*Indian J. pure appl. Math.*, **27**(10) : 1005-1016, October 1996

# A UNIFORMLY ACCURATE DIFFERENCE SCHEME FOR SINGULAR PERTURBATION PROBLEM

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(*Received* 31 *March* 1995; *after revision* 19 *March* 1996; *accepted* 30 *April* 1996)

For the problem $\varepsilon y'' + p(x)y' - d(x)y = f(x)$, $y(0) = \alpha_0$, $y(1) = \alpha_1$, $p(x) > 0$ and $d(x) \geq 0$ a difference scheme is derived. It is proved that the errors at the grid points are bounded by $Mh^4/(\varepsilon^2 + h^2)$ where $M$ is a constant independent of $\varepsilon$ and step size $h$, for $d(x) \equiv 0$. The numerical results show that the estimate is valid when $d(x) \neq 0$. The scheme is derived via exponential spline from $C^1[0, 1]$. A modification of the scheme giving better results for very small $\varepsilon$ is also presented.

## 1. INTRODUCTION

Let us consider the following singularly perturbed problem

$$\begin{cases} Ly = \varepsilon y'' + p(x)y' - d(x)y = f(x), \quad x \in I = [0, 1], \\ y(0) = \alpha_0, \quad y(1) = \alpha_1, \end{cases} \qquad \dots (1.1)$$

where $\varepsilon$ is a small positive parameter, $\alpha_0$ and $\alpha_1$ are given numbers, $p(x)$, $d(x)$ and $f(x)$ are sufficiently smooth functions and $p(x) \geq p > 0$, $p \in R, d(x) \geq 0$. By using the exponential spline $e(x)$ from Hess and Schmidt[4], $e(x) \in C^1(I)$, as collocation function a family of difference schemes has been derived in Surla and Uzelac[8]. When $h \to 0$, the family reduces to the one derived in Surla and Uzelac[7] via cubic spline. In section 2 we will briefly present the derivation of the schemes. The well known Allen-Southwell-Ilin and El Mistikawy-Werle (EMW) schemes are members of that family. In Surla and Uzelac[9] two schemes from the family having second order accuracy (EMW scheme and new one called IEMW scheme) are analysed and compared. For that purpose the following definitions are introduced. Let $\tau_j(y)$ denote the truncation error of a difference scheme.

*Definition* 1 — The difference scheme has the accuracy of degree $s$ if $\tau_j(y) = 0$ when $y \in P_s$, where $P_s$ is the set of polynomials of degree less than or equal to $s$.

*Definition* 2 — The difference scheme has the ε-accuracy of degree $s$ if $\lim_{\varepsilon \to 0} \tau_j(y) = 0$ when $y \in P_s$, where $P_s$ is the set of polynomials of degree less than or equal to $s$.

The EMW and IEMW schemes have the accuracy of the first degree, while the scheme from Sakai and Usmani[6] has the accuracy of the second degree. Besides that, the IEMW scheme has ε-accuracy of the second order. In this paper we give a new scheme which is a linear combination of the two mentioned schemes (EMW and IEMW). The scheme has the form :

$$R_1 u_j = Q_1 f_j \qquad \text{... (1.2)}$$

where

$$R_1 u_j = r_1^- u_{j-1} + r_1^c u_j + r_1^+ u_{j+1}$$

$$Q_1 f_j = q_1^- f_{j-1} + q_{1/2}^- f_{j-1/2} + q^c f_j + q_{1/2}^+ f_{j+1/2} + q_1^+ f_{j+1}, \quad j = 1(1)n-1,$$

$$u_0 = \alpha_0, \quad u_1 = \alpha_1.$$

The corresponding error estimate is $O(h^4/(\varepsilon^2 + h^2))$. In Berger *et al.*[1] a question has been raised : is there a scheme of the form

$$r_1^- u_{j-1} + r_1^c u_j + r_1^+ u_{j+1} = q_1^- f_{j-1} + q^c f_j + q_1^+ f_{j+1}, \quad j = 1(1)n-1,$$

$$u_0 = \alpha_0, \quad u_1 = \alpha_1,$$

which would exhibit an error behaviour combining the best aspects of the generalized OCI schemes and the uniformly converging $O(h^2)$ scheme, perhaps like $O(h^4/(\varepsilon^2 + h^2))$ ? The scheme given in Lynch and Rice[5] has these properties but under the assumption that the derivative and definite integral of the function $p(x)$ are known. The discrete approximations to these which yield the same uniform order of convergence are given in Gartland[3], but our scheme has simpler form. The collocation functions which have been used for the construction of the schemes are not of the same type. So, a comparative analysis of the two schemes would be of interest. Our scheme requires calculation of the coefficients at the middle of the intervals, since it has a form (1.2). The scheme loses ε-accuracy in the sense of Definition 2, which has IEMW. In order to compensate for that loss, we introduce a parameter λ which allows us to go on IEMW scheme when ε is very small. In section 5 we shall present numerical experiments which support the theoretical results presented in section 4.

