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ON COMMON FIXED POINTS OF SEMIGROUPS IN COMPACT METRIC SPACES

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It is shown that if S is a left reversible semigroup of continuous selfmaps on a compact metric space (M, d) such that there exists an s in S satisfying that, for any x, y in M with $x \neq y$, $d(sx, sy) <$ diam $(\{hu : u \in O(x, y), h \in H_S\})$, then it has a unique common fixed point, where $H_S = \left\{ h : M \to M : h\left(\bigcap_{s \in S} sM \right) \subseteq \bigcap_{s \in S} sM \right\}$.

1. INTRODUCTION

Suppose S is a semigroup of selfmaps on a metric space (M, d). For any x in M the orbit of x under S is the set $O(x) := \{x\} \cup \{g(x) : g \in S\}$. If $f \in S$, then the orbit of x under f is the set $O_f(x) := \{x\} \cup \{f^n(x) : n \in \mathbb{N}\}$. The set $O(x, y)$ is the union of $O(x)$ and $O(y)$, and $O_f(x, y)$ is the union of $O_f(x)$ and $O_f(y)$. For a subset A of M, $\delta(A)$ denotes the diameter of A, that is, $\delta(A) := \sup\{d(x, y) : x, y \in A\}$. The semigroup S is said to have diminishing orbital diameters (d.o.d.), if for any x in M with $\delta(O(x)) > 0$ there is $s \in S$ such that $\delta(O(sx)) < \delta(O(x))$. A member f in S is said to have d.o.d. if the subsemigroup generated by f has d.o.d., namely, for any x in M with $\delta(O_f(x)) > 0$ there is $n \in \mathbb{N}$ such that $\delta(O_f(f^n(x)) < \delta(O_f(x))$. The notation H_S is used to denote the set $\left\{ h : M \to M : h\left(\bigcap_{s \in S} sM \right) \subseteq \bigcap_{s \in S} sM \right\}$, and H_f is the set $\left\{ h : M \to M : h\left(\bigcap_{n \in \mathbb{N}} f^n M \right) \subseteq \bigcap_{n \in \mathbb{N}} f^n M \right\}$.

Recently, Liu[7] established that if f is a continuous selfmap on a compact metric space (M, d) such that $d(f(x), f(y)) < \delta(\{hu : u \in O_f(x, y), h \in H_f\})$ for any x, y in M with $x \neq y$, then f has both a fixed point and d.o.d. This result subsumes some theorems in Edelstein[2], Fisher[3] and Hardy and Rogers[4] and answers positively a question raised by Rogers and Hardy[8]. In this paper, we shall generalize Liu's result on a single map to the case of semigroup of selfmaps.

2. The Results

To begin with, recall that a semigroup S is said to be left reversible if for any f, g in S there are a, b in S such that $fa = gb$. We call a semigroup S near-commutative if for any f, g in S there is t in S such that $fg = gt$. It is apparent that every near-commutative semigroup is left reversible.

Let S be a left reversible semigroup. For a, b in S the relation $a \geq b$ iff $a \in bS \cup \{b\}$ renders $(S, \geq)$ a directed set.

Lemma 2.1 — Let S be a left reversible semigroup of continuous selfmaps on a compact metric space (M, d) and let $A = \bigcap_{s \in S} sM$. Then

(i) $\lim_{s \in S} \delta(sM) = \delta(A)$, and

(ii) A is nonempty and $sA = A$ for any s in S.

