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COINCIDENCE THEOREMS AND EQUILIBRIA OF GENERALIZED GAMES*

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It is well known that F-KKM theorems and coincidence theorems are very useful in different fields of nonlinear analysis. However their applications are restricted by the closedness and compactness assumptions of maps and sets. For relaxing the restrictions, we introduce the notions of compact closure and compact interior for sets in topological spaces and the notions of transfer compactly closed-valued (resp., open-valued) maps. A new H-KKM type theorem is proved in H-spaces. As applications, some coincidence theorems, fixed point theorems and some existence theorems of equilibria for generalized games are obtained.

1. INTRODUCTION

The classical Knaster-Kuratowski-Mazurkiewicz (KKM) theorem[24] is a basic result in nonlinear analysis which is equivalent to many important theorems such as Sperner Lemma, Browder fixed point theorem and Fan's minimax inequality. Since KKM theorem was given, the theorem has been generalized in various directions and has become a very useful tool in treating many sophisticated nonlinear problems from different fields. The most important generalization is the infinitely dimensional F-KKM theorem obtained by Fan[16-18]. Horvath[20, 21], Bardaro and Ceppitelli[1-3], Ding and Tan[12], Tarafdar[29], Ding[9-11], Ding and Tarafdar[13] and Chang and Ma[6] have proved some H-KKM type theorems in the H-spaces without linear structure and given many applications in various area.

However, in the most of above F-KKM and H-KKM type theorems, the closedness and compactness assumptions for mappings and sets always restrain their applications. Recently, by introducing transfer closedness and transfer compact closedness for mappings, Tian[32] and Ding[11] offered some further generalizations of F-KKM and H-KKM type theorems respectively, and gave some applications of their results to minimax inequality, geometric properties of sets, coincidence theorems and maximal elements of preference relations.

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In this paper, we introduce the notions of compact closure and compact interior for sets in topological spaces and the notions of the transfer compactly closed-valued (transfer compactly open-valued) mappings. A new H-KKM type is proved. As applications, some coincidence theorems and fixed point theorems are obtained in H-spaces. Some existence theorems of equilibria of generalized gemes are also proved in H-spaces.

2. Preliminaries

Let X and Y be topological spaces, $\mathcal{F}(X)$ denote the family of all finite subsets of X and 2^Y denote the family of all subsets of Y. Let $F : X \to 2^Y$ be a set-valued mapping. For $A \subset X$ and $y \in Y$, let

$$F(A) = \bigcup \{F(x) : x \in A\} \text{ and } F^{-1}(y) = \{x \in X : y \in F(x)\}.$$

A subset A of a topological space X is said to be compactly closed (resp., compactly open) if for each nonempty compact subset K of X, $A \cap K$ is closed (resp., open) in K. A set $A \subset X$ is called an k-text set if it is compactly closed in X. A topological space X is called an k-space if each k-text set of X is closed in X (or equivalently, a subset B of X is open in X if and only if B is compactly open in X), e.g. see Wilansky[33] (p. 142) or Dugundji[14] (p. 248). However the topological vector space R^R is not an k-space, e.g. see Kelley[23] (p. 240) or Wilansky[33] (p. 143). Hence the notions compact closedness and compact openness for sets are true generalizations of the notions of closedness and openness for sets in topological spaces. We define the compact closure and compact interior of a set $A \subset X$, denote by $ccl(A)$ and $cint(A)$, as

$$ccl(A) = \bigcap \{B \subset X : A \subset B \text{ and } B \text{ is compact closed in } X\}, \text{ and}$$

$$cint(A) = \bigcup \{B \subset X : B \subset A \text{ and } B \text{ is compact open in } X\}.$$

It is easy to see that $ccl(A)$ is the smallest compactly closed subset containing A and $cint(A)$ is the largest compactly open subset which is contained in A. Clearly, for each nonempty compact subset K of X, $K \cap ccl(A) = cl_K(K \cap A)$ and if A is compactly closed (resp., open), then $ccl(A) = A$ (resp., $cint(A) = A$).

A mapping $G : X \to 2^Y$ is said to be transfer compactly closed-valued (resp., transfer compactly open-valued) if for each $x \in X$ and for each compact set $K \subset Y$, $y \notin G(x) \cap K$ (resp., $y \in G(x) \cap K$) implies that there exists $x' \in X$ such that $y \notin cl_K(G(x') \cap K)$ (resp., $y \in int_K(G(x') \cap K)$). Clearly each closed-valued (resp., open-valued) mapping is transfer closed-valued (resp., transfer open-valued) (see Tian[32]) and is also compactly closed-valued (resp., compactly open-valued). Each transfer closed-valued (resp., transfer open-valued) mapping is transfer compactly closed-valued (resp., transfer compactly open-valued) and the inverse is not true in general.

The following notions were introduced by Bardaro and Ceppitelli[1-3]. A pair $(X, \{\Gamma_A\})$ is said to be an H-space if X is a topological space and $\{\Gamma_A\}$ is a family of contractible subsets of X indexed by $A \in \mathcal{F}(X)$ such that $\Gamma_A \subset \Gamma_{A'}$, whenever $A \subset A'$. A subset D of an H-space $(X, \{\Gamma_A\})$ is said to be (i) H-convex if $\Gamma_A \subset D$ for each $A \in \mathcal{F}(D)$; (ii) weakly H-convex if $\Gamma_A \bigcap D$ is contractible for each $A \in \mathcal{F}(D)$; (iii) H-compact if for each $A \in \mathcal{F}(X)$, there exists a compactly weakly H-convex subset D_A of X such that $D \bigcup A \subset D_A$.

