Indian J. pure appl. Math., **27**(10) : 1017-1028, October 1996

CHEBYSHEV POLYNOMIALS IN THE STUDY OF VIBRATIONS OF NON-UNIFORM RECTANGULAR PLATES

ROSHAN LAL, U. S. GUPTA AND SANGEETA RASTOGI

Department of Mathematics, University of Roorkee, Roorkee 247 667

(*Received* 28 *March* 1995; *after revision* 27 *January* 1996; *accepted* 22 *March* 1996)

The fourth order differential equation with variable coefficients governing the free transverse vibrations of thin, isotropic rectangular plate of exponentially varying thickness along one direction and resting on an elastic foundation of Winkler type has been solved by using the Chebyshev polynomials. Following Levy's approach, the characteristic equations for three different combinations of boundary conditions at the other two edges have been obtained and numerically solved for different values of the taper constant, the aspect ratio and the foundation moduli for the first three modes of vibration. Mode shapes have been computed for two different values of taper constant and keeping fixed the other plate parameters. A comparison of results with those available in literature has been presented.

INTRODUCTION

Numerous studies have been made dealing with free vibrations of rectangular plates of variable thickness due to their increasing use in civil, mechanical, aerospace, ocean engineering systems, optical and electronic equipments etc. and are reported earlier[1-8]. Recently Ng and Araar[9] studied the free vibrations of clamped rectangular plates of variable thickness by Galerkin's method. Bhat *et al.*[10] used characteristic orthogonal polynomials in determination of natural frequencies of non-uniform rectangular plates. Sonzogni *et al.*[11] analysed free vibrations of rectangular plates of exponentially varying thickness using optimized Kantorovich approach and finite element method.

Various methods such as Frobenius'[12] Ritz method[13], finite-difference method[14] and a spline technique method of solution[15, 16] have been employed for obtaining the natural frequencies of rectangular plates of variable thickness. The Frobenious' method is found to have a very slow convergence in case of variable thickness plates. Further, quintic splines technique requires handling of determinants of the order more than 30 providing not more than 4 decimals accuracy for appreciable thickness variation. The method of characteristic orthogonal polynomials also requires large number of terms i.e. 36 terms of the expansion[17]. Keeping the above facts in view, Chebyshev polynomials which have min-max property i.e. of all the monic polynomials, the maximum error is minimum has been employed to study the effect of Winkler type foundation on the natural frequencies of virbation of rectangular plates of exponentially varying thickness due to its importance in modern technological and foundation engineering such as reinforced concrete pavements of high runways, foundation of deep wells and storage tanks, slabs of buildings etc. (Sizlard[18], p. 136).

2. Mathematical Formulation

The differential equation which governs the transverse vibrations of a homogeneous thin rectangular plate of length a, breadth b, thickness h, density ρ and Poisson's ratio ν resting on Winkler type foundation is given by

$$\nabla^2 (D\nabla^2 w) - (1-\nu)\ \{D_{,xx}\, w_{,yy} - 2D_{,xy}\, w_{,xy} + D_{,yy}\, w_{,xx}\} + \rho h w_{,tt} + k_f w = 0 \text{ in } R \quad \text{... (1)}$$

where $R = \{0 \le x \le a$ and $0 \le y \le b\}$, $w(x, y, t)$ is the transverse displacement function, t the time, ∇^2 the Laplacian operator in Cartesian coordinates (x, y), k_f the foundation modulus, E the Young's modulus of elasticity, D $(= Eh^3/12(1 - \nu^2))$ with $h = h(x, y)$ the flexural rigidity and a comma followed by a suffix represents partial differentiation with respect to that variable.

For harmonic vibrations, the deflection function w (Levy approach) satisfying the simply supported edge conditions at $y = 0$ and $y = b$ is

$$w(x, y, t) = \bar{w}(x) \sin (p\pi y/b) e^{i\omega t}, \quad \text{... (2)}$$

where p is a positive integer and ω is the radian frequency of vibration. The thickness variation is taken along x-direction only[5,6,8,19-22] and h is thus independent of y.

