions for $\bar{u}$ and $\bar{\theta}$ given by eqns. (24) and (25), we find that the regularity condition (33), and the boundary conditions (31) and (32), will be satisfied, if

$$(A_1, A_2) = \frac{T_0}{p} \frac{1}{B_1\lambda_2 - B_2\lambda_1} (\lambda_2, -\lambda_1). \qquad \text{... (34)}$$

Then from eqns. (24)-(26), we find that

$$\bar{u} = -\frac{T_0 \lambda_1 \lambda_2}{p(B_1\lambda_2 - B_2\lambda_1)} \{\exp(-\lambda_1 x) - \exp(-\lambda_2 x)\} \qquad \text{... (35)}$$

$$\bar{\theta} = \frac{T_0}{p(B_1\lambda_2 - B_2\lambda_1)} \{B_1\lambda_2 \exp(-\lambda_1 x) - B_2\lambda_1 \exp(-\lambda_2 x)\} \qquad \text{... (36)}$$

$$\bar{\sigma} = \frac{\rho\alpha_1 p T_0}{B_1\lambda_2 - B_2\lambda_1} \{\lambda_2 \exp(-\lambda_1 x) - \lambda_1 \exp(-\lambda_2 x)\}. \qquad \text{... (37)}$$

Theoretically, we can take the inverse Laplace transform of eqns. (35)-(37), and find the expressions for the quantities, but it is impossible to find the inverse transforms of these equations in the present form. We shall try to find the inverse transforms for small values of time (large values of p) by expanding the above expressions in the inverse powers of p to a few terms.

To find the small time approximate solution, by expanding the right side of eqn. (23) in ascending powers of $1/p$ and retaining only necessary terms, we find

$$\lambda_1 \sim b_{10} p + b_{11} \qquad \lambda_2 \sim b_{20} p^{1/2} \qquad (38)$$

where

$$b_{10} = (b_1/\alpha_4)^{1/2} = b_5\,(\alpha_2 + 1)^{1/2} \qquad b_{20} = b_3^{1/2}$$

$$b_{11} = \frac{1}{2}\,(b_1 b_2\,\alpha_4 - b_3\,\alpha_4^2 - b_1^2)/(b_1\,\alpha_4)^{3/2}.$$

In the same way, we find

$$\frac{1}{B_1\lambda_2 - B_2\,\lambda_1} \approx \frac{1}{p^3\,C_2\,b_{10}}\,(1 - D_1 p^{-1/2} - D_2\,p^{-1}).$$

Using the same method to find the expansions for the following expressions

$$\frac{\lambda_1\lambda_2}{B_1\lambda_2 - B_2\lambda_1},\ \frac{B_1\lambda_2}{B_1\lambda_2 - B_2\lambda_1},\ \frac{B_2\lambda_1}{B_1\lambda_2 - B_2\lambda_1},$$

$$\frac{\lambda_2}{B_1\lambda_2 - B_2\lambda_1},\ \frac{\lambda_1}{B_1\lambda_2 - B_2\lambda_2}$$

and only keeping the necessary terms, we arrive at

$$\bar{u} \approx \frac{T_0}{C_2 b_{10}}\,(E_1 p^{-5/2} + E_2\,p^{-3} + E_3\,p^{-7/2}) \times \{\exp\,(-\lambda_1 x) - \exp\,(-\lambda_2 x)\} \qquad \text{... (39)}$$

$$\bar{\theta} \approx \frac{T_0}{C_2\,b_{10}}\,\{(F_1\,p^{-3/2} + F_2\,p^{-2} + F_3\,p^{-5/2})\exp\,(-\lambda_1 x) - (F_4\,p^{-1} + F_5\,p^{-3/2} + F_6\,p^{-2})\exp\,(-\lambda_2 x)\} \qquad \text{... (40)}$$

$$\bar{\sigma} \approx \frac{\rho\alpha_1\,T_0}{C_2\,b_{10}}\,\{(G_1 p^{-3/2} + G_2 p^{-2} + G_3 p^{-5/2})\exp\,(-\lambda_1 x) - (G_4\,p^{-1} + G_5\,p^{-3/2} + G_6\,p^{-2})\exp\,(-\lambda_2 x)\} \qquad \text{... (41)}$$

where λ_1, λ_2 are given by eqn. (38) and

$$E_1 = -\,b_{10}\,b_{20},\ \ E_2 = b_{10}\,b_{20}\,D_1,\ \ E_3 = D_2\,b_{10}\,b_{20} - b_{11}\,b_{20},$$