When it is clear from the context, the $j$ subscripts will be omitted. $M$ denotes different constants independent of $h$ and ε.

## 2. Derivation of the Schemes

We seek an approximate solution of the problem (1.1) in the form of the exponential spline given in Hess and Schmidt[4]. The spline $e(x)$ has the form :

$$e(x) = e_j(x) = u_j + hm_j t + a_j (\operatorname{ch} \mu_j t - 1)/\rho_j + l_j (\operatorname{sh} \mu_j t - \mu_j)/\rho_j,$$

$$x \in [x_j, x_{j+1}],$$

where $t = (x - x_j)/h$, $x_j = jh$, $h = 1/n$, $\mu_j = h\rho_j, j = O(1)n$, $\rho_j$ are tension parameters and $m_j = e'(x_j)$. The values $a_j$ and $l_j$ are determined from the requirement $e(x) \in C^1(I)$. From the collocation conditions

$$\varepsilon e''(x) + p^- e'(x) + d^- y = f^-, \quad x = x_j,\ x = x_{j-1},$$

$$\varepsilon e''(x) + p^+ e'(x) + d^+ y = f^+, \quad x = x_j,\ x = x_{j+1},$$

where $p^-, d^-$ and $f^-$ are constant approximations to $p(x)$, $d(x)$ and $f(x)$ for $x \in [x_{j-1}, x_j]$ and similarly $p^+, d^+$ and $f^+$ are constant approximations to $p(x)$, $d(x)$ and $f(x)$ on the interval $[x_j, x_{j+1}]$ for fixed $j$, we obtain the following family of the difference schemes (see Surla and Uzelac[8]) :

$$\varepsilon h^{-2} Ru_j = Q(Ly)_j, \quad j = 1(1)n - 1, \qquad \text{... (2.1)}$$

$$u_0 = \alpha_0, \quad u_1 = \alpha_1,$$

where

$$Ru_j = r^- u_{j-1} + r^c u_j + r^+ u_{j+1},$$

$$Qf_j = q^- f^- + q^+ f^+,$$

$$r^- = \exp(n_1)/g(n_1 - k_1), \quad r^+ = \exp(-k_2)/g(n_2 - k_2),$$

$$r^c = -n_1 + k_2 - 1/g(n_1 - k_1) - 1/g(n_2 - k_2),$$

$$q^- = v_1(g(n_1) - \exp(n_1)\, g(-k_1)), \quad q^+ = v_2(g(-k_2) - \exp(-k_2)\, g(n_2)),$$

$$g(x) = (\exp(x) - 1)/x; \ g(0) = 1, \quad v_i = (1 - \exp(n_i - k_i))^{-1}, \quad i = 1, 2.$$

Further, $n_1 = hn_1^-$, $k_1 = hk_1^-$, $n_2 = hn_2^+$, $k_2 = hk_2^+$ and $n_1^-$ and $k_1^-$ are the roots of $\varepsilon w^2 + p^- y' - d^- y = 0$ and $n_2^+$, $k_2^+$ are roots of $\varepsilon w^2 + p^+ y' - d^+ y = 0$, $(n_i < k_i,\ i = 1, 2)$.

By determining

$$p^{\pm} = (p(x_{j\pm 1}) + p(xj))/2, \ d^{\pm} = (d(x_{j\pm 1}) + d(x_j))/2, \ f^{\pm} = (f(x_{j\pm 1}) + f(x_j))/2$$

we obtain the EMW scheme. When we take

$$p^{\pm} = p(x_j \pm h/2),\ d^{\pm} = d(x_j \pm h/2),\ f^{\pm} = f(x_j \pm h/2)$$

we obtain the IEMW scheme that was analysed in Surla and Uzelac[9]. In this paper we shall analyse a scheme that is obtained as a linear combination :

$$(4 \cdot \mathrm{IEMW} - \mathrm{EMW})/3. \qquad \text{... (2.2)}$$

## 3. Truncation Error

Let us consider the problem (1.1) with $d(x) \equiv 0$. In that case the scheme (2.2) has the form (1.2) with :

$$\begin{cases} q_1^- = -a(\mu_1^-)/2,\ q_{1/2}^- = 4a(\mu_2^-),\ q^c = -a(\mu_1^-)/2 - b(\mu_1^+)/2, \\ q_{1/2}^+ = 4b(\mu_2^+),\ q_1^+ = -b(\mu_1^+)/2, \\ r_1^- = (4c(\mu_2^-) - c(\mu_1^-))\,\varepsilon/h^2,\ r_1^+ = (4s(\mu_2^+) - s(\mu_1^+))\,\varepsilon/h^2, \\ r_1^c = -r_1^- - r_1^+,\ \mu_1^{\pm} = \dfrac{(p_{j\pm 1} + p_j)h}{2\varepsilon},\ \mu_2^{\pm} = \dfrac{p(x_j \pm h/2)/h}{\varepsilon}. \end{cases} \qquad \text{... (3.1)}$$

where

$$s(t) = t/(1 - \exp(-t)),$$

$$c(t) = t\exp(-t)/(1 - \exp(-t)),$$

$$a(t) = (1 - c(t))/t,$$

$$b(t) = (s(t) - 1)/t.$$

In this section we shall analyse the truncation error of the scheme (1.2), (3.1).