Let $\{X, \{\Gamma_A\})$ be an H-space. For each $A \in \mathcal{F}(X)$, Γ_A is said to be polytope in X. $(X, \{\Gamma_A\})$ is said to be an H-space with compact polytopes if each polytope in X is compact. If X is a convex subset of a vector space with finite topology, then X becomes a convex space (see, Lassonde[25]). For each $A \in \mathcal{F}(X)$, let $\Gamma_A = co(A)$, then it is easy see that $(X, \{\Gamma_A\})$ becomes an H-space with compact polytopes.

Following Tarafdar[29], for a nonempty D of an H-space $(X, \{\Gamma_A\})$, define the H-convex hull of D, denoted by $H - co(D)$, as

$$H - co(D) = \bigcap \{B \subset X : D \subset B \text{ and } B \text{ is } H\text{-convex}\}$$

and by Lemma 1 of Tarafdar[29],

$$H - co(D) = \bigcup \{H - co(A) : A \in \mathcal{F}(D)\}.$$

The following notion were introduced by Chang and Ma[6].

Definition 2.1 — Let D be a nonempty set and $(X, \{\Gamma_A\})$ be an H-space. A mapping $F : D \to 2^X$ is said to be a generalized H-KKM mapping if for each $N \in \mathcal{F}(D)$, there exists a single-valued mapping $\theta : N \to X$ such that $M \subset N$ implies $\Gamma_{\theta(M)} \subset F(M)$. Moreover, if D is a subset of $(X, \{\Gamma_A\})$ and θ is the identity mapping in the above definition, then F is said to be H-KKM mapping.

3. H-KKM Type Theorems

The following result is Lemma 2 of Ding and Tan[12] which is a variation of Theorem 1 of Horvath[20].

Lemma 3.1 — Let X be a topological space and $\{R_i\}_{i=0}^{n}$ be a family of subsets of X. Suppose that

(i) for each nonempty subset J of $\{0, ..., n\}$, there exists a contractible subset Γ_J of X such that $\Gamma_J \subset \bigcup_{j \in J} R_j$ and $\Gamma_J \subset \Gamma_{J'}$, whenever $J \subset J'$,

(ii) for each $i \in \{0, ..., n\}$, $R_i \bigcap \Gamma_{\{0, ..., n\}}$ is closed in $\Gamma_{\{0, ..., n\}}$.

Then $\bigcap_{i=0}^{n} R_i \neq \phi$.

Theorem 3.1 — Let D be a nonempty set, $(X, \Gamma_A\})$ be an H-space and $G : D \to 2^X$. If G is generalized H-KKM and for each $N \in \mathcal{F}(D)$ and $z \in N$, $G(z) \cap \Gamma_{\theta(N)}$ is closed in $\Gamma_{\theta(N)}$, then $\{G(z) : z \in D\}$ has the finite intersection property. Conversely, if further assume $\Gamma_{\{x\}} = \{x\}$ for each $x \in X$, then the converse is also true.

PROOF : Since G is a generalized H-KKM mapping, for any $N = \{z_0, ..., z_n\} \in \mathcal{F}(D)$, there exists a single-valued mapping $\theta : N \to X$ such that for any $M \subset N$, $\Gamma_{\theta(M)} \subset \bigcup_{z \in M} G(z)$. By the assumption, for each $z \in N$, $G(z) \cap \Gamma_{\theta(N)}$ is closed in $\Gamma_{\theta(N)}$. It is easy to see that the assumptions of Lemma 3.1 are satisfied and hence we have $\bigcap_{z \in N} G(z) \neq \phi$, that is, $\{G(z) : z \in D\}$ has the finite intersection property.

Now suppose that $\Gamma_{\{x\}} = \{x\}$ for each $x \in X$, and $\{G(z) : z \in D\}$ has the finite intersection property. For any $N = \{z_0, ..., z_n\} \in \mathcal{F}(D)$, take an $x^* \in \bigcap_{i=0}^{n} G(z_i)$. Let $\theta : N \to X$ be the constant mapping to x^*, i.e. $\theta(z) = x^*$ for all $z \in N$. Then for each $M \subset N$, we have

$$\Gamma_{\theta(M)} = \{x^*\} \subset \bigcap_{i=0}^{n} G(z_i) \subset G(M)$$

this shows that G is generalized H-KKM. Since for each $z \in N$, $G(z) \cap \Gamma_{\theta(N)} = \{x^*\} = \Gamma_{\theta(N)}$, therefore $G(z) \cap \Gamma_{\theta(N)}$ is closed in $\Gamma_{\theta(N)}$.

Remark 3.1 : Theorem 3.1 generalized Theorem 3.1 of Chang and Zhang[7] and Theorem 1.2 of Dugundji and Granas[15] to H-spaces. Theorem 3.1 also improves Theorem 1 of Chang and Ma[6] and Lemma of Park[27].