Introducing the non-dimensional variables

$$X = x/a,\ Y = y/b,\ H = h/a,\ W = \bar{w}/a \quad \text{... (3)}$$

and assuming the thickness variation as

$$H = h_0\, e^{\alpha X} \quad \text{... (4)}$$

i.e. an exponential function of the space co-ordinate[15,16,23] due to their practical importance, h_0 being the thickness of the plate at $X = 0$ and α is the taper parameter. Moreover this type of thickness variation is interesting since it gives a reasonable approximation to linear variation situation for small values of taper parameter. Equation (1) now reduces to

$$A_0\, W^{iv} + A_1\, W''' + A_2\, W'' + A_3\, W' + A_4\, W = 0 \quad \text{... (5)}$$

where $A_0 = 1, A_1 = 6\alpha, A_2 = 9\alpha^2 - 2\lambda^2, A_3 = -6\alpha\lambda^2,$

$$A_4 = \lambda^4 - 9\nu\alpha^2 \lambda^2 + \eta e^{-3\alpha X} - \Omega^2 e^{-2\alpha X},\ \lambda^2 = p^2 \pi^2 a^2/b^2,$$

$$\eta = 12(1 - \nu^2)\, k_f a/Eh_0^3,\ \ \Omega^2 = 12\rho a^2\, \omega^2\, (1 - \nu^2)/Eh_0^2,$$

and primes denote differentiation with respect to X.

The solution of equation (5) together with the boundary conditions at the edges $X = 0$ and $X = 1$ constitutes a well defined boundary value problem in the range [0, 1]. Due to the presence of variable coefficients in equation (5), its closed form solution is not possible. Keeping this in view an approximate solution is obtained by applying the Chebyshev collocation technique. The above method is preferred due to its mini-max property as it gives faster convergence with lesser number of terms as compared to other methods such as Frobenius' method, quintic splines and polynomial coordinate functions widely used in the present day literature.

3. Method of Solution

As the present technique is applicable only in the interval –1 to 1 of the independent variable X, introduce a new independent variable ξ as follows :

$$\xi = 2X - 1 \qquad \text{... (6)}$$

which transforms the range $0 \le X \le 1$ into the applicability range $-1 \le \xi \le 1$. Equation (5) now reduces to

$$V_0\, W^{iv} + V_1\, W''' + V_2\, W'' + V_3\, W' + V_4\, W = 0 \qquad \text{... (7)}$$

where $V_i = 2^{4-i} A_i,\ i = 0(1)\ 4.$

According to Chebyshev collocation technique[24-26, 29] we assume the solution as

$$W = c_1 + c_2\, T_1 + c_3\, T_1^1 + c_4\, T_1^2 + \sum_{k=0}^{m-5} c_{k+5}\, T_k^4, \qquad \text{... (8)}$$

where $c_j\ (j = 1, 2, ..., m)$ are unkonwn constants, $T_j\ (j = 0, 1, ..., m - 5)$ are Chebyshev polynomials and T_k^j represents the jth integral of T_k.

Substitution of W and its derivatives in equation (7) gives an equation in terms of the unknown constants c's and Chebyshev polynomials T_j. The satisfaction of this resultant equation at $(m - 4)$ collocation points given by

$$\xi_k = \cos\left(\frac{2k+1}{m-4}\,\frac{\pi}{2}\right),\ k = 0, 1, 2, ..., m - 5, \qquad \text{... (9)}$$

provides a set of $(m - 4)$ equations in terms of unknowns $c_j\ (j = 1, 2, ..., m)$, which can be denoted by the matrix equation,

$$[B]\ \{C^*\} = \{0\}, \qquad \text{... (10)}$$

where B is a matrix of order $(m - 4) \times m$ and C^* and 0 are column vectors.

4. Boundary Condition

The following three sets of boundary conditions have been considered :

(i) C-C : Clamped at the edges $X = 0$ and $X = 1$;

(ii) C-S : Clamped at $X = 0$ and simply supported at $X = 1$;

(iii) C-F : Clamped at $X = 0$ and free at $X = 1$.

The relations which should be satisfied at a clamped, simply supported and free edge are

$$\left.\begin{aligned} & W = \frac{dW}{d\xi} = 0, \quad W = 4\frac{d^2 W}{d\xi^2} + \nu\lambda^2 W = 0, \\ & 4\frac{d^2 W}{d\xi^2} + \nu\lambda^2 W = 4\frac{d^3 W}{d\xi^3} + (2-\nu)\lambda^2 \frac{dW}{d\xi} = 0, \end{aligned}\right] \quad \text{... (11)}$$

respectively.