Let $h \le \varepsilon$. The truncation error $(\tau_j(y))$ of the family (2.1) :

$$\tau_j(y) = Ry_j - Q(Ly_j),$$

can be written in the form

$$\tau_j(y) = T_{j0}\, y_j + T_{j1}\, y_j' + T_{j2}\, y_j'' + T_{j3}\, y_j''' + R_{j4}(y),$$

where $T_{j0} = T_{j1} = 0$ for both IEMW and EMW schemes. Further, for IEMW,

$$T_{j2} = \frac{h^2}{2}(r^- + r^+) - \varepsilon(q^- + q^+) + \frac{h}{2}(p^- q^- - p^+ q^+),$$

$$T_{j3} = \frac{h^3}{6}(r^+ - r^-) + \frac{h}{2}\varepsilon(q^- - q^+) - \frac{h^2}{8}(p^+ q^+ + p^- q^-),$$

$$R_{j4}(y) = T_{rj} + T_{qj},$$

$$T_{jr} = \frac{c(\mu^-)\varepsilon}{h^2} R_3(x_j, x_{j-1}, y) + \frac{s(\mu^+)\varepsilon}{h^2} R_3(x_j, x_{j+1}, y),$$

$$T_{jq} = -q^- \varepsilon R_1'(x_j, x_{j-1/2}, y'') - q^+ \varepsilon R_1(x_j, x_{j+1/2}, y''),$$

$$- q^- p^- R_2(x_j, x_{j-1/2}, y') - q^+ p^+ R_2(x_j, x_{j+1/2}, y'),$$

where

$$R_n(a, b, g) = \frac{1}{n!} \int_a^b (b-s)^n g^{(n+1)}(s)\, ds$$

$$= g^{(n+1)}(\xi) \frac{(b-a)^{n+1}}{(n+1)!}, \quad a \le \xi \le b.$$

The corresponding expressions for the EMW scheme can be found in Berger *et al.*[1].

In the case of $h \le \varepsilon$, after some Taylor's expansions, we obtain

$$T_{j2} = \frac{-h^2}{6} (p'(\beta_1) + p'(\beta_2)) + O\left(\frac{h^3}{\varepsilon}\right)$$

for the EMW scheme and

$$T_{j2} = \frac{-h^2}{24} (p'(\beta_3) + p'(\beta_4)) + O\left(\frac{h^3}{\varepsilon}\right)$$

for the IEMW scheme, where

$$x_{j-1} < \beta_1 < x_j < \beta_2 < x_{j+1}, \; x_{j-1/2} < \beta_3 < x_j < \beta_4 < x_{j+1/2}.$$

When $p(x) = p =$ const. we have

$$T_{j3} = \frac{-h^2}{6} p + O\left(\frac{h^3}{\varepsilon}\right)$$

for the EMW scheme and

$$T_{j3} = \frac{-h^2}{24} p + O\left(\frac{h^3}{\varepsilon}\right) \qquad \ldots (3.2)$$

for the IEMW scheme. These facts indicate that the IEMW scheme is four times better than the EMW scheme when $h \le \varepsilon$, which is in accordance with numerical results in Surla and Uzelac[9]. Besides, this fact indicates the mentioned linear combination. Also, it was proved in Surla and Uzelac[9] that in the case $\varepsilon < h$ the following estimates are valid :

$$| T_{j2} | \le Mh\varepsilon, \qquad \ldots (3.3)$$

for the IEMW scheme and

$$| T_{j2} | \leq Mh^2$$

for the EMW scheme. The estimate (3.2) gives an ε-accuracy of the second degree for the IEMW scheme. The behaviour of the values $T_{j3}$ for both schemes is very interesting. Namely, when $p(x) = p =$ const. we obtain

$$\lim_{\varepsilon \to 0} T_{j3} = -h^2 p/12$$

for the EMW scheme and

$$\lim_{\varepsilon \to 0} T_{j3} = h^2 p/24 \qquad \text{... (3.4)}$$

for the IEMW scheme. From (3.1) and (3.3) one can see that $T_{j3}$ for the IEMW scheme changes the sign when ε goes to zero, for fixed $h$. As $T_{j3}$ is a continuous function of ε, at a certain point it becomes zero which may contribute to error decreasing for small ε. The good behaviour for small ε of the IEMW scheme is lost in the combination with EMW scheme. If we want to improve that, we may use the scheme :

$$[(3 + \lambda) \cdot \text{IEMW} - \lambda \cdot \text{EMW}]/3. \qquad \text{... (3.5)}$$

When $\lambda = 1$ the scheme obtains the form (1.2), (3.1). When ε is very small we can take $\lambda = \varepsilon$ or $\lambda = M\varepsilon$ for some fixed $M$.