Corollary 3.1 — Let D, $(X, \{\Gamma_A\})$ and G be the same as in Theorem 3.1. Suppose that for some $M \in \mathcal{F}(D)$, the set $\bigcap \{G(z) : z \in M\}$ is compact. If G is generalized H-KKM and for each $N \in \mathcal{F}(D)$ and $z \in N$, $G(z) \cap \Gamma_{\theta(N)}$ is closed in $\Gamma_{\theta(N)}$, then $\bigcap \{G(z) : z \in D\} \neq \phi$. Conversely, if futher assume that $\Gamma_{\{x\}} = \{x\}$ for each $x \in X$, then the inverse is also true.

Remark 3.2 : Corollary 3.1 improves and generalized Theorem 3.2 of Chang and Zhang[7] and Theorem 1 of Chang and Ma[6].

Theorem 3.2 — Let D be a nonempty subset of an H-space $(X, \{\Gamma_A\})$ with compact polytopes and $G : D \to 2^X$ be a generalized H-KKM mapping with compactly closed-valued. Suppose that there exists a nonempty compact subset K of X such that either

(i) $\bigcap_{x \in X} G(x) \subset K$ for some $M \in \mathcal{F}(D)$; or

(ii) for each $N \in \mathcal{F}(D)$, there exists an compact weakly H-convex subset L_N of X with $\theta(N) \subset L_N$ such that

$$L_N \cap \bigcap \{G(z) : z \in L_N \cap D\} \subset K$$

where θ is the single-valued mapping involving in the definition of generalized H-KKM mappings.

Then $\quad K \cap \bigcap \{G(z) : z \in D\} \neq \phi.$

PROOF : Suppose the condition (i) holds. Since for each $N \in \mathcal{F}(D)$ and for any single-valued mapping $\theta : N \to X$, the polytope $\Gamma_{\theta(N)}$ is compact in X and G is compactly closed-valued, we must have that for each $N \in \mathcal{F}(D)$ and $z \in N, G(z) \cap \Gamma_{\theta(N)}$ is closed in $\Gamma_{\theta(N)}$. Thus the conclusion follows from Corollary 3.1.

Now suppose the condition (ii) holds. Define the mapping $G_1 : D \to 2^X$ by

$$G_1(x) = G(x) \cap K.$$

We claim that $\{G_1(z) : z \in D\}$ has the finite intersection property. For any $N \in \mathcal{F}(D)$, let L_N be the set in the condition (ii). Define $G_2 : L_N \cap D \to 2^{L_N}$ by $G_2(z) = G(z) \cap L_N$ for each $z \in L_N \cap D$. Then each $G_2(z)$ is closed in L_N and hence it is compact. For any $A \in \mathcal{F}(L_N \cap D)$ and $z \in L_N \cap D$, $G_2(z) \cap \Gamma_A$ is closed in Γ_A. Moreover, G_2 is generalized H-KKM. In fact, for any $N \in \mathcal{F}(L_N \cap D)$, since $N \in \mathcal{F}(D)$ and G is generalized H-KKM, we have that $M \subset N$ implies

$$\Gamma_{\theta(M)} \subset G(M).$$

Note that $(L_N, \{\Gamma_A \cap L_N\})$ is also an H-space and $\theta(M) \subset \theta(N) \subset L_N$, we have

$$\Gamma_{\theta(M)} \cap L_N \subset L_N \cap G(M) = G_2(M),$$

i.e. G_2 is generalized H-KKM. By Corollary 3.1, $\bigcap \{G_2(z) : z \in L_N \cap D\} \neq \phi$. Let $y \in \bigcap \{G_2(z) : z \in L_N \cap D\} = L_N \cap \bigcap \{G(z) : z \in L_N \cap D\}$. Then $y \in K$ by the assumption (ii) and thus

$$y \in K \cap \bigcap \{G(z) : z \in L_N \cap D\} \subset K \cap \bigcap \{G(z) : z \in N\}$$

$$= \bigcap \{G_1(z) : z \in N\}.$$

This shows that $\{G_1(z) : z \in D\}$ has the finite intersection property. Since K is compact, we must have $K \cap \bigcap \{G(z) : z \in D\} \neq \phi$.

Theorem 3.3 — Let D be a nonempty subset of an H-space $(X, \{\Gamma_A\})$ with compact polytopes, $G : D \to 2^X$ and K be a nonempty compact subset of X such that

(i) G is transfer compactly closed-valued on D,

(ii) the mapping $cclG : D \to 2^X$ defined by $(cclG)(z) = ccl(G(z))$ is generalized H-KKM,

(iii) for each $N \in \mathcal{F}(D)$, there exists a compact weakly H-convex subset L_N of X with $N \bigcup \theta(N) \subset L_N$ such that

$$L_N \bigcap \bigcap \{ccl(G(z)) : z \in L_N \bigcap D\} \subset K.$$

Then $\quad K \bigcap \bigcap \{G(z) : z \in D\} \neq \phi.$

PROOF : Since each $ccl(G(z))$ is compactly closed in X by the definition of compact closure. By applying Theorem 3.2 to $cclG$, we have

$$K \bigcap \bigcap \{ccl(G(z)) : z \in D\} \neq \phi.$$

Now we prove that

$$K \bigcap \bigcap_{z \in D} ccl(G(z)) = \bigcap_{z \in D} [K \bigcap ccl(G(z))]$$

$$= \bigcap_{z \in D} cl_K (K \bigcap G(z))$$

$$= K \bigcap \bigcap_{z \in D} G(z).$$

It is clear that $K \bigcap \bigcap_{z \in D} G(z) \subset K \bigcap \bigcap_{z \in D} ccl(G(z)) = \bigcap_{z \in D} cl_K (K \bigcap G(z))$. So we only need to show that $\bigcap_{x \in D} cl_K (K \bigcap G(z)) \subset \bigcap_{z \in D} (K \bigcap G(z))$. Suppose, by the way of contradiction, that there is some $y \in \bigcap_{z \in D} cl_K (K \bigcap G(z))$ such that $y \notin \bigcap_{z \in D} (K \bigcap G(z))$. By the assumption (i), there exists an $z' \in D$ such that $y \notin cl_K (K \bigcap G(z'))$ which is a contradiction. Hence we have

$$K \bigcap \bigcap_{x \in D} G(z) = K \bigcap \bigcap_{z \in D} ccl(G(z)) \neq \phi.$$