Applying the boundary condition C-C to the displacement function (8), one obtains a set of four homogeneous equations. These equations together with the field equations (10) give a complete set of m equations, which can be written in matrix form as

$$\left[\frac{B}{B_{CC}}\right] \{C^*\} = \{0\},$$

where B_{CC} is a matrix of order $4 \times m$. For a non-trivial solution of the above equation, the frequency determinant must vanish and hence

$$\left|\frac{B}{B_{CC}}\right| = 0. \quad \text{... (12)}$$

Similarly, for C-S and C-F plates, the frequency determinants are

$$\left|\frac{B}{B_{CS}}\right| = 0, \quad \text{... (13)}$$

$$\left|\frac{B}{B_{CF}}\right| = 0, \quad \text{... (14)}$$

respectively.

Numerical Evaluation and Discussion

The characteristic equations (12), (13) and (14) are transcendental in nature and can give infinitely many values of the frequency parameter Ω. In the work reported here, numerical results have been computed for the first three modes (i.e.

doubly-symmetric, antisymmetric-symmetric and second symmetric-symmetric, obtained by writing $p = 1$ in the frequency equations) for various values of foundation parameter $K(= k_f a/E = 0.0, 0.01, 0.02)$, taper constant $\alpha(= -0.5\ (0.2)\ 0.5, 0.0)$, the aspect ratio a/b (= 0.5, 1.0) and Poisson's ratio $\nu = 0.3$ for three boundary conditions C-C, C-S and C-F. In all the computations, we have fixed $m = 15$, because a further increase in m does not improve the results except in the fourth place of decimal. Figure 1 shows the convergence of the solution with the number of collocation points.

The results are presented in Tables I-IV and Figs. 2-4. From Tabes I-III, it is evident that the frequencies for a C-S plate are higher than those for a C-F plate but less than those for a C-C plate and increase with the increasing values of aspect ratio for the same set of plate parameters for all the three boundary conditions and in all the three modes. The frequency parameter is found to increase continuously with increase in taper parameter α for all the three modes and the boundary conditions for all the plate parameters except for a C-F plate when $a/b = 0.5$. However the rate of increase of frequency parameter with increasing α increases rapidly with the increase in number of modes.

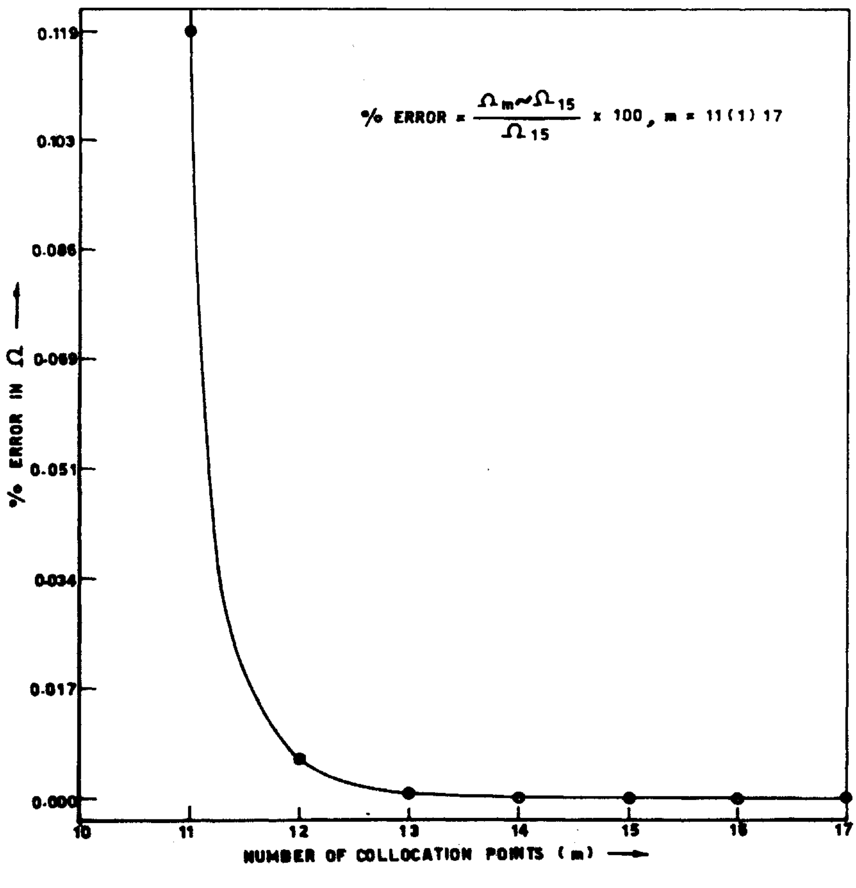

FIG. 1. % Error in Ω for C-C plate of $a/b = 1.0$, $K = 0.02$ and $\alpha = -0.5$.

