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MEAN CONVERGENCE OF LAGRANGE INTERPOLATION

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The purpose of the paper is to investigate weighted L^p convergence of Lagrange interpolation taken at the zeros of Hermite polynomials. It is shown that if a continuous function satisfies some conditions, then the corresponding Lagrange interpolation process converges in every L^p $(1 < p < \infty)$ and the rate of convergence is estimated.

Let $\{H_n\}_{n=0}^{\infty}$ denote the system of Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

and let $x_{1n} > x_{2n} > \ldots > x_{nn}$ be the zeros of H_n. Then for a given function f the Lagrange interpolation polynomial $L_n(f)$ corresponding to H_n is defined to be the unique algebraic polynomial of degree at most $n-1$ which satisfies

$$L_n(f, x_{kn}) = f(x_{kn}) \qquad (k = 1, 2, \ldots, n).$$

It is well known that $L_n(f)$ can be written in the form

$$L_n(f, x) = \sum_{k=1}^{n} f(x_{kn})\, l_{kn}(x),$$

where the fundamental polynomials l_{kn} are defined by

$$l_{kn}(x) = \frac{H_n(x)}{H_n'(x_{kn})(x - x_{kn})}.$$

We write briefly x_k, $l_k(x)$. The purpose of this paper is to investigate weighted L^p convergence properties of $L_n(f)$. We refer the interested reader to different

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authors[2, 4, 9-11]. Denote $W(x) := e^{-x^2/2}$ and

$$\|f\|_p := \left\{ \int_{-\infty}^{\infty} |f(x)|^p \, dx \right\}^{1/p} \qquad (1 \le p < \infty),$$

$$\|f\|_\infty := \text{vrai max} \, |f(x)| \qquad (p = \infty).$$

For a function $f : \mathbb{R} \to \mathbb{R}$ we shall use the generalized continuity modulus $\omega_p(f, \delta)$ (see e.g. Freud[5, 6])

$$\omega_p(f, \delta) := \sup_{|t| \le \delta} \| W(x+t) f(x+t) - W(x) f(x) \|_p + \| \tau(\delta x) W(x) f(x) \|_p,$$

where

$$\tau(x) := \begin{cases} |x|, & \text{if } |x| \le 1 \\ 1, & \text{if } |x| > 1. \end{cases}$$

Our result is the following

Theorem — If $Wf \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $(1 \le p < \infty)$, f is continuous, $L_n(f, x)$ is the Lagrange interpolation process, $\omega_p\left(f, \frac{1}{\sqrt{n}}\right)$ and $\omega_\infty\left(f, \frac{1}{\sqrt{n}}\right)$ are the generalized continuity modulus, then we have

$$\left(\int_{-\infty}^{\infty} (e^{-x^2/2} |f(x) - L_{n+1}(f, x)|)^p \, dx \right)^{1/p}$$

$$= O(1) \, \omega_p\left(f, \frac{1}{\sqrt{n}}\right) + O(1) \, n^{1/(2p)} \log n \, \omega_\infty\left(f, \frac{1}{\sqrt{n}}\right).$$

Remark : If p = 2, then using orthogonality property

$$\int_{-\infty}^{\infty} e^{-x^2} \, l_k(x) \, l_j(x) \, dx = 0 \; (k \ne j)$$

we can eliminate the log n factor in the theorem.

For the proof we need a lemma.

Lemma — If $Wf \in L^p(\mathbb{R})$ and $1 \le p \le \infty$ then there exist polynomials p_n of degree $\le n$ such that

$$\| W(f - p_n) \|_p \le c(p) \, \omega_p\left(f, \frac{1}{\sqrt{n}}\right),$$

where the constant $c(p) > 0$ depends only on p.

PROOF : see Freud[5,6].

Remark : The polynomials p_n do not depend on p, because in the proofs was used the de La Vallée-Poussin means which is independent on p.