## 4. Convergence of the Scheme

*Theorem* 1 — Let in (1.1) $d(x) \equiv 0$ and $y(x) \in C^6(I)$. Let $u_j$ be approximation to $y(x_j)$ obtained using scheme (1.2), (3.1). Then

$$| y(x_j) - u_j | \leq Mh^4/(\varepsilon^2 + h^2)$$

where $M$ is a constant independent of ε and $h$.

Proof : Let $\varepsilon \leq h$. The proof follows from the proof for the EMW scheme given in Berger *et al.*[1] and the facts that $(p(x_j) + p(x_{j\pm 1}))/2 = p(xj \pm h/2) + R_1$ and $(f(x_j) + f(x_{j\pm 1}))/2 = f(xj \pm h/2) + R_2$ where $| R_1 |, | R_2 | \leq Mh^2$.

Let $h \leq \varepsilon$. The proof requires the Taylor expansions up to sixth degree in the truncation error. The proof follows the logic of the proof for the EMW scheme from Berger *et al.*[1]. We can obtain the expression for truncation error by using linear combination of truncation errors for the EMW and IEMW schemes. Then, in the first part which corresponds to Theorem 1.1 from Berger *et al.*[1], we obtain :

$$| y(x_j) - u_j | \leq M(h^4 + h^4/\varepsilon^3 \exp(-\sigma x_j/\varepsilon)).$$

In the second part, which corresponds to the final part of the proof of Theorem 1.1 of Berger *et al.*[1], we use the same technique but more Taylor expansion terms must be carried along. Then we have $|\tau_j(E(x)/p(x))| \leq Mh^4/\varepsilon^3 \exp(-\beta h/\varepsilon)$ where $E(x) = \exp(-\varepsilon^{-1}\int_0^x p(x)\,dx)$ and $\beta$ is a constant independent of $\varepsilon$ and $h$. By applying the comparison functions we get that contribution to the error from this term is bounded by $Mh^4/\varepsilon^2$ and by Lemma 3.2 of Berger *et al.*[1] we prove the statement. □

The numerical results presented in Table V indicate that the error estimate given in Theorem 1 holds for $d(x) \neq 0$.

## 5. Numerical Results

In this section we present the results of some numerical experiments using the EMW, IEMW schemes and the new scheme (2.2). We denote by $E_n$ the maximum of $|y(x_j) - u_j^n|$, $j = O(1)n$, by $I_n$ the maximum of $|u_j^n - u_{2j}^{2n}|$, $j = O(1)n$, where $u_j^n$ and $u_j^{2n}$ denote approximate solutions at the mesh points for two successive values on $n$. The order of convergence *Ord*, we define in the usual way :

$$Ord = \frac{\log I_n - \log I_{2n}}{\log 2}.$$

Different values of $\varepsilon = 2^{-k}$ and $n$ are considered.

As numerical exmaple we shall consider two boundary value problems :

*Example 1 (Van Veldhuizen*[10]*)*

$$\varepsilon y''(x) + 2\varepsilon(1+x)/(1+x)^2 y' = f(x, \varepsilon)$$

with

$$y(x) = \cos(\pi x/(1+x)) + (\exp(-1/\varepsilon) - \exp(-2x/(\varepsilon(1+x))))/(1-\exp(-1/\varepsilon)),$$

the solution $y(x)$ determines $f(x, \varepsilon)$, $\alpha_0$ and $\alpha_1$, the derivatives of $f(x, \varepsilon)$ are bounded functions of $\varepsilon$.

*Example 2 (Berger et al.*[1]*)*

$$\varepsilon y''(x) + (x+1)^3 y'(x) + 0.31(x+1)^5 y(x) = -0.43 - 0.29x - 23x^2$$

$$y(0) = 2.7, \quad y(1) = 0.53.$$

Table I and Table II contain numerical order of the convergence *Ord* and the maximum errors $E_n$ at the mesh points obtained using the IEMW scheme and EMW scheme, respectively. Table III presents the corresponding results obtained using the new scheme (1.2), (3.1). Table IV shows that the results from Table III can be improved for $\varepsilon << h$ by the variation of the parameter $\lambda$. Table V contains numerical order of the convergence *Ord* for the scheme (2.2) and Example 2.