$$F_1 = b_{20}\,(C_1\,b_{10}^2 - C_2)\,,\ \ F_2 = (C_2 - C_1\,b_{10}^2)\,D_1\,b_{20},$$

$$F_3 = (2C_1\,b_{10}\,b_{11} - (C_1\,b_{10}^2 - C_2)\,D_2)\,b_{20},\ \ F_4 = -\,C_2\,b_{10},$$

$$F_5 = D_1\,C_2\,b_{10},\ \ F_6 = b_{10}\,C_2\,D_2 + C_1\,b_{10}\,b_{20}^2 - C_2\,b_{11},$$

$$G_1 = b_{20},\ \ G_2 = -\,b_{20}\,D_1,\ \ G_3 = -\,b_{20}\,D_2,$$

$$G_4 = b_{10}, \quad G_5 = -b_{10} D_1, \quad G_6 = -b_{10} D_2 + b_{11}$$

$$D_1 = \frac{(C_1 b_{10}^2 - C_2)\, b_{20}}{C_2\, b_{10}}$$

$$D_2 = \frac{b_{11}}{b_{10}} + \frac{C_1}{C_2} b_{20}^2 - \frac{b_{20}^2}{b_{10}^2} - \frac{C_1^2}{C_2^2} b_{10}^2 b_{20}^2 .$$

We note that

$$\exp\,(-\lambda_1 x) = \exp\,(-b_{10}\, px) \cdot \exp\,(-b_{11} x) \qquad \text{... (42)}$$

$$\exp\,(-\lambda_2 x) = \exp\,(-b_{20}\, p^{1/2}\, x). \qquad \text{... (43)}$$

Now, to obtain the inverse Laplace transforms of eqns. (39)-(41), we will need the following results (Carslaw and Jaeger[9], p. 494)

$$L^{-1}\,[p^{-\nu-1}] = \frac{t^{\nu}}{\Gamma(\nu+1)}, \quad \nu > -1$$

$$L^{-1}\,[\exp\,(-ap)] = \delta(t-a), \qquad a > 0$$

$$L^{-1}\,[p^{-\frac{2+n}{2}} \exp\,(-ap^{-1/2}\, x)] = (4t)^{1/2n}\, i^n\, erfc\left(\frac{ax}{2\sqrt{t}}\right), \qquad n = 0, 1, 2, \ldots \qquad \text{... (44)}$$

where $L^{-1}[\,]$ denotes the Laplace transform, $\delta(\,)$ is the Dirac delta function, and *erf*( ) is the usual error function, and also we have the following notations

$$erfc\,(x) = 1 - erf\,(x)$$

$$i\, erfc(x) = i^1\, erfc(x) = \int_x^{\infty} erfc(\xi)\, d\xi$$

$$i^n\, erfc(x) = \int_x^{\infty} i^{n-1}\, erfc(\xi)\, d\xi, \qquad n = 2, 3, 4, \ldots \qquad \text{... (45)}$$

Using the results of eqns. (44) and the following convolution theorem

$$L^{-1}\,[g_1(p) \cdot g_2(p)] = \int_0^t f_1\,(t-z)\, f_2\,(z)\, dz \qquad \text{... (46)}$$

where

$$L^{-1}\,[g_1(p)] = f_1(t), \quad L^{-1}\,[g_2(p)] = f_2(t) \qquad \text{... (47)}$$

to eqns. (39)-(41) and evaluating the integrals, we obtian

$$u = \frac{T_0}{C_2 b_{10}} \sum_{j=1}^{3} E_j \left\{ \exp(-X_{11}) H(t - X_{10}) \frac{(t - X_{10})^{(1 + (j/2))}}{\Gamma\left(1 + \frac{j}{2}\right)} - ((4t)^{(1 + j/2)} i^{(2 + j)} erfc \left(\frac{X_{20}}{2\sqrt{t}} \right) \right\} \quad \ldots (48)$$

$$\theta = \frac{T_0}{C_2 b_{10}} \left\{ \exp(-X_{11}) H(t - X_{10}) \sum_{j=1}^{3} F_j \frac{(t - X_{10})^{j/2}}{\Gamma\left(1 + \frac{j}{2}\right)} - \sum_{j=4}^{6} F_j (4t)^{(j-4)/2} i^{(j-4)} erfc \left(\frac{X_{10}}{2\sqrt{t}} \right) \right\} \quad \ldots (49)$$

$$\sigma = \frac{\rho \alpha_1 T_0}{C_2 b_{10}} \left\{ \exp(-X_{11}) H(t - X_{10}) \sum_{j=1}^{3} G_j \frac{(t - X_{10})^{j/2}}{\Gamma\left(1 + \frac{j}{2}\right)} - \sum_{j=4}^{6} G_j (4t)^{(j-4)/2} i^{(j-4)} erfc \left(\frac{X_{10}}{2\sqrt{t}} \right) \right\} \quad \ldots (50)$$

where

$$X_{11} = b_{11} x \qquad X_{10} = b_{10} x \qquad X_{20} = b_{20} x.$$

Due to the presence of the error function in eqns. (48)-(50), we conclude that this theory predicts an infinite speed for heat propagation.