PROOF OF THE THEOREM : Using the Lemma we obtain

$$\left(\int_{-\infty}^{\infty} (e^{-x^2/2} \,|\, f(x) - L_{n+1}(f, x) \,|\,)^p \, dx \right)^{1/p}$$

$$\leq \left(\int_{-\infty}^{\infty} (e^{-x^2/2} \,|\, f(x) - p_n(x) \,|\,)^p \, dx \right)^{1/p} + \left(\int_{-\infty}^{\infty} (e^{-x^2/2} \,|\, L_{n+1}(p_n - f, x) \,|\,)^p \, dx \right)^{1/p}$$

$$= O(1)\, \omega_p\left(f, \frac{1}{\sqrt{n}}\right) + O(1)\, \omega_\infty\left(f, \frac{1}{\sqrt{n}}\right)$$

$$\times \left(\int_{-\infty}^{\infty} \left(e^{-x^2/2} \sum_{k=1}^{n} e^{x_k^2/2} \,|\, l_k(x) \,|\, \right)^p dx \right)^{1/p} \qquad \dots (1)$$

Using (6.1) and (6.2) of Joó[8] we obtain

$$e^{-x^2/2} \sum_{k=1}^{n} e^{x_k^2/2} \,|\, l_k(x) \,|\, \asymp e^{-x^2/2} \frac{|\, H_n(x) \,|}{\sqrt{2^n\, n!}} \sum_{k=1}^{n} \sqrt{\varphi_n(x_k)} \frac{1}{|\, x - x_k \,|},$$

where $\varphi_n(x_k) = x_k - x_{k+1}$. Hence

$$\left(\int_{-\infty}^{\infty} \left(e^{-x^2/2} \sum_{k=1}^{n} e^{x_k^2/2} \,|\, l_k(x) \,|\, \right)^p dx \right)^{1/p}$$

$$\leq \left(\int_{0}^{\infty} \left(e^{-x^2/2} \frac{|\, H_n(x) \,|}{\sqrt{2^n\, n!}} \sum_{k=1}^{n} \sqrt{\varphi_n(x_k)} \frac{1}{|\, x - x_k \,|} \right)^p dx \right)^{1/p}$$

$$+ \left(\int_{-\infty}^{0} \left(e^{-x^2/2} \frac{|\, H_n(x) \,|}{\sqrt{2^n\, n!}} \sum_{k=1}^{n} \sqrt{\varphi_n(x_k)} \frac{1}{|\, x - x_k \,|} \right)^p dx \right)^{1/p} . \qquad \dots (2)$$

Since $|\, H_n(-x) \,| = |\, H_n(x) \,|$, the zeros are symmetrical with respect to the origin, $x_k = -x_{n-k}$, therefore $\varphi_n(x_k) \asymp \varphi_n(x_{n-k})$ and $\frac{1}{|\, x - x_k \,|} \geq \frac{1}{|\, x - x_{n-k} \,|}$ if $x \geq 0$, thus it is enough to estimate

$$\int_0^\infty \left(e^{-x^2/2} \frac{|H_n(x)|}{\sqrt{2^n n!}} \sum_{k=1}^{n/2} \sqrt{x_k - x_{k+1}} \frac{1}{|x - x_k|} \right)^p dx. \qquad \dots (3)$$

Here

$$\int_0^\infty \left(e^{-x^2/2} \frac{|H_n(x)|}{\sqrt{2^n n!}} \sum_{k=1}^{n/2} \sqrt{x_k - x_{k+1}} \frac{1}{|x - x_k|} \right)^p dx = \int_0^{x_{n/2}} + \sum_{j=1}^{n/2-1} \int_{x_{j+1}}^{x_j} + \int_{x_1}^\infty \qquad \dots (4)$$