TABLE I
*Example 1, IEMW scheme*

| $k$ | $n$ = 16 | 32 | 64 | 128 | 256 | 512 | 1024 | |
|---|---|---|---|---|---|---|---|---|
| 1 | 3.65(–4) | 9.15(–5) | 2.29(–5) | 5.72(–6) | 1.43(– 6) | 3.57(–7) | 8.94(–8) | $E_n$ |
| | 2.49 | 2.69 | 2.82 | 2.88 | 2.89 | | | *Ord* |
| 2 | 1.77(–4) | 4.47(–5) | 1.12(–5) | 2.80(–6) | 7.01(– 7) | 1.75(–7) | 4.38(–8) | $E_n$ |
| | 2.15 | 2.03 | 2.15 | 2.44 | 2.62 | | | *Ord* |
| 3 | 5.67(–4) | 1.40(–4) | 3.44(–5) | 8.56(–6) | 2.14(– 6) | 5.34(–7) | 1.34(–7) | $E_n$ |
| | 2.04 | 2.06 | 2.30 | 2.51 | 2.65 | | | *Ord* |
| 4 | 1.12(–3) | 2.63(–4) | 6.51(–5) | 1.62(–5) | 4.06(– 6) | 1.01(–6) | 2.54(–7) | $E_n$ |
| | 2.10 | 2.02 | 2.13 | 2.41 | 2.60 | | | *Ord* |
| 5 | 1.27(–3) | 3.46(–4) | 9.08(–5) | 2.28(–5) | 5.70(– 6) | 1.42(–6) | 3.56(–7) | $E_n$ |
| | 1.85 | 1.92 | 2.00 | 2.19 | 2.46 | | | *Ord* |
| 6 | 7.04(–4) | 3.51(–4) | 1.04(–4) | 2.73(–5) | 6.89(– 6) | 1.73(–6) | 4.33(–7) | $E_n$ |
| | 0.72 | 1.68 | 1.92 | 1.98 | 2.20 | | | *Ord* |
| 7 | 4.99(–5) | 1.69(–4) | 9.15(–5) | 2.84(–5) | 7.58(– 6) | 1.93(–6) | 4.83(–7) | $E_n$ |
| | 0.94 | 0.41 | 1.59 | 1.89 | 1.98 | | | *Ord* |
| 8 | 5.36(–4) | 3.71(–5) | 4.10(–5) | 2.33(–5) | 7.52(– 6) | 2.02(–6) | 5.14(–7) | $E_n$ |
| | 2.76 | 2.06 | 0.18 | 1.53 | 1.87 | | | *Ord* |
| 9 | 8.51(–4) | 1.81(–4) | 1.42(–5) | 1.00(–5) | 5.92(– 6) | 1.94(–6) | 5.23(–7) | $E_n$ |
| | 2.06 | 2.83 | 2.54 | 0.04 | 1.49 | | | *Ord* |
| 10 | 1.02(–3) | 2.67(–4) | 5.21(–5) | 4.21(–6) | 2.48(– 6) | 1.50(–6) | 4.94(–7) | $E_n$ |
| | 1.87 | 2.19 | 2.86 | 2.77 | –0.06 | | | *Ord* |
| 11 | 1.11(–3) | 3.14(–4) | 7.44(–5) | 1.39(–5) | 1.14(– 6) | 6.16(–7) | 3.79(–7) | $E_n$ |
| | 1.79 | 2.01 | 2.26 | 2.88 | 2.89 | | | *Ord* |
| 12 | 1.16(–3) | 3.38(–4) | 8.65(–5) | 1.96(–5) | 3.59(– 6) | 2.95(–7) | 1.53(–7) | $E_n$ |
| | 1.76 | 1.93 | 2.07 | 2.29 | 2.88 | | | *Ord* |
| 13 | 1.18(–3) | 3.50(–4) | 9.28(–5) | 2.27(–5) | 5.03(– 6) | 9.13(–7) | 7.52(–8) | $E_n$ |
| | 1.75 | 1.90 | 2.00 | 2.11 | 2.30 | | | *Ord* |
| 14 | 1.19(–3) | 3.56(–4) | 9.60(–5) | 2.43(–5) | 5.81(– 6) | 1.27(–6) | 2.30(–7) | $E_n$ |
| | 1.74 | 1.89 | 1.98 | 2.03 | 2.12 | | | *Ord* |
| 15 | 1.20(–3) | 3.60(–4) | 9.76(–5) | 2.51(–5) | 6.21(– 6) | 1.47(–6) | 3.21(–7) | $E_n$ |
| | 1.73 | 1.88 | 1.95 | 2.00 | 2.05 | | | *Ord* |