Corollary 3.2 — Let D be a nonempty subset of an H-space $(X, \{\Gamma_A\})$ with compact polytopes, $G : D \to 2^X$ and K be a nonempty compact subset of X such that

(i) G is transfer compact closed-valued on D,

(ii) $cclG$ is an H-KKM mapping,

(iii) for each $N \in \mathcal{F}(D)$, there exists a compact weakly H-convex subset L_N of X with $N \subset L_N$ such that

$$L_N \bigcap \bigcap_{z \in L_N \bigcap D} ccl(G(z)) \subset K.$$

Then $\quad K \bigcap \bigcap_{z \in D} G(z) \neq \phi.$

Corollary 3.3 — Let D be a nonempty subset of a topological vector space X, $G : D \to 2^X$ and K be a nonempty compact subset of X such that

(i) G is transfer compactly closed-valued on D,

(ii) $cclG$ is generalized KKM mapping,

(iii) for each $N \in \mathcal{F}(D)$, there exist a compact convex subset L_N of X with $\theta(N) \subset L_N$ such that

$$L_N \bigcap \bigcap_{z \in L_N \bigcap D} ccl(G(z)) \subset K.$$

Then $\quad K \bigcap \bigcap_{z \in D} G(z) \neq \phi.$

PROOF : For each $A \in \mathcal{F}(X)$, let $co(A) = \Gamma_A$, then $(X, \{\Gamma_A\})$ is an H-space. The conclusion follows from Theorem 3.3.

Remark 3.3 : The coercivity condition (iii) of Corollary 3.3 can be replaced by any one of the following conditions without affecting its conclusion.

$(iii)_1$ there exists a nonempty subset D_0 of D such that $\bigcap_{z \in D_0} ccl(G(z)) \subset K$ and D_0 is contained in a compact convex subset of X (see Fan[18]),

$(iii)_2$ for some $D_0 \in \mathcal{F}(D)$, $\bigcap_{z \in D_0} ccl(G(z)) \subset K$ (see Granas[19]),

$(iii)_3$ for some $x_0 \in D$, $ccl(G(x_0))$ is compact (see Fan[16]),

$(iii)_4$ X itself is compact.

It is easy to see that $(iii)_4 \Rightarrow (iii)_3 \Rightarrow (iii)_2 \Rightarrow (iii)_1 \Rightarrow (iii)$ and the condition (iii) due to Chang[5].

Remark 3.4 : Corollary 3.2 improves and generalizes Theorem 2.1 of Chang[5], Theorems 2 and 3 of Tian[32], Corollaries 3.1 and 3.2 of Ding[11], Theorem III of Lassonde[25]. Note that Corollary 3.2 also includes Lemma 2 of Fan[16], Theorem 4 of Fan[18] and Lemma 1 of Brezis *et al.*[4]. Hence Theorem 3.3 further generalizes above results to H-spaces.

4. Coincidence and Fixed Point Theorems

In this section, we shall prove some coincidence theorems and fixed point theorems by using our generalized H-KKM type theorem.

Theorem 4.1 — Let D be a nonempty subset of an H-space $(X, \{\Gamma_A\})$ with compact polytopes, K be compact subset of X, Y be a topological space and S, T : $D \to 2^Y$ be such that

(1) for each $x \in D$, the set $\{y \in D : S(y) \bigcap T(x) \neq \phi\}$ is H-convex,

(2) the mapping $G : D \to 2^D$ defined by $G(x) = \{y \in D : S(x) \cap T(y) = \phi\}$ is transfer compactly closed-valued,

(3) for each $N \in \mathcal{F}(D)$, there exists a compact weakly H-convex subset L_N of X with $N \subset L_N$ such that for each $y \in L_N \setminus K$ there is $x \in L_N \cap D$ satisfying $y \notin cclG(x)$,

(4) for each $x \in K$, $S(D) \cap T(x) \neq \phi$.

Then there exists $x^* \in K$ such that $S(x^*) \cap T(x^*) \neq \phi$.

PROOF : By (2), the mapping $G : D \to 2^D$ defined by

$$G(x) = \{y \in D : S(x) \cap T(y) = \phi\}, \quad \forall \quad x \in D$$

is transfer compactly closed-valued on D. The condition (3) implies that the condition (iii) of Corollary 3.2 holds. If the mapping $cclG$ is H-KKM, then it follows from Corollary 3.2 that $K \cap \bigcap_{x \in D} G(x) \neq \phi$. Therefore there exists $y \in K$ such that $S(D) \cap T(y) = \phi$ which contradicts the condition (4). Hence $cclG$ is not H-KKM and so there exist $N \in \mathcal{F}(D)$ and $x^* \in \Gamma_N$ such that $x^* \notin \bigcup_{x \in N} cclG(x)$. It follows that $S(x) \cap T(x^*) \neq \phi$ for all $x \in N$ and $N \subset \{z \in D : S(z) \cap T(x^*) \neq \phi\}$. By the condition (1), we have

$$x^* \in \Gamma_N \subset \{z \in D : S(z) \cap T(x^*) \neq \phi\}.$$

Hence $\quad S(x^*) \cap T(x^*) \neq \phi$.