To analyze the results given above, we let $b_1 = 2.8$, $b_2 = 5.2$, $b_3 = 3.35$, $b_4 = 2.25$, $\alpha_1 = 1.25$, $\alpha_2 = 0.25$, $\alpha_3 = 0.25$, $\alpha_4 = 3.1$, $\rho_0 = 1.0$.

The numerical values of temperature and stress distribution at $t = 0.02, 0.04, 0.06$ and $0.0 \le x \le 1.0$ are displayed graphically in Figs. 1 and 2.

STATEMENT AND SOLUTUON OF PROBLEM B

For this problem, we consider the halfspace $x \ge 0$ at a uniform temperature with its boundary $x = 0$, free of stress and subjected to sudden heating so that the boundary conditions are

$$\sigma(x, t) = 0 \qquad \text{at} \quad x = 0 \quad \ldots (51)$$

$$\theta(x, t) = T_0 H(t) \qquad \text{at} \quad x = 0 \quad \ldots (52)$$

and

$$\{\sigma(x, t), \theta(x, t)\} \to 0 \text{ as } x \to \infty \qquad t > 0. \quad \ldots (53)$$

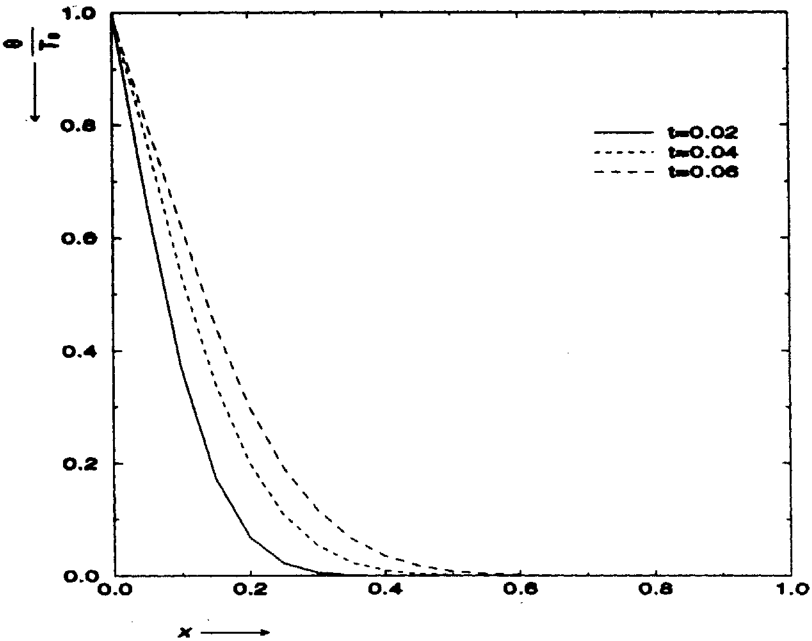

FIG. 1. Numerical values of temperature θ/T_0 against x for a fixed boundary.

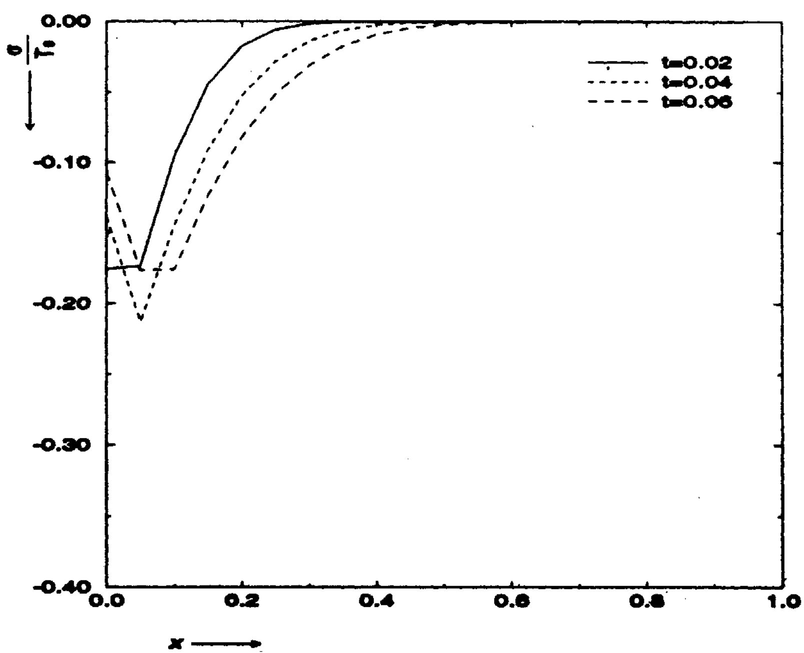

FIG. 2. Numerical values of stress σ/T_0 against x for a fixed boundary.