TABLE II

*Example 1, EMW scheme*

| $k$ | $n$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 16 | 32 | 64 | 128 | 256 | 512 | 1024 | |
| 1 | 1.43(−3) | 3.65(−4) | 9.14(−5) | 2.29(−5) | 5.72(−6) | 1.43(−6) | 3.57(−7) | $E_n$ |
| | 2.46 | 2.68 | 2.81 | 2.87 | 2.89 | | | *Ord* |
| 2 | 7.02(−4) | 1.78(−4) | 4.48(−5) | 1.12(−5) | 2.80(−6) | 7.01(−7) | 1.75(−7) | $E_n$ |
| | 2.13 | 2.02 | 2.15 | 2.44 | 2.61 | | | *Ord* |
| 3 | 2.19(−3) | 5.54(−4) | 1.37(−4) | 3.42(−5) | 8.55(−6) | 2.14(−6) | 5.34(−7) | $E_n$ |
| | 2.00 | 2.05 | 2.29 | 2.51 | 2.65 | | | *Ord* |
| 4 | 4.27(−3) | 1.04(−3) | 2.60(−4) | 6.49(−5) | 1.62(−5) | 4.06(−6) | 1.01(−6) | $E_n$ |
| | 2.05 | 2.00 | 2.13 | 2.41 | 2.60 | | | *Ord* |
| 5 | 5.60(−3) | 1.41(−3) | 3.65(−4) | 9.12(−5) | 2.28(−5) | 5.70(−6) | 1.43(−6) | $E_n$ |
| | 1.98 | 1.95 | 2.00 | 2.19 | 2.46 | | | *Ord* |
| 6 | 5.24(−3) | 1.68(−3) | 4.37(−4) | 1.10(−4) | 2.77(−5) | 6.92(−6) | 1.73(−6) | $E_n$ |
| | 1.61 | 1.93 | 1.98 | 2.00 | 2.21 | | | *Ord* |
| 7 | 4.44(−3) | 1.57(−3) | 4.61(−4) | 1.21(−4) | 3.08(−5) | 7.74(−6) | 1.94(−6) | $E_n$ |
| | 1.49 | 1.74 | 1.92 | 1.97 | 2.00 | | | *Ord* |
| 8 | 3.87(−3) | 1.35(−3) | 4.30(−4) | 1.21(−4) | 3.23(−5) | 8.21(−6) | 2.06(−6) | $E_n$ |
| | 1.53 | 1.65 | 1.80 | 1.89 | 1.97 | | | *Ord* |
| 9 | 3.52(−3) | 1.20(−3) | 3.74(−4) | 1.12(−4) | 3.15(−5) | 8.38(−6) | 2.13(−6) | $E_n$ |
| | 1.57 | 1.70 | 1.72 | 1.80 | 1.89 | | | *Ord* |
| 10 | 3.34(−3) | 1.11(−3) | 3.35(−4) | 9.81(−5) | 2.87(−5) | 8.07(−6) | 2.13(−6) | $E_n$ |
| | 1.59 | 1.75 | 1.80 | 1.76 | 1.80 | | | *Ord* |
| 11 | 3.24(−3) | 1.07(−3) | 3.12(−4) | 8.83(−5) | 2.51(−5) | 7.26(−6) | 2.04(−6) | $E_n$ |
| | 1.61 | 1.79 | 1.84 | 1.83 | 1.78 | | | *Ord* |
| 12 | 3.19(−3) | 1.04(−3) | 3.00(−4) | 8.26(−5) | 2.27(−5) | 6.36(−6) | 1.83(−6) | $E_n$ |
| | 1.62 | 1.80 | 1.88 | 1.88 | 1.85 | | | *Ord* |
| 13 | 3.17(−3) | 1.03(−3) | 2.94(−4) | 7.94(−5) | 2.12(−5) | 5.74(−6) | 1.60(−6) | $E_n$ |
| | 1.62 | 1.81 | 1.89 | 1.92 | 1.91 | | | *Ord* |
| 14 | 3.16(−3) | 1.02(−3) | 2.90(−4) | 7.78(−5) | 2.04(−5) | 5.38(−6) | 1.44(−6) | $E_n$ |
| | 1.63 | 1.82 | 1.90 | 1.94 | 1.94 | | | *Ord* |
| 15 | 3.15(−3) | 1.02(−3) | 2.89(−4) | 7.70(−5) | 2.00(−5) | 5.18(−6) | 1.35(−6) | $E_n$ |
| | 1.63 | 1.82 | 1.91 | 1.95 | 1.96 | | | *Ord* |

TABLE III

*Example* 1, *New scheme* (1.2), (3.1)