Corollary 4.1 — Let D be a nonempty H-convex subset of an H-space $(X,\{\Gamma_A\})$ with compact polytopes, Y be a topological space and S, $T : X \to 2^Y$ be such that

(1) for each $x \in D$, $S(D) \cap T(x) \neq \phi$ and the set $\{y \in D : S(y) \cap T(x) \neq \phi\}$ is H-convex,

(2) for each $x \in D$, the set $\{y \in D : S(x) \cap T(y) \neq \phi\}$ is compactly open in D,

(3) there exists an H-compact set $L \subset D$ such that for each $y \in D \setminus L, S(L) \cap T(y) \neq \phi$.

Then there exists $x^* \in D$ such that $S(x^*) \cap T(x^*) \neq \phi$.

PROOF : By (2), the mapping $G : D \to 2^D$ defined by

$$G(x) = \{y \in D : S(x) \cap T(y) = \phi\}, \quad \forall \quad x \in D$$

is compactly closed-valued. Since L is H-compact, for any fixed $y_0 \in L$, there exists a compact weakly H-convex subset $L_{\{y_0\}}$ of X satisfying $L \subset L_{\{y_0\}}$. Let $K = L_{\{y_0\}}$. For each $N \in \mathcal{F}(D)$, there exists a compact weakly H-convex subset L_N satisfying $L \bigcup N \subset L_N \subset D$ since D is H-convex. By (3), for each $y \in L_N \setminus K \subset D \setminus L$, there exists $x \in L \subset L_N$ such that $S(x) \bigcap T(y) \neq \phi$. By the definition of G and G is compactly closed-valued, we must have $y \notin G(x) = cclG(x)$. The conclusion follows from Theorem 4.1.

Corollary 4.2 — Let $(X, \{\Gamma_A\})$ be an H-space with compact polytopes, C and D be a nonempty subset and a nonempty H-convex subset of X, respectively and Y be a topological space. Let $S : X \to 2^Y$, $H : C \to 2^Y$ and $g : D \to 2^C$ with $g(x) = \{x\}$ for each $x \in D \bigcap C$ such that

(1) for each $x \in D$, $S(D) \bigcap H(g(x)) \neq \phi$, and the set $\{y \in X : H(g(x)) \bigcap S(y) \neq \phi\}$ is H-convex,

(2) for each $x \in D$, the set $\{y \in D : S(x) \bigcap H(g(y)) \neq \phi\}$ is compactly open in D,

(3) there exists an H-compact set $L \subset D$ such that for each $y \in D \setminus L$, $S(L) \bigcap H(g(y)) \neq \phi$,

(4) for each $x \in D \setminus C$, $S(x) \bigcap H(g(x)) = \phi$.

Then for exists a point $x^* \in C$ such that $S(x^*) \bigcap H(x^*) \neq \phi$.

PROOF : Define a mapping $T : D \to 2^Y$ by

$$T(x) = H(g(x)), \ \forall \ x \in D.$$

Since D is H-convex, by (1), we have that for each $x \in D$, the set

$$\{y \in D : T(x) \bigcap S(y) \neq \phi\} = D \bigcap \{y \in X : T(x) \bigcap S(y) \neq \phi\}$$

is H-convex. Hence, by the assumptions (1), (2) and (3), all conditions of Corollary 4.1 are satisfied. There exist $x^* \in D$ such that $S(x^*) \bigcap H(g(x^*)) = S(x^*) \bigcap T(x^*) \neq \phi$. The assumption (4) implies $x^* \in C$ and hence $x^* \in D \bigcap C$ and $g(x^*) = \{x^*\}$. Therefore $S(x^*) \bigcap H(x^*) \neq \phi$.

Remark 4.1 : Corollary 4.2 generalized Theorem 1 of Huang[22] in the following aspect : (1) S may be a set-valued mapping, (2) S may be discontinuous and (3) D may not be closed.

Corollary 4.3 — Let D be a nonempty H-convex subset of an H-space $(X, \{\Gamma_A\})$ with compact polytopes and $T : D \to 2^X$ be such that

(1) for each $x \in D$, $T(x) \bigcap D$ is nonempty, $T(x)$ is H-convex and for each $y \in D$, $T^{-1}(y)$ is compactly open in D,

(2) there exists an H-compact subset L of D such that for each $x \in D \setminus L$, $T(x) \cap L \neq \phi$. Then there exists a point $x^* \in D$ such that $x^* \in T(x^*)$.

PROOF : Let $X = Y$, and S be the identity mapping in Corollary 4.1. Then all conditions of Corollary 4.1 are satisfied. The conclusion follows from Corollary 4.1.

Remark 4.2 : Corollary 4.3 improves Corollary 1 of Huang[22] in the following : (1) T may not be self-mapping, (2) D may not be closed. Letting $D = L$ is nonempty compact H-convex subset of X and $T : D \to 2^D$, Corollary 4.3 reduces to Corollary 2 of Huang[22].