Applying the Laplace transform to eqns. (51)-(53) and then utilizing the transformed boundary conditions alongwith eqns. (24)-(26), we find

$$\bar{u} = -\frac{T_0}{p(B_1 - B_2)}\{\lambda_1 \exp(-\lambda_1 x) - \lambda_2 \exp(-\lambda_2 x)\} \quad \text{... (54)}$$

$$\bar{\theta} = \frac{T_0}{p(B_1 - B_2)}\{B_1 \exp(-\lambda_1 x) - B_2 \exp(-\lambda_2 x)\} \quad \text{... (55)}$$

$$\bar{\sigma} = \frac{\rho\alpha_1 p T_0}{B_1 - B_2}\{\exp(-\lambda_1 x) - \exp(-\lambda_2 x)\}. \quad \text{... (56)}$$

Expanding the following terms

$$\frac{1}{B_1 - B_2}\left(1, \frac{\lambda_1}{p}, \frac{\lambda_2}{p}, \frac{B_1}{p}, \frac{B_2}{p}\right)$$

in powers of $1/p$ by using the method similar to Problem A, we arrive at

$$\bar{u} \sim \frac{T_0}{C_1 b_{10}^2}\{(\bar{E}_1 p^{-2} + \bar{E}_2 p^{-3} + \bar{E}_3 p^{-4}) \exp(-\lambda_1 x)$$

$$- (\bar{E}_4 p^{-5/2} + \bar{E}_5 p^{-7/2} + \bar{E}_6 p^{-9/2}) \exp(-\lambda_2 x)\} \quad \text{... (57)}$$

$$\bar{\theta} \sim \frac{T_0}{C_1 b_{10}^2}\{\bar{F}_1 p^{-1} + \bar{F}_2 p^{-2} + \bar{F}_3 p^{-3}) \exp(-\lambda_1 x)$$

$$- (\bar{F}_4 p^{-1} + \bar{F}_5 p^{-2} + \bar{F}_6 p^{-3}) \exp(-\lambda_2 x)\} \quad \text{... (58)}$$

$$\bar{\sigma} \sim \frac{\rho\alpha_1 T_0}{C_1 b_{10}^2}\{\bar{G}_1 p^{-1} + \bar{G}_2 p^{-2} + \bar{G}_3 p^{-3})$$

$$\times \{\exp(-\lambda_1 x) - \exp(-\lambda_2 x)\} \quad \text{... (59)}$$

where

$$\bar{E}_1 = -b_{10}, \quad \bar{E}_2 = \bar{D}_1 b_{10} - b_{11}, \quad \bar{E}_3 = b_{11}\bar{D}_1 + \bar{D}_2 b_{10}$$

$$\bar{E}_4 = -b_{20}, \quad \bar{E}_5 = \bar{D}_1 b_{20}, \quad \bar{E}_6 = \bar{D}_2 b_{20}$$

$$\bar{F}_1 = C_1 b_{10}^2 - C_2, \quad \bar{F}_2 = 2C_1 b_{10} b_{11} - \bar{D}_1 (C_1 b_{10}^2 - C_2),$$

$$\bar{F}_3 = C_1 b_{11}^2 - 2C_1 \bar{D}_1 b_{10} b_{11} - \bar{D}_2(C_1 b_{10}^2 - C_2), \quad \bar{F}_4 = -C_2$$

$$\bar{F}_5 = C_1 b_{20}^2 + C_2\bar{D}_1, \quad \bar{F}_6 = C_2\bar{D}_2 - C_1\bar{D}_1 b_{20}^2$$

$$\bar{G}_1 = 1, \quad \bar{G}_2 = -\bar{D}_1, \quad \bar{G}_3 = -\bar{D}_2$$

$$\bar{D}_1 = \frac{2b_{10}b_{11} - b_{20}^2}{b_{10}^2}, \quad \bar{D}_2 = \frac{4b_{10}b_{11}b_{20}^2 - 3b_{11}^2 b_{10}^2 - b_{20}^4}{b_{10}^4}$$

and C_1, C_2 are given by eqn. (27).