| k | n | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 16 | 32 | 64 | 128 | 256 | 512 | 1024 | |
| 1 | 8.17(−6) | 5.11(−7) | 3.19(−8) | 1.99(−9) | 1.25(−10) | 7.77(−12) | 2.84(−13) | $E_n$ |
| | 4.05 | 4.27 | 4.47 | 4.61 | 4.68 | | | *Ord* |
| 2 | 2.99(−6) | 1.75(−7) | 1.08(−8) | 6.72(−10) | 4.19(−11) | 2.72(−12) | 5.22(−13) | $E_n$ |
| | 3.76 | 3.87 | 3.99 | 4.16 | 4.43 | | | *Ord* |
| 3 | 3.51(−5) | 2.33(−6) | 1.48(−7) | 9.35(−9) | 5.85(−10) | 3.66(−11) | 2.07(−12) | $E_n$ |
| | 3.91 | 3.98 | 4.00 | 4.25 | 4.48 | | | *Ord* |
| 4 | 6.20(−5) | 6.68(−6) | 4.39(−7) | 2.78(−8) | 1.75(−9) | 1.09(−10) | 6.85(−12) | $E_n$ |
| | 3.20 | 3.93 | 3.98 | 3.99 | 4.20 | | | *Ord* |
| 5 | 1.71(−4) | 1.08(−5) | 7.34(−7) | 5.35(−8) | 3.51(−9) | 2.22(−10) | 1.38(−11) | $E_n$ |
| | 3.99 | 3.88 | 3.77 | 3.93 | 3.96 | | | *Ord* |
| 6 | 8.12(−4) | 9.43(−5) | 6.78(−6) | 4.50(−7) | 2.84(−8) | 1.78(−9) | 1.11(−10) | $E_n$ |
| | 3.07 | 3.79 | 3.91 | 4.10 | 4.95 | | | *Ord* |
| 7 | 1.50(−3) | 2.98(−4) | 3.19(−5) | 2.46(−6) | 1.63(−7) | 1.03(−8) | 6.49(−10) | $E_n$ |
| | 2.26 | 3.18 | 3.68 | 3.91 | 4.04 | | | *Ord* |
| 8 | 2.01(−3) | 5.01(−4) | 8.86(−5) | 9.44(−6) | 7.38(−7) | 4.89(−8) | 3.10(−9) | $E_n$ |
| | 1.93 | 2.42 | 3.18 | 3.66 | 3.91 | | | *Ord* |
| 9 | 2.31(−3) | 6.44(−4) | 1.43(−4) | 2.41(−5) | 2.60(−6) | 3.01(−7) | 1.34(−8) | $E_n$ |
| | 1.80 | 2.09 | 2.49 | 3.16 | 3.68 | | | *Ord* |
| 10 | 2.48(−3) | 7.28(−4) | 1.81(−4) | 3.83(−5) | 6.27(−6) | 6.82(−7) | 5.29(−8) | $E_n$ |
| | 1.75 | 1.97 | 2.17 | 2.53 | 3.15 | | | *Ord* |
| 11 | 2.57(−3) | 7.74(−4) | 2.03(−4) | 4.80(−5) | 9.89(−6) | 1.60(−6) | 1.75(−7) | $E_n$ |
| | 1.72 | 1.91 | 2.04 | 2.21 | 2.54 | | | *Ord* |
| 12 | 2.61(−3) | 7.98(−4) | 2.15(−4) | 5.37(−5) | 1.23(−5) | 2.51(−6) | 4.04(−7) | $E_n$ |
| | 1.71 | 1.88 | 1.98 | 2.08 | 2.22 | | | *Ord* |
| 13 | 2.64(−3) | 8.11(−4) | 2.22(−4) | 5.67(−5) | 1.38(−5) | 3.13(−6) | 6.33(−7) | $E_n$ |
| | 1.70 | 1.86 | 1.95 | 2.02 | 2.10 | | | *Ord* |
| 14 | 2.65(−3) | 8.17(−4) | 2.25(−4) | 5.83(−5) | 1.46(−5) | 3.49(−6) | 7.88(−7) | $E_n$ |
| | 1.70 | 1.86 | 1.94 | 1.99 | 2.04 | | | *Ord* |
| 15 | 2.65(−3) | 8.20(−4) | 2.26(−4) | 5.92(−5) | 1.50(−5) | 3.69(−6) | 8.79(−7) | $E_n$ |
| | 1.69 | 1.86 | 1.93 | 1.98 | 2.01 | | | *Ord* |

TABLE IV

*Example 1, New scheme (3.5) with $\lambda = 1$ for $\varepsilon > h/4$ and $\lambda = \varepsilon$ for $\varepsilon \le h/4$*