Corollary 4.4 — Let D be a nonempty H-convex subset of an H-space $(X, \{\Gamma_A\})$ with compact polytopes, $T : D \to 2^X$ compactly open lower section, i.e., for each $y \in X$, $T^{-1}(y)$ is compactly open in D, and for each $x \in D$, $T(x)$ is nonempty H-convex. If there exists a compact H-convex subset $L \subset D$ such that for each $x \in L$, $T(x) \cap L \neq \phi$, then there exists a point $x^* \in L$ such that $x^* \in T(x^*)$.

PROOF : Define a mapping $F : L \to 2^L$ by

$$F(x) = T(x) \cap L, \quad \forall\ x \in L.$$

For each $y \in L$, we have $y \in D$ and

$$F^{-1}(y) = \{x \in L : y \in F(x)\} = \{x \in L : y \in T(x) \cap L\} = L \cap T^{-1}(y)$$

and hence $F^{-1}(y)$ is open in L. Clearly, each $F(x)$ is nonempty H-convex by the assumptions. By Remark 4.2, the conclusion follows from Corollary 4.3.

Remark 4.3 : Corollary 4.4 improves Theorem 2 of Huang[22].

5. Equilibria of Generalized Games

Following Debreu[8] and Shafer and Sonnenschein[28] we shall describe a generalized game with the utility functions by $\varepsilon = (D_i, X_i, F_i, u_i)_{i \in I}$ where $I = \{1, ..., n\}$ be the set of agents, D_i is the choice set of the ith agent, $F_i : D \to 2^{X_i}$ is the constraint correspondence and $u_i : D \to \mathbf{R}$ is the utility function, where $D = \Pi_{i \in I} D_i$ and D_i is a nonempty H-convex subset of the H-space $(X_i, \{\Gamma^i_{A_i}\})$. We denote the product $\Pi_{j \in I, j \neq i} D_i$ by D^i and a generic element of D^i by x^i. A point $\bar{x} \in D$ is called an equilibrium point or a generalized Nash equilibrium point of the generalized game ε if for each $i \in I$,

$$u_i(\bar{x}) = u_i(\bar{x}_i, \bar{x}^i) = \sup_{z_i \in F_i(\bar{x})} u_i(z_i, \bar{x}^i)$$

where $\bar{x}_i$ and $\bar{x}^i$ are respectively projections of $\bar{x}$ onto D_i and D^i. If for all $i \in I$, $F_i(x) = D_i$ for all $x \in D$, the generalized game reduces to the conventional game and the equilibrium is called a Nash equilirbium[26].

A generalized game instead of being given by $\varepsilon = (D_i, X_i, F_i, u_i)_{i \in I}$ may be given by $\Gamma = (D_i, X_i, F_i, P_i)_{i \in I}$ where for each $i \in I$, $P_i : D \to 2^{X_i}$ is the preference correspondence of the ith agent. The relationship between the utility function u_i and the preference correspondence P_i can be exhibited by the definition

$$P_i(x) = \{y_i \in D_i : u_i(y_i, x^i) > u_i(x)\}$$

where (y_i, x^i) is the point of D whose ith co-ordinate is y_i. A point $\bar{x} \in D$ is called an equilibrium point of the generalized game Γ if for each $i \in I$, $\bar{x}_i \in F_i(\bar{x})$ and $P_i(\bar{x}) \cap F_i(\bar{x}) = \phi$. It can be easily checked that a point $\bar{x} \in D$ is an equilibrium point of ε if and only if $\bar{x}$ is an equilibrium point of Γ.

Theorem 5.1 — Let $\Gamma = (D_i, (X_i, \{\Gamma^i_{A_i}\}), F_i, P_i)_{i \in I}$ be a generalized game such that for each $i \in I$,

(1) D_i is a nonempty H-convex subset of an H-space $(X_i, \{\Gamma^i_{A_i}\})$ with compact polytopes,

(2) $F_i, P_i : D = \Pi_{i \in I} D_i \to 2^{X_i}$ is such that for each $x \in D$, $F_i(x) \cap D_i \neq \phi$ and $F_i(x)$ is H-convex,

(3) for each $y_i \in X_i$, the set $[(H-coP_i)^{-1}(y_i) \cup G_i] \cap F_i^{-1}(y_i)$ is compact open in D where $G_i = \{x \in D : F_i(x) \cap P_i(x) = \phi\} = \{x \in D : F_i(x) \cap P_i(x) \cap D_i = \phi\}$.,

(4) there exist a nonempty H-compact set $L_i \subset D_i$ such that for each $x \in D \setminus L$, there is an $y \in L$ satisfying that for each $i \in I, y_i \in F_i(x)$ if $x \in G_i$ and $y_i \in F_i(x) \cap H-coP_i(x)$ if $x \notin G_i$, where $L = \Pi_{i \in I} L_i$.

(5) for each $x \in X$, $x_i \notin H-coP_i(x)$.

Then Γ has an equilibrium point in D.