Proof : For any s, t in S with $s \geq t$ we have $sM \subseteq tM$, and so $\delta(sM) \leq \delta(tM) \leq \delta(M) < \infty$, that is, $\{\delta(sM)\}_{s \in S}$ is a bounded decreasing net in $\mathbb{R}$. Therefore $\lim_{s \in S} \delta(sM)$ exists in $\mathbb{R}$. Obviously, $\delta(A) \leq \lim_{s \in S} \delta(sM)$. On the other hand, for any s in S there are x_s and y_s in sM such that $d(x_s, y_s) = \delta(sM)$. By the compactness of M we can choose two subnets $\{x_{s_j}\}$ and $\{y_{s_j}\}$ of $\{x_s\}$ and $\{y_s\}$ respectively such that $x_{s_j} \to x$ and $y_{s_j} \to y$ for some x, y in M. For simplicity, assume that $x_s \to x$ and $y_s \to y$. Then $d(x_s, y_s) \to d(x, y)$. Now, for any t in S, since tM is closed and x_s, y_s are in tM for any $s \geq t$, we see that x, y are in tM. Since t in S is arbitrary, we conclude that $x, y \in A$. Consequently,

$$\begin{aligned} \lim_{s \in S} \delta(sM) &= \lim_{s \in S} d(x_s, y_s) \\ &= d(x, y) \\ &\leq \delta(A). \end{aligned}$$

The left reversibility of S shows that for any $n \in \mathbb{N}$ and for any $s_1, s_2, \ldots, s_n$ in S there are $a_1, a_2, \ldots, a_n$ in S such that $s_1 a_1 = \ldots = s_n a_n = b$ for some $b \in S$. So $\phi \neq bM \subseteq \bigcap_{i=1}^{n} s_i M$, which implies that $A \neq \phi$.

To show that $tA \subseteq A$ for any $t \in S$, let x be any point in A. Then for any s in S we choose a, b in S such that $ta = sb$. Since $x \in aM$, $x = ay$ for some $y \in M$. Hence $tx = tay = sby \in sM$. This shows that $tx \in \bigcap_{s \in S} sM$. Also, since x in A is arbitrary, we obtain that $tA \subseteq A$ for any $t \in S$. For the reverse inclusion, let $t \in S$

and $y \in A$. That $y \in tsM$ for any $s \in S$ assures the existence of $x_s \in sM$ such that $tx_s = y$. The compactness of M then renders a subnet $\{x_{s_j}\}$ of $\{x_s\}$ such that $x_{s_j} \to x$ for some $x \in M$. It then follows from the continuity of t that $tx = \lim_j t(x_{s_j}) = y$. Now, note that, for any $r \in S$, rM is closed and $x_s \in rM$ for any $s \geq r$. So the limit point x of $\{x_s\}$ lies in rM, and thus $x \in A$. Hence $y = tx \in tA$. Since y in A is arbitrary, we conclude $A \subseteq tA$. This completes the proof of the lemma.

Without the assumption of left reversibility for S, the set A in the above lemma may be empty. For example, let (M, d) be any compact metric space consisting of at least two distinct points, and for any a in M let θ_a denote the map on M with constant value a. Then $S := \{\theta_a : a \in M\}$ is a semigroup under composition. Due to S is not left reversible, $A := \bigcap_{s \in S} sM = \phi$.

Theorem 2.2 — Let S be a left reversible semigroup of continuous selfmaps on a compact metric space (M, d) such that the following property (*) is satisfied :

(*) There exists an $s \in S$ such that for any x, y in M with $x \neq y$ one has $d(sx, sy) < \delta(\{hu : u \in O(x,y),\ h \in H_S\})$.

Then S has a unique common fixed point. Moreover, if S is near-commutative, it has d.o.d.

PROOF : Let $A = \bigcap_{s \in S} sM$. We claim that $\delta(A) = 0$. Otherwise, there are a, b in A such that $\delta(A) = d(a, b) > 0$. Now let s be a member in S with property (*). In view of Lemma 2.1, we can choose x, y in A such that $sx = a$ and $sy = b$. Obviously $x \neq y$. So (*) implies that

$$\delta(A) = d(a, b) = d(sx, sy) < \delta(\{hu : u \in O(x, y),\ h \in H_S\})$$
$$\leq \delta(A),$$

which is a contradiction. Therefore, $\delta(A) = 0$ and so A is a singleton, say $A = \{\xi\}$. Then that $sA = A$ for any $s \in S$ shows that ξ is a common fixed point of S.