Following the same method as for problem A, we obtain

$$u \approx \frac{T_0}{C_1 b_{10}^2} \left\{ \exp(-X_{11}) H(t - X_{10}) \sum_{j=1}^{3} \bar{E}_j \frac{(t - X_{10})^j}{(j+1)!} - \sum_{j=4}^{6} \bar{E}_j (4t)^{(j-5/2)} i^{(2j-5)} erfc\left(\frac{X_{20}}{2\sqrt{t}} \right) \right\} \quad \text{... (60)}$$

$$\theta \approx \frac{T_0}{C_1 b_{10}^2} \left\{ \exp(-X_{11}) H(t - X_{10}) \sum_{j=1}^{3} \bar{F}_j \frac{(t - X_{10})^{(j-1)}}{j!} - \sum_{j=4}^{6} \bar{F}_j (4t)^{(j-4)} i^{2(j-4)} erfc\left(\frac{X_{20}}{2\sqrt{t}} \right) \right\} \quad \text{... (61)}$$

$$\sigma \approx \frac{\rho \alpha_1 T_0}{C_1 b_{10}^2} \left\{ \exp(-X_{11}) H(t - X_{10}) \sum_{j=1}^{3} \bar{G}_j \frac{(t - X_{10})^{(j-1)}}{j!} - \sum_{j=1}^{3} \bar{G}_j (4t)^{(j-1)} i^{2(j-1)} erfc\left(\frac{X_{20}}{2\sqrt{t}} \right) \right\} \quad \text{... (62)}$$

Due to the presence of the error function in eqns. (60)-(62), we conclude this theory predicts an infinite heat propagation speed.

The numerical values of temperature and stress distribution for the same values of the elastic and thermal parameters as used in Problem A at $t = 0.02, 0.04, 0.06$ for $0.0 \le x \le 1.0$ are displayed graphically in Figs. 3 and 4.

A Special Case of Problem A

Because the constitutive equations include a diffusion type of equation for heat conductivity, generally, this theory predicts a infinite speed for the heat propagation. But for a special case, when $k^* >> k$, that is, $\alpha_4 \approx 0$, $b_1 \approx 0$, eqn. (23) becomes

$$\lambda_i \approx c_i p, \qquad i = 1, 2 \quad \text{... (63)}$$

where

$$c_i = \sqrt{\frac{b_2 + (-1)^{i+1} (b_2^2 - 4b_3)^{1/2}}{2}}$$

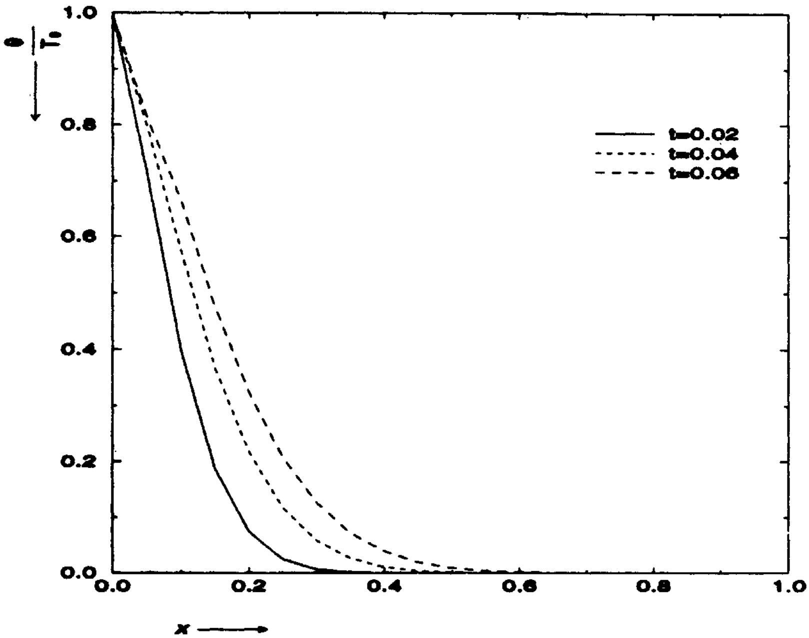

FIG. 3. Numerical values of temperature θ/T_0 against x for a free boundary.