PROOF : For each $i \in I$, let $G_i = \{x \in D : F_i(x) \cap P_i(x) = \phi\}$ and define the mapping $T_i : D \to 2^{X_i}$ by

$$T_i(x) = \begin{cases} F_i(x) \cap H-coP_i(x), & \text{if } x \notin G_i, \\ F_i(x), & \text{if } x \in G_i. \end{cases}$$

Then, by the conditions (2) and (3), $T_i(x) \cap D$ is nonempty H-convex for each $x \in D$. By the condition (3), for each $y_i \in X_i$, we have

$$\begin{aligned} T_i^{-1}(y_i) &= \{x \in D : y_i \in T_i(x)\} \\ &= \{x \in D \setminus G_i : y_i \in F_i(x) \cap H-coP_i(x)\} \cup \{x \in G_i : y_i \in F_i(x)\} \\ &= [(D \setminus G_i) \cap F_i^{-1}(y_i) \cap (H-coP_i)^{-1}(y_i)] \cup [G_i \cap F_i^{-1}(y_i)] \end{aligned}$$

$$= [F_i^{-1}(y_i) \cap (\dot{H} - coP_i)^{-1}(y_i)] \cup [G_i \cap F_i^{-1}(y_i)]$$

$$= [(H - coP_i)^{-1}(y_i) \cup G_i] \cap F_i^{-1}(y_i)$$

is compactly open in D.

Define the mapping $T: D \to 2^X$ by

$$T(x) = \prod_{i \in I} T_i(x), \quad \forall\ x \in D.$$

Then, D is a nonempty H-convex subset of the H-space $(X, \{\Gamma_A\})$ with compact polytopes where $\Gamma_A = \Pi_{i \in I} \Gamma^i_{A_i}$ and for each $x \in D$, $T(x)$ is nonempty H-convex. For each $y \in X$, we have

$$T^{-1}(y) = \{x \in D : y \in T(x)\} = \bigcap_{i \in I} \left\{x \in D : y_i \in T_i(x)\right\} = \bigcap_{i \in I} T_i^{-1}(y_i)$$

and hence $T^{-1}(y)$ is compact open in D for each $y \in X$.

By (4), $L = \Pi_{i \in I} L_i$ is an H-compact subset of D and for each $x \in D \setminus L$ there is a point $y \in L$ such that for each $i \in I$, $y_i \in T_i(x)$. It follows that $L \cap T(x) \neq \phi$. Now all conditions of Corollary 4.3 are satisfied. By Corollary 4.3, there exists a point $x^* \in D$ such that $x^* \in T(x^*)$ and so $x_i^* \in T_i(x^*)$ for each $i \in I$. If for some $i_0 \in I$, $x_i^* \notin G_{i0}$, then $x_{i0}^* \in F_i(x^*) \cap H - coP_i(x^*) \subset H - coP_i(x^*)$ which contradicts the condition (5). Thus, for each $i \in I$, we must have $x^* \in G_i$. So that for each $i \in I$, $x_i^* \in F_i(x^*)$ and $F_i(x^*) \cap P_i(x^*) = \phi$.

Remark 5.1 : Theorem 5.1 is closely related to Theorem 3.1 of Tarafdar[30] and Theorem 4.1 of Tarafdar[31].

Corollary 5.1 — Let $\Gamma = (D_i, (X_i, \{\Gamma^i_{A_i}\}), F_i, P_i)_{i \in I}$ be a generalized game such that for each $i \in I$,

(1) D_i be a compact H-convex subset of X_i with compact polytopes,

(2) $F_i, P_i : D = \Pi_{i \in I} D_i \to 2^{D_i}$ is such that for each $x \in D$, $F_i(x)$ is nonempty and H-convex,

(3) for each $y_i \in D_i$, thet set $[(H - coP_i)^{-1}(y_i) \cup G_i] \cap F_i^{-1}(y_i)$ is compactly open in D where $G_i = \{x \in D : F_i(x) \cap P_i(x) = \phi\}$.

(4) for each $x \in D$, $x_i \notin H - coP_i(x)$.

Then Γ has an equilibrium point in D.

PROOF : Note that $F_i(x), P_i(x) \subset D_i$ for all $x \in D$, the conclusion follows from Theorem 5.1 with $L_i = D_i$.

For each $i \in I$, let D_i be a nonempty H-convex subset of an H-space $(X_i, \{\Gamma^i_{A_i}\})$ and $D = \Pi_{i \in I} D_i$. A functional $\phi_i : D \times D_i \to \mathbf{R} \cup \{\pm\infty\}$ is said to be 0-generalized diagonally quasiconcave (0-GDQCV) in $y_i \in D_i$ if for any finite set $A_i = \{y_{i1}, ..., y_{im}\} \subset D_i$ and any $x \in D$ with $x_i \in H-coA_i$, we have

$$\min_{1 \le k \le m} \phi(x, y_{i_k}) \le 0.$$

When I contains only one element and D_i is a nonempty convex subset of a topological vector space, the notion of 0-GDQCV coincide with the notion of 0-DQCV introduced by Zhou and Chen[34].

Proposition 5.1 — Let $\phi_i : D \times D_i \to \mathbf{R} \cup \{\pm\infty\}$ be a functional. Then the following conditions are equivalent :

(1) $\phi_i(x, y_i)$ is 0-GDQCV in $y_i \in D_i$,

(2) for each $x \in D$, $x_i \notin H-co(\{y_i \in D_i : \phi_i(x, y_i) > 0\})$.