We know[3]

$$e^{-x^2} \sum_{k=1}^{n} e^{x_k^2} l_k^2(x) \le 1 \quad (x \in \mathbb{R});$$

this means that finitely many members of the sums (2), (3) and (4) can be estimated by $O(1)$. Hence

$$\int_{x_{j+1}}^{x_j} \left(e^{-x^2/2} \frac{|H_n(x)|}{\sqrt{2^n n!}} \right)^p \left(\sum_{k=1}^{n/2} \sqrt{x_k - x_{k+1}} \frac{1}{|x - x_k|} \right)^p dx$$

$$= \int_{x_{j+1}}^{x_j} \left(e^{-x^2/2} \frac{|H_n(x)|}{\sqrt{2^n n!}} \right)^p \left(\sum_{\substack{k=1 \\ k \ne j, j\pm1, j+2}}^{n/2} \sqrt{x_k - x_{k+1}} \frac{1}{|x - x_k|} \right)^p dx$$

$$+ \int_{x_{j+1}}^{x_j} O(1)\, dx. \qquad \dots (5)$$

We know [Joó[7], (7)]

$$x_k - x_{k+1} \asymp n^{-1/6} k^{-1/3}, \quad 1 \le k \le n/2. \qquad \dots (6)$$

If $x_{j+1} \le x \le x_j$ and $k \ne j,\ j \pm 1,\ j+2$ then

$$|x - x_k| \asymp |x_j - x_k| \asymp n^{-1/6} (j^{-1/3} + \dots + k^{-1/3}) \asymp \frac{|k^{2/3} - j^{2/3}|}{n^{1/6}}. \qquad \dots (7)$$

We know that (Askey and Wainger[1], p. 700)

$$\frac{e^{-x^2/2} |H_n(x)|}{\sqrt{2^n n!}} \le c n^{-1/8} (\sqrt{2n+1} - x_j)^{-1/4}, \quad x_{j+1} \le x \le x_j. \qquad \dots (8)$$

We know[12] that $\sqrt{2n+1} - x_1 \asymp n^{-1/6}$, hence by (6)

$$\sqrt{2n+1} - x_j \asymp \sum_{i=1}^{j} n^{-1/6} i^{-2/3} \asymp n^{-1/6} j^{2/3} \quad (1 \le j \le n/2). \qquad \dots (9)$$

Using (6), (7), (8) and (9) we obtain

$$\int_{x_{j+1}}^{x_j}\left(e^{-x^2/2}\frac{|H_n(x)|}{\sqrt{2^n n!}}\right)^p\left(\sum_{\substack{k=1\\k\neq j,j\pm 1,j+2}}^{n/2}\sqrt{x_k-x_{k+1}}\,\frac{1}{|x-x_k|}\right)^p dx$$

$$\asymp n^{p/12}\left(\sum_{\substack{k=1\\k\neq j,j\pm 1,j+2}}^{n/2}\frac{k^{-1/6}}{|k^{2/3}-j^{2/3}|}\right)^p\int_{x_{j+1}}^{x_j}\left(e^{-x^2/2}\frac{|H_n(x)|}{\sqrt{2^n n!}}\right)^p dx$$

$$= O(1)\,n^{-1/6}j^{-(p+2)/6}\left(\sum_{\substack{k=1\\k\neq j,j\pm 1,j+2}}^{n/2}\frac{k^{-1/6}}{|k^{2/3}-j^{2/3}|}\right)^p. \qquad \dots (10)$$

Obviously

$$\sum_{\substack{k=1\\k\neq j,j\pm 1,j+2}}^{n/2}\frac{k^{-1/6}}{|k^{2/3}-j^{2/3}|}=\sum_{k=1}^{j/2}+\sum_{\substack{k=j/2\\k\neq j,j\pm 1,j+2}}^{3j/2}+\sum_{\substack{k=3j/2\\k\neq j,j\pm 1,j+2}}^{n/2}.$$

Here

$$\sum_{k=1}^{j/2}\frac{k^{-1/6}}{|k^{2/3}-j^{2/3}|}\asymp\sum_{k=1}^{j/2}\frac{k^{-1/6}}{j^{2/3}}\asymp j^{1/6},$$

and

$$\sum_{\substack{k=3j/2\\k\neq j+1,j+2}}^{n/2}\frac{k^{-1/6}}{|k^{2/3}-j^{2/3}|}\asymp\sum_{k=3j/2}^{n/2}\frac{k^{-1/6}}{j^{2/3}}=O(1)\,j^{1/6}.$$