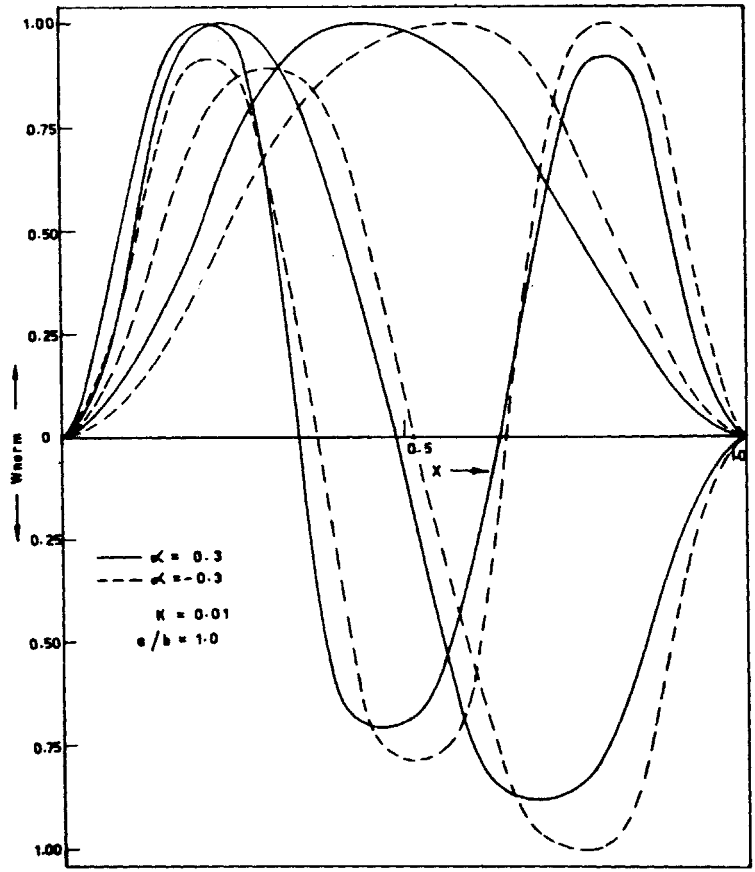

FIG. 2 Normalized displacements of C-C plate for the first three modes of vibration.

The presence of the elastic foundation increases the frequency parameter in all the cases. The increase becomes more and more pronounced as $\alpha(< 0)$ decreases i.e. as the plate gets thinner and thinner towards the edge $X = 1$. Further, this increase becomes less and less as a/b increases. The same is true with increasing number of modes. This can be attributed to the fact that an increase in a/b and α amounts to an increase in the stiffness of the plate.

A comparison of results with those available by spline technique[27] has been presented. Our results (Table IV) agree with those of Soni and Rao[15] for $K = 0$. For $\alpha = 0$ (plate of uniform thickness) and $K = 0$, the results agree very well with those of Soni[28]. The only available fundamental frequency for a square plate (a/b = 1.0) and C-F boundary obtained by two methods i.e. by finite element method and by optimized Kantorovich method[11] has been reported, which shows a reasonably good agreement.

Mode shapes have been computed for $\alpha = \pm 0.3$, $K = 0.01$ and $a/b = 1.0$ for all the boundary conditions. Normalized displacements W_{norm} $(= W/W_{max})$ are shown in Figs. 2-4 for the first three above mentioned modes.