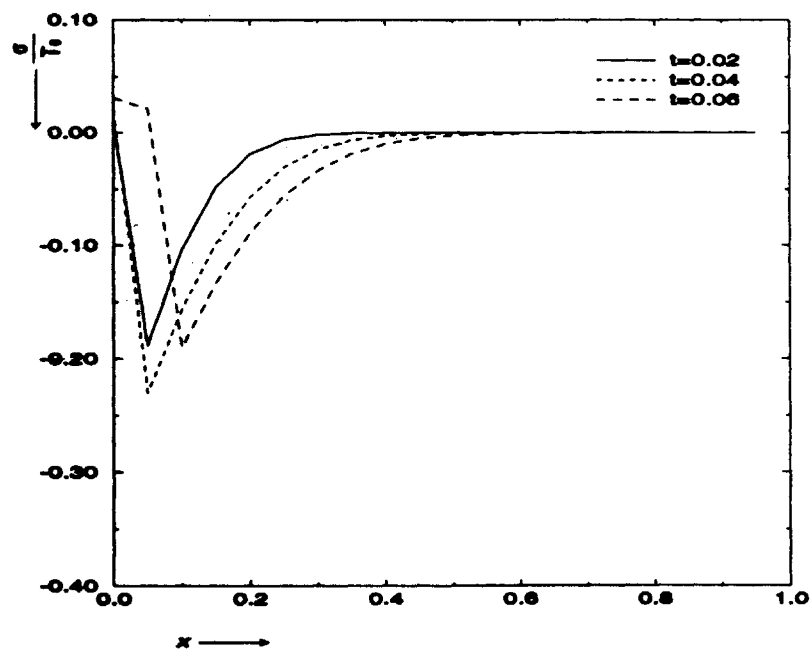

FIG. 4. Numerical values of stress σ/T_0 against x for a free boundary.

Let

$$M = \frac{1}{C_1c_1^2c_2 - C_2c_2 - C_1c_1c_2^2 + C_2c_1}$$

then

$$\frac{1}{B_1\lambda_2 - B_2\lambda_1} = \frac{M}{p^3}. \qquad \text{... (64)}$$

Substituting results of eqns. (63) and (64) into eqns. (35)-(37), we arrive at

$$\bar{u} = -\frac{T_0 M c_1 c_2}{p^2} \{\exp(-\lambda_1 x) - \exp(-\lambda_2 x)\} \qquad \text{... (65)}$$

$$\bar{\theta} = \frac{T_0 M}{p} \{M_1 \exp(-\lambda_1 x) - M_2 \exp(-\lambda_2 x)\} \qquad \text{... (66)}$$

$$\bar{\sigma} = \frac{\rho\alpha_1 T_0 M}{p} \{c_2 \exp(-\lambda_1 x) - c_1 \exp(-\lambda_2 x)\} \qquad \text{... (67)}$$

where

$$\{M_1, M_2\} = (C_1c_1^2 - C_2)\ \{c_2, c_1\}.$$

Taking the inverse Laplace transform of eqns. (65)-(67), we find

$$u = -\frac{T_0 c_1 c_2 M}{2} \{H(t - c_1x)(t - c_1x) - H(t - c_2x)(t - c_2x)\} \qquad \text{... (68)}$$

$$\theta = T_0 M \{M_1 H(t - c_1x) - M_2 H(t - c_2x)\} \qquad \text{... (69)}$$

$$\sigma = \rho\alpha_1 T_0 M \{c_2 H(t - c_1x) - c_1 H(t - c_2x)\}. \qquad \text{... (70)}$$

The numerical values of temperature and stress at $t = 0.15, 0.25, 0.5$ have been displayed against x in Fig. 5 and 6. The jumps in temperature and stress fields occur at x_1, x_2 as given below :