Finally

$$\sum_{\substack{k=j/2\\k\neq j,j\pm 1,j+2}}^{3j/2}\frac{k^{-1/6}}{|k^{2/3}-j^{2/3}|}\asymp\sum_{\substack{k=j/2\\k\neq j,j\pm 1,j+2}}^{3j/2}\frac{j^{-1/6}}{j^{-1/3}|k-j|}\asymp j^{1/6}\log(j+1).$$

Hence

$$n^{-1/6}j^{-(p+2)/6}\left(\sum_{\substack{k=1\\k\neq j,j\pm 1,j+2}}^{n/2}\frac{k^{-1/6}}{|k^{2/3}-j^{2/3}|}\right)^p=O(1)\,n^{-1/6}j^{-1/3}\log^p(j+1).$$

$$\dots (11)$$

Using (11) we obtain from (10)

$$\sum_{j=1}^{n/2-1} \int_{x_{j+1}}^{x_j} \left(e^{-x^2/2} \frac{|H_n(x)|}{\sqrt{2^n n!}} \sum_{\substack{k=1 \\ k \neq j, j \pm 1, j+2}}^{n/2} \sqrt{x_k - x_{k+1}} \frac{1}{|x - x_k|} \right)^p dx$$

$$= O(1) \sqrt{n} \log^p n. \qquad \text{... (12)}$$

Using the above estimation and (5) we have

$$\int_{x_{n/2}}^{x_1} \left(e^{-x^2/2} \frac{|H_n(x)|}{\sqrt{2^n n!}} \sum_{k=1}^{n/2} \sqrt{x_k - x_{k+1}} \frac{1}{|x - x_k|} \right)^p dx = O(1) \sqrt{n} \log^p n. \qquad \text{... (13)}$$

Similarly

$$\int_{0}^{x_{n/2}} \left(e^{-x^2/2} \frac{|H_n(x)|}{\sqrt{2^n n!}} \sum_{k=1}^{n/2} \sqrt{x_k - x_{k+1}} \frac{1}{|x - x_k|} \right)^p dx = O(1) \sqrt{n} \log^p n \text{ ... (14)}$$

We know that (Askey and Wainger[1], p.700)

$$e^{-x^2/2} \frac{|H_n(x)|}{\sqrt{2^n n!}} \leq cn^{-1/8} (x - \sqrt{2n+1})^{-1/4}$$

$$\times \exp [-\xi (2n+1)^{1/4} (x - \sqrt{2n+1})^{3/2}]$$

$$\leq cn^{-1/8} (x - \sqrt{2n+1})^{-1/4}, \quad x_1 \leq x \leq \sqrt{2} x_1, \qquad \text{... (15)}$$

where ξ is some positive number. Introduce the numbers x_j^*, $(1 \leq j \leq n/2)$ in the following way :

$$x_1^* = x_1, \quad x_j + x_j^* = 2x_1, \qquad (2 \leq j \leq n/2).$$

Then we can write

$$\int_{x_1}^{\infty} \left(e^{-x^2/2} \frac{|H_n(x)|}{\sqrt{2^n n!}} \sum_{k=1}^{n/2} \sqrt{x_k - x_{k+1}} \frac{1}{|x - x_k|} \right)^p dx = \sum_{j=1}^{n/2-1} \int_{x_j^*}^{x_{j+1}^*} + \int_{x_{n/2}^*}^{\infty}.$$

Similarly as we proved (13) we obtain

$$\int_{x_1^*}^{x_{n/2}^*} \left(e^{-x^2/2} \frac{|H_n(x)|}{\sqrt{2^n n!}} \sum_{k=1}^{n/2} \sqrt{x_k - x_{k+1}} \frac{1}{|x - x_k|} \right)^p dx = O(1) \sqrt{n} \log^p n. \qquad \text{... (16)}$$