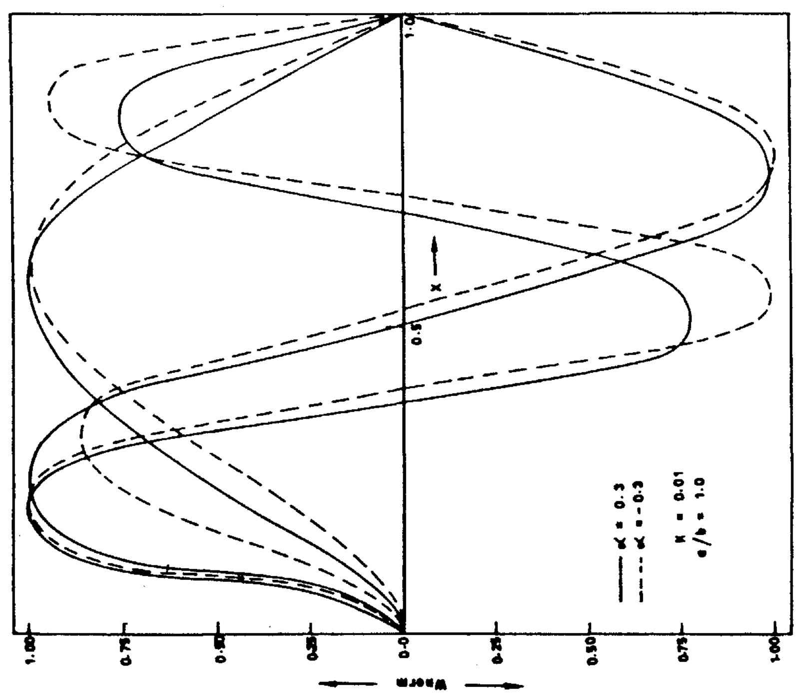

FIG. 3 Normalized displacements of C-S plate for the first three modes of vibration.

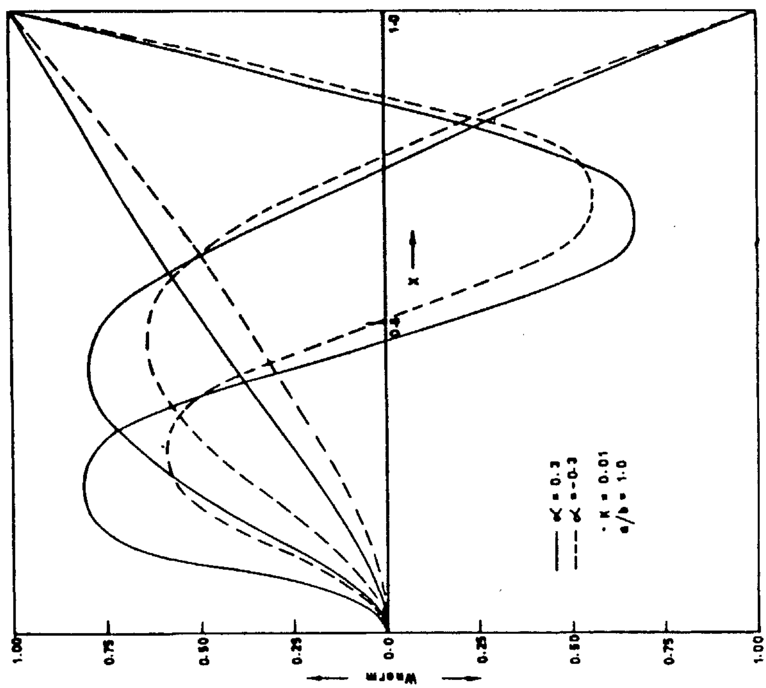

FIG. 4 Normalized displacements of C-F plate for the first three modes of vibration.