t	0.15	0.25	0.50
x_1	.0711	.1186	.2371
x_2	.1728	.2882	.5760

A Special Case of Problem B

For this case, we find that

$$\frac{1}{B_1 - B_2} = \frac{N}{p^2} \qquad \text{... (71)}$$

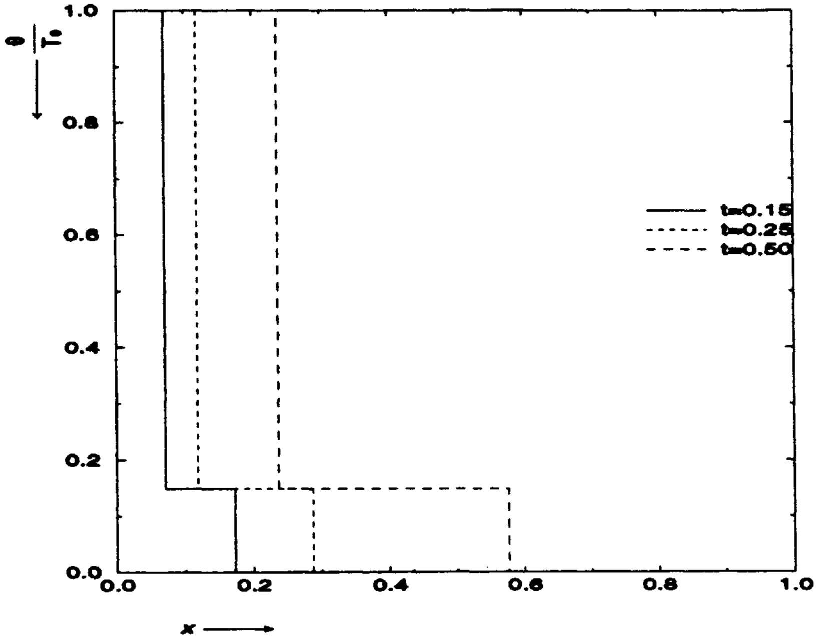

FIG. 5. Numerical values of temperature θ/T_0 against x for a fixed boundary problem in special case A.

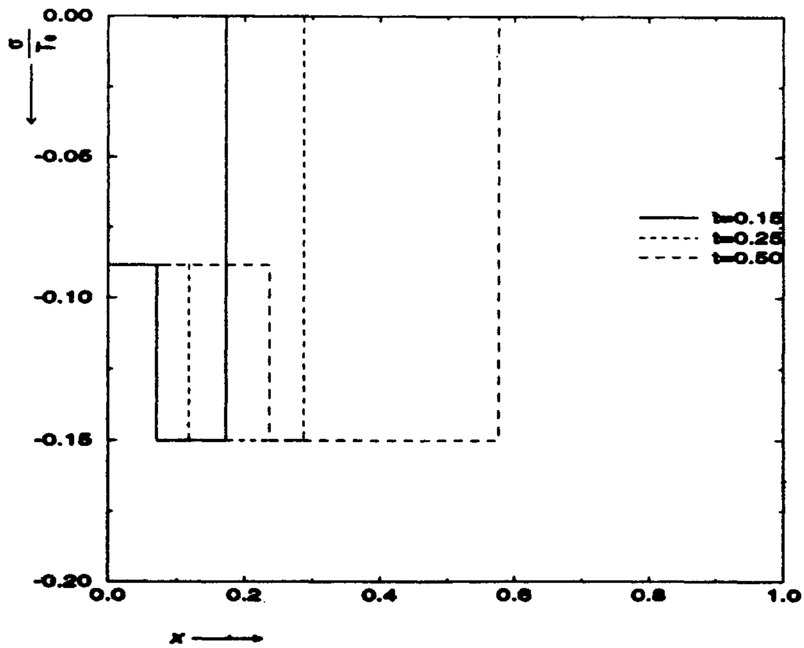

FIG. 6. Numerical values of stress σ/T_0 against x for a fixed boundary problem in special case A.

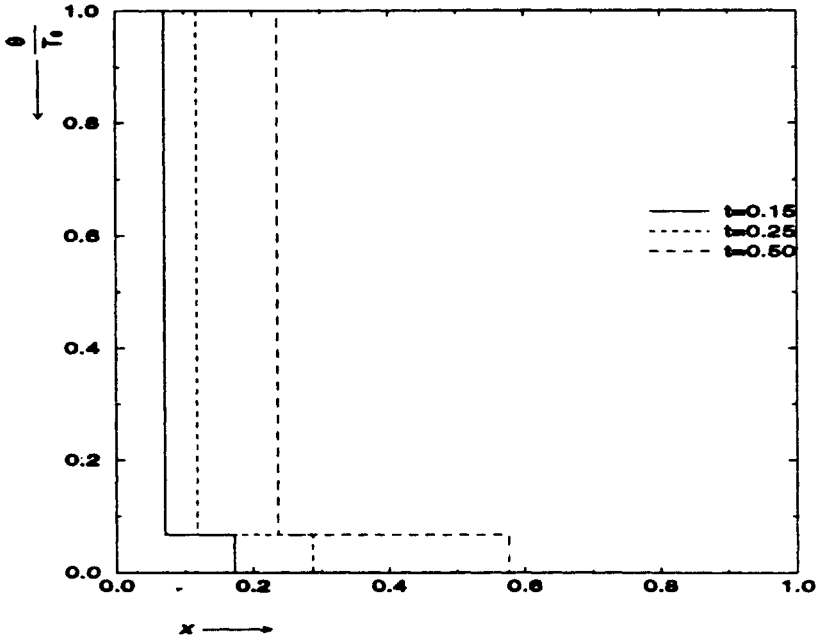

FIG. 7. Numerical values of temperature θ/T_0 against x for a free boundary in special case B.