We know that (Askey and Wainger[1], p.700)

$$e^{-x^2/2}\frac{|H_n(x)|}{\sqrt{2^n\, n!}} = O(1)\exp(-\xi x^2), \quad x \ge \sqrt{2}x_1.$$

Using this we obtain easily

$$\int_{x^*_{n/2}}^{\infty}\left(e^{-x^2/2}\frac{|H_n(x)|}{\sqrt{2^n\, n!}}\sum_{k=1}^{n/2}\sqrt{x_k - x_{k+1}}\,\frac{1}{|x-x_k|}\right)^p dx = O(1). \qquad \ldots (17)$$

From (2), (3), (12), (14), (16), (17) and (5) we obtain

$$\left(\int_{-\infty}^{\infty}\left(e^{-x^2/2}\sum_{k=1}^{n} e^{x_k^2/2}\,|l_k(x)|\right)^p dx\right)^{1/p} = O(1)\, n^{1/(2p)}\log n. \qquad \ldots (18)$$

Now we prove that the estimation (18) is sharp.
From (2), (3) and (10) we obtain

$$\int_{-\infty}^{\infty}\left(e^{-x^2/2}\sum_{k=1}^{n} e^{x_k^2/2}\,|l_k(x)|\right)^p dx$$

$$\ge c\int_0^{\infty}\left(e^{-x^2/2}\frac{|H_n(x)|}{\sqrt{2^n\, n!}}\right)^p\left(\sum_{k=1}^{n/2}\sqrt{x_k - x_{k+1}}\,\frac{1}{|x-x_k|}\right)^p dx$$

$$\ge c\sum_{j=1}^{n/2-1}\int_{x_{j+1}}^{x_j}\left(e^{-x^2/2}\frac{|H_n(x)|}{\sqrt{2^n\, n!}}\right)^p\left(\sum_{\substack{k=1\\ k\neq j,\, j\pm 1,\, j+2}}^{n/2}\sqrt{x_k - x_{k+1}}\,\frac{1}{|x-x_k|}\right)^p dx$$

$$\ge cn^{p/12}\sum_{\substack{j=1\\ \sqrt{n}/2\le x_j\le\sqrt{n}}}^{n/2-1}\int_{x_{j+1}}^{x_j}\left(e^{-x^2/2}\frac{|H_n(x)|}{\sqrt{2^n\, n!}}\right)^p\left(\sum_{\substack{k=1\\ k\neq j,\, j\pm 1,\, j+2}}^{n/2}\frac{k^{-1/6}}{|k^{2/3}-j^{2/3}|}\right)^p dx$$

$$\ge cn^{p/12}\sum_{\substack{j=1\\ \sqrt{n}/2\le x_j\le\sqrt{n}}}^{n/2-1}(j^{1/6}\log j)^p\int_{x_{j+1}}^{x_j}\left(e^{-x^2/2}\frac{|H_n(x)|}{\sqrt{2^n\, n!}}\right)^p dx$$

$$\ge cn^{p/4}\log^p n\int_{\sqrt{n}/2}^{\sqrt{n}}\left(e^{-x^2/2}\frac{|H_n(x)|}{\sqrt{2^n\, n!}}\right)^p dx. \qquad \ldots (19)$$

We know that [Szegö[12], (8.22.12)]

$$e^{-\frac{x^2}{2}} H_n(x) = 2^{\frac{n}{2}+\frac{1}{4}} (n!)^{\frac{1}{2}} (\pi n)^{-\frac{1}{4}} (\sin\varphi)^{-\frac{1}{2}}$$
$$\times \left\{ \sin\left[\left(\frac{n}{2}+\frac{1}{4}\right)(\sin 2\varphi - 2\varphi) + \frac{3\pi}{4}\right] + O\left(\frac{1}{n}\right)\right\}, \qquad \text{... (20)}$$

where $x = \sqrt{2n+1}\cos\varphi$, $0 < \varepsilon \le \varphi \le \pi - \varepsilon$, ε is an arbitrary but fixed number, the error term is uniform in φ.