TABLE I

Values of frequency parameter Ω for C-C plate

Mode	α / K	− 0.5	− 0.3	− 0.1	0.0	0.1	0.3	0.5
					***a/b* = 0.5**			
I	0.00	18.6566	20.5412	22.6594	23.8156	25.0424	27.7277	30.7595
		*18.652	20.546	22.671	23.830	25.055	27.738	30.754
	0.01	22.1516	23.4435	25.0665	26.0074	27.0384	29.3837	32.1352
		*22.148	23.448	25.076	———	27.048	29.389	32.132
	0.02	25.1638	26.0236	27.2617	28.0283	28.8967	30.9511	33.4543
II	0.00	49.4994	54.6921	60.4368	63.5346	66.7930	73.8267	81.6107
		*49.594	54.811	60.586	63.709	66.838	74.000	81.814
	0.01	50.9226	55.8478	61.3799	64.3882	67.5667	74.4648	82.1400
		*51.027	55.969	61.541	---------	67.751	74.634	82.330
	0.02	52.3076	56.9802	62.3087	65.2306	68.3317	75.0977	82.6660
III	0.00	95.5469	105.7304	116.9245	122.9291	129.2216	142.7211	157.5302
		*96.107	106.364	117.659	123.766	130.0320	143.559	158.528
	0.01	96.2923	106.3330	117.4147	123.3725	129.6232	143.0524	157.8052
		*96.835	106.940	118.265	--------	130.420	144.097	159.272
	0.02	97.0322	106.9322	117.9030	123.8143	130.0235	143.3829	158.0798
					***a/b* = 1.0**			
I	0.00	22.6091	24.9427	27.5419	28.9509	30.4385	33.6691	37.2761
		*22.595	24.936	27.540	28.949	30.434	33.660	37.255
	0.01	25.5735	27.3836	29.5540	30.7791	32.1007	35.0462	38.4208
		25.562	27.378	29.552	---------	32.099	35.038	38.401
	0.02	28.2267	29.6237	31.4376	32.5046	33.6810	36.3710	39.5323
II	0.00	53.9764	59.6642	65.9452	69.3270	72.8806	80.5382	88.9922
		*54.057	59.762	66.066	69.461	73.021	80.678	89.108
	0.01	55.2844	60.7253	66.8106	70.1101	73.5903	81.1237	89.4778
		*55.278	60.821	66.929	---------	73.710	81.272	89.527
	0.02	56.5625	61.7682	67.6649	70.8846	74.2933	81.7048	89.9608
III	0.00	100.3214	111.0266	122.7884	129.0950	135.7022	149.8702	165.4020
		*100.835	111.622	123.487	129.696	136.474	150.620	166.277
	0.01	101.0316	111.6005	123.2553	129.5174	136.0847	150.1856	165.6640
		*101.532	112.198	124.055	----------	136.834	150.887	166.490
	0.02	101.7368	112.1715	123.7205	129.9382	136.4661	150.5004	165.9255

*Values taken from Lal[27].

TABLE II

Values of frequency parameter Ω, for C-S plate

Mode	K \ α	− 0.5	− 0.3	− 0.1	0.0	0.1	0.3	0.5
					***a/b* = 0.5**			
I	0.00	14.1614	15.3470	16.6414	17.3317	18.0523	19.5886	21.2622
		*14.161	15.352	16.649	17.340	18.050	19.593	21.253
	0.01	18.6396	19.1198	19.8131	20.2383	20.7155	21.8245	23.1404
		*18.640	19.124	19.820	---------	20.723	21.823	23.133
	0.02	22.2294	22.2612	22.5427	22.7769	23.0734	23.8516	24.8768
II	0.00	41.2319	45.2852	49.7238	52.0979	54.5813	59.8934	65.6981
		*41.319	45.391	49.848	52.231	54.715	59.997	65.822
	0.01	42.9492	46.6835	50.8683	53.1356	55.5234	60.6737	66.3483
		*43.027	46.787	50.990	---------	55.657	60.767	66.462
	0.02	44.6016	48.0414	51.9877	54.1534	56.4498	61.4441	66.9923
III	0.00	83.4278	92.0375	101.4509	106.4786	111.7311	122.9448	135.1613
		*83.862	92.606	102.083	106.857	112.396	123.710	135.987
	0.01	84.2856	92.7315	102.0162	106.9901	112.1949	123.3281	135.4805
		*84.722	93.283	102.634	---------	112.909	124.106	136.322
	0.02	85.1348	93.4202	102.5783	107.4992	112.6568	123.7101	135.7989
					***a/b* = 1.0**			
I	0.00	18.6666	20.5050	22.5441	23.6463	24.8082	27.3264	30.1346
		*18.662	20.504	22.545	23.647	24.808	27.320	30.116
	0.01	22.2546	23.4600	24.9758	25.8525	26.8100	28.9765	31.4974
		*22.251	23.461	24.977	---------	26.810	28.971	31.483
	0.02	25.3366	26.0816	27.1909	27.8846	28.6724	30.5375	32.8034
II	0.00	46.0939	50.7645	55.8951	58.6464	61.5294	67.7158	74.5075
		*46.161	50.850	56.002	58.748	61.635	67.817	74.585
	0.01	47.6367	52.0164	56.9159	59.5701	62.3665	68.4061	75.0801
		*47.714	52.099	57.019	---------	62.467	68.504	75.151
	0.02	49.1322	53.2389	57.9187	60.4797	63.1925	69.0896	75.6486
III	0.00	88.5142	97.7362	107.8311	113.2280	118.8704	130.9307	144.0931
		*88.934	98.250	108.382	114.033	119.493	131.611	144.696
	0.01	89.3233	98.3902	108.3631	113.7092	119.3036	131.2905	144.3920
		*89.741	98.885	108.915	----------	119.941	131.950	145.014
	0.02	90.1253	99.0397	108.8926	114.1884	119.7408	131.6492	144.6905

*Values taken from Lal[27].