| k | n = 16 | 32 | 64 | 128 | 256 | 512 | 1024 | |
|---|---|---|---|---|---|---|---|---|
| 1 | 8.17(–6) | 5.11(–7) | 3.19(–8) | 1.99(–9) | 1.25(–10) | 7.77(–12) | 2.84(–13) | $E_n$ |
| | 4.05 | 4.27 | 4.47 | 4.61 | 4.68 | | | *Ord* |
| 2 | 2.99(–6) | 1.75(–7) | 1.08(–8) | 6.72(–10) | 4.19(–11) | 2.72(–12) | 5.22(–13) | $E_n$ |
| | 3.76 | 3.87 | 3.99 | 4.16 | 4.43 | | | *Ord* |
| 3 | 3.51(–5) | 2.33(–6) | 1.48(–7) | 9.35(–9) | 5.85(–10) | 3.66(–11) | 2.07(–12) | $E_n$ |
| | 3.91 | 3.98 | 4.00 | 4.25 | 4.48 | | | *Ord* |
| 4 | 6.20(–5) | 6.68(–6) | 4.39(–7) | 2.78(–8) | 1.75(–9) | 1.09(–10) | 6.85(–12) | $E_n$ |
| | 3.20 | 3.93 | 3.98 | 3.99 | 4.20 | | | *Ord* |
| 5 | 1.71(–4) | 1.08(–5) | 7.34(–7) | 5.35(–8) | 3.51(–9) | 2.22(–10) | 1.38(–11) | $E_n$ |
| | 3.99 | 3.88 | 3.77 | 3.93 | 3.96 | | | *Ord* |
| 6 | 8.12(–4) | 9.43(–5) | 6.78(–6) | 4.50(–7) | 2.84(–8) | 1.78(–9) | 1.11(–10) | $E_n$ |
| | 3.07 | 3.79 | 3.91 | 4.10 | 4.95 | | | *Ord* |
| 7 | 9.92(–4) | 2.98(–4) | 3.19(–5) | 2.46(–6) | 1.63(–7) | 1.03(–8) | 6.49(–10) | $E_n$ |
| | 2.19 | 3.18 | 3.68 | 3.91 | 4.04 | | | *Ord* |
| 8 | 4.44(–4) | 2.72(–4) | 8.86(–5) | 9.44(–6) | 7.38(–7) | 4.89(–8) | 3.10(–9) | $E_n$ |
| | –0.65 | 2.14 | 3.18 | 3.66 | 3.91 | | | *Ord* |
| 9 | 1.11(–4) | 1.23(–4) | 7.10(–5) | 2.41(–5) | 2.60(–6) | 2.01(–7) | 1.34(–8) | $E_n$ |
| | 0.19 | –0.67 | 2.12 | 3.16 | 3.68 | | | *Ord* |
| 10 | 1.46(–4) | 3.35(–5) | 3.21(–5) | 1.80(–5) | 6.27(–6) | 6.82(–7) | 5.29(–8) | $E_n$ |
| | 3.08 | –0.03 | –0.69 | 2.11 | 3.15 | | | *Ord* |
| 11 | 1.92(–4) | 3.86(–5) | 9.00(–6) | 8.13(–6) | 4.53(–6) | 1.60(–6) | 1.75(–7) | $E_n$ |
| | 2.28 | 3.64 | –0.52 | –0.72 | 2.10 | | | *Ord* |
| 12 | 2.30(–4) | 5.06(–5) | 1.00(–6) | 2.28(–6) | 2.03(–6) | 1.13(–6) | 4.04(–7) | $E_n$ |
| | 2.11 | 2.40 | 4.36 | –1.26 | –0.74 | | | *Ord* |
| 13 | 2.54(–4) | 5.98(–5) | 1.30(–5) | 2.54(–6) | 5.61(–7) | 5.05(–7) | 2.82(–7) | $E_n$ |
| | 2.05 | 2.15 | 2.51 | 4.86 | –1.81 | | | *Ord* |
| 14 | 2.70(–4) | 6.55(–5) | 1.53(–5) | 3.32(–6) | 6.42(–7) | 1.36(–7) | 1.25(–7) | $E_n$ |
| | 2.02 | 2.07 | 2.20 | 2.57 | 5.17 | | | *Ord* |
| 15 | 2.80(–4) | 6.92(–5) | 1.67(–5) | 3.88(–6) | 8.40(–7) | 1.62(–7) | 3.30(–8) | $E_n$ |
| | 2.00 | 2.03 | 2.08 | 2.23 | 2.59 | | | *Ord* |

TABLE V

*Example 2, New scheme (2.2)*

| k | n | | | | | |
|---|---|---|---|---|---|---|
| | 16 | 32 | 64 | 128 | 256 | |
| 1 | 3.86 | 4.27 | 4.49 | 4.62 | 4.00 | *Ord* |
| 2 | 4.15 | 4.46 | 4.60 | 4.70 | 4.36 | *Ord* |
| 3 | 3.93 | 4.42 | 4.63 | 4.72 | 4.75 | *Ord* |
| 4 | 3.35 | 4.10 | 4.51 | 4.70 | 4.79 | *Ord* |
| 5 | 2.72 | 3.49 | 4.15 | 4.55 | 4.74 | *Ord* |
| 6 | 2.39 | 2.91 | 3.54 | 4.15 | 4.55 | *Ord* |
| 7 | 2.28 | 2.57 | 3.03 | 3.55 | 4.13 | *Ord* |
| 8 | 1.82 | 2.42 | 2.72 | 3.07 | 3.54 | *Ord* |
| 9 | 1.66 | 2.07 | 2.48 | 2.77 | 3.10 | *Ord* |
| 10 | 1.59 | 1.91 | 2.18 | 2.51 | 2.80 | *Ord* |
| 11 | 1.56 | 1.85 | 2.02 | 2.24 | 2.53 | *Ord* |
| 12 | 1.55 | 1.82 | 1.96 | 2.08 | 2.26 | *Ord* |
| 13 | 1.54 | 1.80 | 1.93 | 2.01 | 2.10 | *Ord* |
| 14 | 1.54 | 1.79 | 1.91 | 1.99 | 2.04 | *Ord* |
| 15 | 1.54 | 1.79 | 1.90 | 1.96 | 2.00 | *Ord* |

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