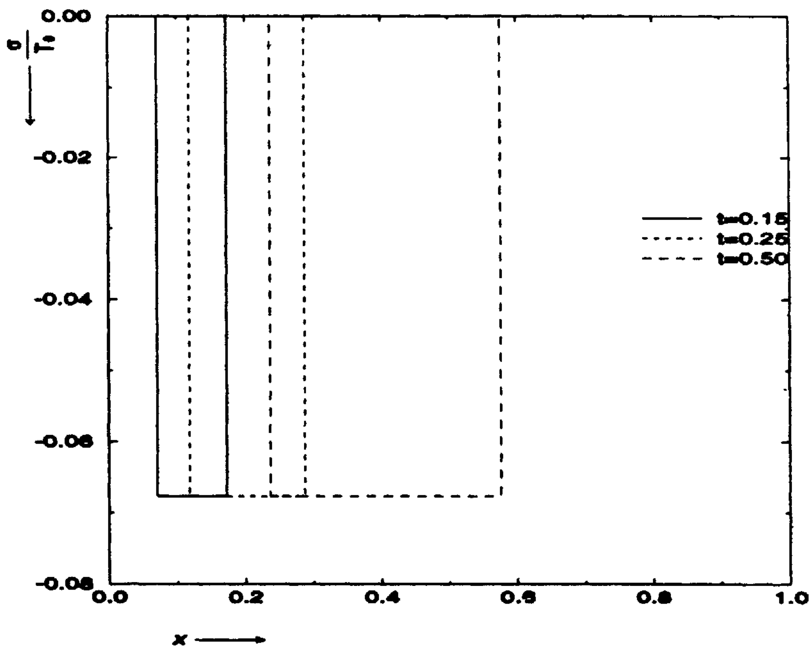

FIG. 8. Numerical values of stress σ/T_0 against x for a free boundary in special case B.

where $N = \dfrac{1}{C_1 c_1^2 - C_1 c_1^2}$.

Substitution of results of eqns. (63) and (71) into eqns. (54)-(56) leads us to

$$\bar{u} = -\frac{T_0 N}{p^2}\{c_1 \exp(-\lambda_1 x) - c_2 \exp(-\lambda_2 x)\} \quad \text{... (72)}$$

$$\bar{\theta} = \frac{T_0 N}{p}\{N_1 \exp(-\lambda_1 x) - N_2 \exp(-\lambda_2 x)\} \quad \text{... (73)}$$

$$\bar{\sigma} = \frac{\rho\alpha_1 T_0 N}{p}\{\exp(-\lambda_1 x) - \exp(-\lambda_2 x)\} \quad \text{... (74)}$$

where $N_i = C_1 c_i^2 - C_2, \quad i = 1, 2.$

Taking the inverse Laplace transforms of eqns. (72)-(74), we obtain

$$u = -\frac{T_0 N}{2}\{c_1 H(t - c_1 x)(t - c_1 x) - c_2 H(t - c_2 x)(t - c_2 x)\} \quad \text{... (75)}$$

$$\theta = T_0 N \{N_1 H(t - c_1 x) - N_2 H(t - c_2 x)\} \quad \text{... (76)}$$

$$\sigma = \rho\alpha_1 T_0 N \{H(t - c_1 x) - H(t - c_2 x)\}. \quad \text{... (77)}$$

The numerical values of temperature and stress for this case are displayed in Figs. 7 and 8. The jumps in temperature and stress occur at x_1, x_2, the values of x_1, x_2 are the same as in a special case for Problem A.

REFERENCES

1. H. Lord and Y. Shulman, *J. Mech. Phys. Solids.* **15** (1967), 299-309.
2. A. Green and K. Kindey, *J. Elast.* **2** (1972), 1-7.
3. D. S. Chandrasekharaiah, *Appl. Mech. Rew.* **39**, no. 3(1986).
4. A. E. Green and P. M. Naghdi, *Proc. Roy. Soc. London Ser. A*, **432** (1991), pp. 171-94.
5. A. E. Green and P. M. Naghdi, *J. Elasticity* **31** (1993), 189-208.
6. A. E. Green and P. M. Naghdi, *J. Thermal Stresses* **15** (1992), 253-64.
7. H. H. Sherief and R. S. Dhaliwal, *J. Therm. Stresses* **4** (1981), 421-33..
8. R. S. Dhaliwal and J. G. Rokne, *J. Therm. Stresses* **11** (1988), 257-71.
9. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Oxford University Press, 1962, p. 494.