TABLE III

Values of frequency parameter Ω, for C-F plate

Mode	K \ α	− 0.5	− 0.3	− 0.1	0.0	0.1	0.3	0.5
				***a/b* = 0.5**				
I	0.00	5.0021	5.2435	5.5347	5.7039	5.8914	6.3305	6.8698
		*4.996	5.239	5.532	5.702	5.890	6.329	6.868
	0.01	13.7214	12.8994	12.2029	11.9052	11.6433	11.2347	10.9960
		*13.772	12.899	12.202	---------	11.642	11.233	10.993
	0.02	18.7272	17.4673	16.3456	15.8409	15.3757	14.5709	13.9477
II	0.00	20.6390	22.2662	24.0173	24.9439	25.9071	27.9557	30.1903
		*20.624	22.264	24.027	24.953	25.923	27.976	30.200
	0.01	23.9999	25.0453	26.3215	27.0443	27.8234	29.5540	31.5270
		*23.992	25.046	26.330	---------	27.838	29.570	31.536
	0.02	26.9567	27.5480	28.4399	28.9929	29.6159	31.0707	32.8103
III	0.00	51.3828	56.2742	61.5821	64.4018	67.3373	73.5740	80.3292
		*51.449	56.392	61.735	64.578	67.484	73.777	80.520
	0.01	52.7831	57.4112	62.5113	65.2441	68.1022	74.2082	80.8590
		*52.848	57.529	62.666	---------	68.287	74.410	81.064
	0.02	54.1485	58.5263	63.4269	66.0756	68.8586	74.8370	81.3852
				***a/b* = 1.0**				
I	0.00	9.5811	10.6814	11.9662	12.6873	13.4666	15.2177	17.2585
		*9.565	10.669	11.956	12.678	13.458	15.209	17.248
	0.01	15.9619	15.8953	16.1652	16.4368	16.8046	17.8433	19.3082
		*15.956	15.888	16.158	---------	16.798	17.836	19.299
	0.02	20.4256	19.7763	19.4789	19.4774	19.5816	20.1285	21.1588
II	0.00	26.2293	28.7538	31.5513	33.0651	34.6636	38.1417	42.0485
		*26.200	28.735	31.544	33.061	34.662	38.140	42.038
	0.01	28.9418	30.9534	33.3383	34.6771	36.1184	39.3289	43.0190
		*28.905	30.938	33.332	---------	36.117	39.327	43.009
	0.02	31.4275	33.0079	35.0343	36.2174	37.5171	40.4816	43.9689
III	0.00	57.0952	62.8050	69.0543	72.3975	75.8961	83.3909	91.6083
		*57.137	62.893	69.180	72.541	76.049	83.558	91.774
	0.01	58.3608	63.8271	69.8847	73.1478	76.5750	83.9494	92.0706
		*58.363	63.916	70.010	---------	76.729	84.113	92.236
	0.02	59.6004	64.8333	70.7054	73.8904	77.2480	84.5043	92.5707

*Values taken from Lal[27].

TABLE IV

Comparison of frequency parameter Ω, for $\alpha = 0$; $a/b = 0.5, 1.0$; $K = 0.0$ with $\nu = 0.3$ for three different boundary conditions

Boundary conditions	C–C		C–S		C–F	
Mode	$a/b = 0.5$	1.0	0.5	1.0	0.5	1.0
I	23.8156	28.9509	17.3317	23.6463	5.7039	12.6873
	23.8156*	28.9508*	17.3316*	23.6463*	--------	---------
	23.828**	28.949**	17.341**	23.648**	5.702**	12.679**
						12.68***
						12.83****
II	63.5346	69.3270	52.0979	58.6464	24.9439	33.0651
	63.5345*	69.3270*	52.0966*	58.6463*	---------	---------
	63.709**	69.462**	52.231**	58.753**	24.959**	33.063**
III	122.9291	129.0950	106.4786	113.2280	64.4018	72.3975
	122.2295*	129.0956*	106.4785*	113.2281*	---------	---------
	123.702**	129.793**	107.115**	113.808**	64.579**	72.540**

* Values taken from Soni[28].