PROOF : (1) $\Rightarrow$ (2). If (2) does not hold, then, by Lemma 1 of Tarafdar[29], there exist an $x \in D$ and a finite set $A_i = \{y_{i1}, ..., y_{im}\} \subset D_i$ such that $x_i \in H-coA_i$ and $\phi_i(x, y_{ik}) > 0$ for each $k = 1, ..., m$ which contradicts the fact that ϕ_i is 0-GDQCV in $y_i \in D_i$.

(2) $\Rightarrow$ (1). If (1) does not hold, then there exist a finite set $A_i = \{y_{i1}, ..., y_{im}\} \in D_i$ and an $x \in D$ with $x_i \in H-coA_i$ such that $\phi_i(x, y_{i_k}) > 0$ for all $k = 1, ..., m$. It follows that

$$x_i \in H-co(\{y_i \in D_i : \phi_i(x, y_i) > 0\})$$

which contradicts the condition (2).

Theorem 5.2 — Let $\varepsilon = (D_i, (X_i, \{\Gamma^i_{F_i}\}), A_i, u_i)_{i \in I}$ be a generalized game with the utility functions such that for each $i \in I$,

(1) D_i is a nonempty H-convex subset of X_i with compact polytopes,

(2) $F_i : D \to 2^{D_i}$ is such that for each $x \in D$, $F_i(x)$ is nonempty H-convex,

(3) $u_i : D \to \mathbf{R}$ is such that for any finite set $A_i = \{y_{i1}, ..., y_{im}\} \subset D_i$ and for any $x \in D$ with $x_i \in H-coA_i$,

$$\min_{1 \le k \le m} u_i(y_{ik}, x^i) \le u_i(x_i, x^i) = u_i(x),$$

(4) for each $y_i \in D_i$, the set $[(H-coP_i)^{-1}(y_i) \cup G_i] \cap F^{-1}(y_i)$ is compact open in D where $G_i = \{x \in D : u_i(y_i, x^i) > u_i(x)$, for some $y_i \in F_i(x)\}$ and the mapping $P_i : D \to 2^{D_i}$ is defined by $P_i(x) = \{y_i \in D_i : u_i(y_i, x^i) > u_i(x)\}$,

(5) there exist a nonempty H-compact subset L_i of D_i such that for each $x \in D \backslash L$ there is an $y \in L$ such that for each $i \in I$, $y_i \in F_i(x)$ if $x \in G_i$ and $y_i \in F_i(x) \cap H-coP_i(x)$ if $x \notin G_i$,

Then ε has an equilibrium point in D.

PROOF : For each $i \in I$, define the functional $\phi_i : D \times D_i \to \mathbf{R}$ by

$$\phi_i(x, y_i) = u_i(y_i, x^i) - u_i(x_i, x^i), \ \forall \ (x, y_i) \in D \times D_i.$$

By the condition (3), the functional ϕ_i is 0-GDQCV in $y_i \in D_i$. It follows from Proposition 5.1 that for each $x \in D$,

$$x_i \notin H - co(\{y_i \in D_i : \phi_i(x, y_i) > 0\})$$

$$= H - co(\{y_i \in D_i : u_i(y_i, x^i) > u_i(x)\})$$

$$= H - coP_i(x).$$

It is easy to check that the generalized game $\Gamma = (D_i, (X_i, \{\Gamma^i_{A_i}\}), F_i, P_i)_{i \in I}$ satisfies all conditions of Theorem 5.1. Hence there exists a point $x^* \in D$ such that for each $i \in I$, $x_i^* \in F_i(x^*)$ and $F_i(x^*) \cap P_i(x^*) = \phi$. By the definition of P_i, we have

$$F_i(x^*) \cap P_i(x^*) = \{y_i \in F_i(x^*) : u_i(y_i, x^i) > u_i(x^*)\} = \phi$$

and hence we must have for each $i \in I$,

$$u_i(x^*) = \sup_{y_i \in F_i(x^*)} u_i(y_i, x^i).$$

This proves that $x^* \in D$ is an equilibrium point of ε.

Remark 5.2 : Theorem 5.2 improves and generalizes Theorem 3 of Huang[22].

Corollary 5.2 — Let $\varepsilon = ((X_i, \{\Gamma^i_{A_i}\}), F_i, u_i)_{i \in I}$ be a generalized game with utility functions such that for each $i \in I$,

(1) X_i is compact H-space,

(2) $F_i : X = \Pi_{i \in I} X_i \to 2^{X_i}$ is such that for each $x \in X, F_i(x)$ is nonempty H-convex,

(3) $u_i : X \to \mathbf{R}$ is such that for any finite set $A_i = \{y_{i1}, ..., y_{im}\} \subset X_i$ and for any $x \in X$ with $x_i \in H - coA_i$,

$$\min_{1 \le k \le m} u_i(y_{ik}, x^i) \le u_i(x),$$

(4) for each $y_i \in X_i$, the set $[(H - coP_i)^{-1}(y_i) \cup G_i] \cap F_i^{-1}(y_i)$ is open in X where $G_i = \{x \in X : u_i(y_i, x^i) > u_i(x),$ for some $y_i \in F_i(x)\}$ and the mapping $P_i : X \to 2^{X_i}$ is defined by $P_i(x) = \{y_i \in X_i : u(y_i, x^i) > u_i(x)\}$.

Then ε has an equilibrium point in X.

PROOF : For each $i \in I$, let $D_i = L_i = X_i$ in Theorem 5.2. The conclusion follows from Theorem 5.2.

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