Using (20) we obtain

$$\int_{\sqrt{n}/2}^{\sqrt{n}} \left(e^{-x^2/2} \frac{|H_n(x)|}{\sqrt{2^n n!}} \right)^p dx$$

$$\ge c\frac{\sqrt{n}}{n^{p/4}} \int_{\varphi_b}^{\varphi_a} \left| \sin\left[\left(\frac{n}{2}+\frac{1}{4}\right)(\sin 2\varphi - 2\varphi) + \frac{3\pi}{4}\right] + O\left(\frac{1}{n}\right)\right|^p d\varphi, \qquad \text{... (21)}$$

where $\sqrt{n} = \sqrt{2n+1}\cos\varphi_b$, $\sqrt{n}/2 = \sqrt{2n+1}\cos\varphi_a$. Take an arbitrary fixed integer r such that $2r \ge p$. Then we have

$$|\sin\alpha|^p \ge \sin^{2r}\alpha.$$

It is well known that

$$\sin^{2r}\alpha = \frac{1}{2^{2r}}\left\{ \sum_{i=0}^{r-1} (-1)^{r-i}\, 2\binom{2r}{i} \cos 2(r-i)\alpha + \binom{2r}{r}\right\}.$$

Using these we obtain

$$\int_{\varphi_b}^{\varphi_a} \left| \sin\left[\left(\frac{n}{2}+\frac{1}{4}\right)(\sin 2\varphi - 2\varphi) + \frac{3\pi}{4}\right]\right|^p d\varphi$$

$$\ge \int_{\varphi_b}^{\varphi_a} \sin^{2r}\left[\left(\frac{n}{2}+\frac{1}{4}\right)(\sin 2\varphi - 2\varphi) + \frac{3\pi}{4}\right] d\varphi$$

$$= \frac{1}{2^{2r}}\binom{2r}{r}\frac{\varphi_a - \varphi_b}{2} + \frac{1}{2^{2r}} \sum_{i=0}^{r-1} (-1)^{r-i}\, 2\binom{2r}{i}$$

$$\times \int_{\varphi_b}^{\varphi_a} \cos\,(r-i)\left[\left(n+\frac{1}{2}\right)(\sin 2\varphi - 2\varphi) + \frac{3\pi}{2}\right] d\varphi.$$

Here

$$\cos(r-i)\left[\left(n+\frac{1}{2}\right)(\sin 2\varphi - 2\varphi) + \frac{3\pi}{2}\right]$$

$$= \pm\binom{\cos}{\sin}(r-i)\left[\left(n+\frac{1}{2}\right)(\sin 2\varphi - 2\varphi)\right].$$

Introducing the new variable $s := \sin 2\varphi - 2\varphi$ and integrating by part we obtain

$$\int_{\varphi_b}^{\varphi_a} \cos(r-i)\left[\left(n+\frac{1}{2}\right)(\sin 2\varphi - 2\varphi) + \frac{3\pi}{2}\right] d\varphi = O\left(\frac{1}{n}\right)$$

Hence from (21) we obtain

$$\int_{\sqrt{n}/2}^{\sqrt{n}} \left(e^{-x^2/2} \frac{|H_n(x)|}{\sqrt{2^n n!}}\right)^p dx \geq c\,\frac{\sqrt{n}}{n^{p/4}}. \qquad \dots (22)$$

From (19) and (22) we get

$$\left(\int_{-\infty}^{\infty} \left(e^{-x^2/2} \cdot \sum_{k=1}^{n} e^{x_k^2/2} |l_k(x)|\right)^p dx\right)^{1/p} \geq c n^{1/(2p)} \log n,$$

which means that (18) is sharp. The Theorem is proved.

Conjecture — There exist continuous functions f_n, $Wf_n \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that

$$\int_{-\infty}^{\infty} e^{-x^2/2} |f_n(x) - L_{n+1}(f_n, x)|\, dx \asymp \log n$$

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