** Values taken from Soni and Rao[15].

*** Value calculated by finite element method Sonzogni *et al.*[11].

**** Value calculated by optimized Kantorovich method Sonzogni *et al.*[11].

CONCLUSION

Numerical results (Tables I-IV) show an agreement of our results with those obtained by quintic splines technique, Frobenius' method, finite element method and optimized Kantorovich method. The agreement between the results is excellent from the engineering view point. The Chebyshev polynomials technique provides the convenience of handling frequency determinants of order fifteen as compared to thirty or more in case of quintic splines. In case of Frobenius' method a series expansion of more than fifty five terms is needed to have the accuracy of 10^{-6} in their absolute values. A 10×10 finite element mesh was used in case of FEM. Thus it appears that Chebyshev polynomials are computationally more efficient than all the above cited numerical techniques.

REFERENCES

1. A. W. Leissa, *Vibration of Plates*, 1969, NASA SP-160.
2. A. W. Leissa, *Shock Vibr. Digest* **10** (1977), 13-24.
3. A. W. Leissa, *Shock Vibr. Digest* **13** (1981), 11-22.
4. A. W. Leissa, *Shock Vibr. Digest* **19** (1987), 11-18.
5. A. W. Leissa, *J. Sound Vibr.* **31** (1973), 257-93.

6. D. J. Gorman, *Free Vibration Analysis of Rectangular Plates*, Elsevier, North-Holland, Amsterdam, 1982.

7. R. B. Bhat, *J. Sound Vibr.* **149** (1991), 170-72.

8. D. V. Bambill, P. A. A. Laura, A. Bergmann and R. Carnicer, *J. Sound Vibr.* **150** (1991), 167-69.

9. S. F. Ng and Y. Araar, *J. Sound Vibr.* **135** (1989), 263-74.

10. R. B. Bhat, P. A. A. Laura, R. G. Gutierrez, V. H. Cortinez and H. C. Sanzi, *J. Sound Vibr.* **138** (1990), 205-19.

11. V. E. Sonzogni, S. R. Idelsohn, P. A. A. Laura and V. H. Cortinez, *J. Sound Vibr.* **140** (1990), 513-22.

12. R. K. Jain and S. R. Soni, *Indian J. pure appl. Math.* **4** (1973), 267-77.

'3. J. E. Ashton, *J. Structural Div. Proc. ASCE* 95-*ST*4 (1969) 787-90.

14. B. B. Raju, *Int. J. Mech. Sci.* **8** (1966), 89-100.

15. S. R. Soni and K. Sankara Rao, *J. Sound Vibr.* **35** (1974), 35-45.

16. U. S. Gupta and R. Lal, *J. Sound Vibr.* **61** (1978), 127-33.

17. R. B. Bhat, *J. Sound Vibr.* **102** (1985), 493-99.

18. R. Sizlard, *Theory and Analysis of Plates*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1974.

19. T. Sakata and Y. Sakata, *J. Acoustical Soc. Am.* **61** (1977), 982-85.

20. S. S. Hwang, *J. appl. Mech. Trans. ASME* **40** (1973), 1127-29.

21. P. A. A. Laura and C. Filipich, *J. Sound Vibr.* **50** (1977), 157-58.

22. P. A. A. Laura and L. E. Luisoni, *Appl. Acoustics* (1979), 333-47.

23. W. Zhixin, *Appl. Math. Mech.* **4** (1983), 103-12.

24. U. S. Gupta, R. Lal and Ritu Sagar, *Indian J. pure appl. Math.* **25** (1994), 1317-26.

25. L. Fox, *The Numerical Solution of Two-point Boundary Value Problems*, Oxford University Press, 1957.

26. M. A. Snyder, *Chebyshev Methods in Numerical Approximations*, Prentice-Hall, Englewood Cliffs, New Jersey, 1969.

27. R. Lal, Vibrations of elastic plates of variable thickness, Ph. D. Thesis, Univ. of Roorkee, 1979.

28. S. R. Soni, Vibrations of elastic plates and shells of variable thickness, Ph. D. Thesis, Univ. of Roorkee, 1975.

29. L. Fox and I. B. Parker, *Chebyshev Polynomials in Numerical* Analysis, Oxford University Press, 1968.