For the uniqueness of the common fixed point of S, let ζ be any common fixed point of S. Then it follows from $s\zeta = \zeta$ for any $s \in S$ that $\zeta \in \bigcap_{s \in M} sM = \{\xi\}$. So $\xi = \zeta$.

Finally, assume that S is near-commutative and let x be any point in M. For any s, t in S choose $r \in S$ such that $ts = sr$. Then $tsx = srx \in sM$, and hence

$$\delta(O(sx)) = \delta(\{sx\} \cup \{tsx : t \in S\})$$
$$\leq \delta(sM),$$

which in conjunction with Lemma 2.1 implies that $\lim_{s \in S} \delta(O(sx)) = 0$. Therefore, if $\delta(O(x)) > 0$, then it follows that $\delta(O(tx)) < \delta(O(x))$ for some $t \in S$, that is, S has d.o.d.

A selfmap t on a metric space (M, d) is said to be contractive if $d(tx, ty) < d(x, y)$ for all x, y in M with $x \neq y$. Obviously, a left reversible semigroup S of contractive selfmaps on a metric space (M, d) possesses property (*). So the existence and uniqueness of common fixed point in the Theorem 1.1 of Huang and Huang[5] follow as a corollary :

Corollary 2.3 — Suppose S is a left reversible semigroup of contractive selfmaps on a compact metric space (M, d). Then S has a unique common fixed point in M.

As an end of this note we give some examples to illustrate Theorem 2.2.

Example 1 — Suppose $M = \{1, 3, 4, 6\}$ with the usual Euclidean metric d. Let s, t, θ, a, b, c be the selfmaps on M defined by $s1 = s3 = s4 = 1$, $s6 = 4$, and $t1 = t4 = 1$, $t3 = 4$, $t6 = 3$, and $\theta 1 = \theta 3 = \theta 4 = \theta 6 = 1$, and $a1 = a4 = 1$, $a3 = a6 = 3$, and $b1 = b4 = 1$, $b3 = b6 = 4$, and $c1 = c3 = c4 = 1$, $c6 = 3$. Then $S := \{s, t, \theta, a, b, c\}$ is a semigroup under composition. It is easy to check that $sS \cap tS \cap \theta S \cap aS \cap bS \cap cS = \{\theta\}$, and so S is left reversible. In addition, for any x, y in M with $x \neq y$ we have $\delta(\{hu : u \in O(x, y),\ h \in H_S\}) = 5$. Thus $d(f(x), f(y)) \leq 3 < \delta(\{hu : u \in O(x, y),\ h \in H_S\})$ for any $f \in S$.

Example 2 — Let (M, d) be just as in Example 1 and let α, β be the selfmaps on M defined by $\alpha 1 = 1$, $\alpha 3 = 4$, $\alpha 4 = \alpha 6 = 3$, and $\beta 1 = 1$, $\beta 3 = \beta 6 = 4$, $\beta 4 = 3$. Then $S := \{\alpha, \alpha^2, \beta, \beta^2\}$ is a semigroup under composition. As is easily seen that $\alpha S = \alpha^2 S = \beta S = \beta^2 S = S$, S is near-commutative. Also, since $\alpha\beta = \beta^2$ and $\beta\alpha = \alpha^2$, S is not commutative. Noting that for any x, y in $\{1, 3, 4\}$ with $x \neq y$ one has $\delta(\{hu : u \in O(1, 3),\ h \in H_S\}) = 3$ and $d(\alpha(1), \alpha(3)) = d(\beta(1), \beta(3)) = d(\alpha^2(1), \alpha^2(4)) = d(\beta^2(1), \beta^2(4)) = 3$, S does not have property (*), but it still has a common fixed point 1.

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