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ON LOCALLY *s*-CLOSED SPACES

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Di Maio and Noiri[7] defined and studied the class of *s*-closed topological spaces. In this paper we define the class of locally *s*-closed spaces and investigate its several properties.

1. INTRODUCTION

As a generalization of *S*-closed spaces due to Thompson[14], Noiri[11] introduced and investigated the notion of locally *S*-closed spaces. Di Maio and Noiri[7] introduced the notion of *s*-closed spaces. The class of *s*-closed spaces is contained in the class of *S*-closed spaces. In the present paper, we introduce a new class of spaces called locally *s*-closed spaces and investigate the fundamental properties. Especially, in section 4 we obtain some preservation theorems by using quasi-irresolute functions and semi θ-closed functions.

2. PRELIMINARIES

Let X be a topological space and A be a subset of X. The closure of A and the interior of A are denoted by $Cl\,(A)$ and $Int\,(A)$, respectively. A subset A is said to be semi-open (Levine[9]) if there exists an open set U of X such that $U \subset A \subset Cl\,(U)$. The complement of a semi-open set is said to be semi-closed. A subset A of X is said to be semi-regular (Di Maio and Noiri[7]) if it is semi-open and semi-closed. The intersection of all semi-closed sets containing A is called the semi-closure (Crossley and Hildebrand[5]) of A and is denoted by $sCl\,(A)$. The union of all semi-open sets contained in A is called the semi-interior of A and is denoted by $sInt\,(A)$. The family of all semi-open (resp. semi-regular) sets of X is denoted by $SO(X)$ (resp. $SR(X)$).

Definition 2.1 — A subset A of a topological space X is said to be

(a) s-closed relative to X (Di Maio and Noiri[7]) (resp. S-closed relative to X (Noiri[10])) if for every cover $\{V_\alpha : \alpha \in \nabla\}$ of A by semi-open sets of X, there exists a finite subfamily ∇_0 of ∇ such that $A \subset \bigcup \{sCl\ (V_\alpha) : \alpha \in \nabla_0\}$ (resp. $A \subset \bigcup \{Cl(V_\alpha) : \alpha \in \nabla_0\}$),

(b) quasi H-closed relative to X (Porter and Thomas[13]) if for every cover $\{V_\alpha : \alpha \in \nabla\}$ of A by open sets of X there exists a finite subset ∇_0 of ∇ such that $A \subset \bigcup \{Cl(V_\alpha) : \alpha \in \nabla_0\}$.

A topological space X is said to be s-closed (Di Maio and Noiri[7]) (resp. S-closed (Thompson[14]), quasi H-closed (Porter and Thomas[13])) if the set X is s-closed relative to X (resp. S-closed relative to X, quasi H-closed relative to X).

Definition 2.2 — A function $f : X \to Y$ is said to be

(a) irresolute (Crossley and Hildebrand[6]) $f^{-1}(V) \in SO(X)$ for every $V \in SO(Y)$,

(b) quasi-irresolute (Di Maio and Noiri[7]) if for each $x \in X$ and each $V \in SO(Y)$ containing $f(x)$, there exists $U \in SO(X)$ containing x such that $f(U) \subset sCl\ (V)$,

(c) semi-closed preserving (Ahmad and Noiri[2]) if $f(A)$ is semi-closed in Y for each semi-closed set A of X,

(d) pre-semi-open (Crossley and Hildebrand[6]) if $f(U) \in SO(Y)$ for every $U \in SO(X)$.

Throughout the present paper, X and Y denote topological spaces and $f : X \to Y$ means a function of X into Y. No separation axioms are assumed unless explicitly stated.

3. Locally s-Closed Spaces

Definition 3.1 — A space X is said to be locally s-closed if each point of X has an open neighbourhood which is an s-closed subspace.

Remark 3.1 : Recently, Basu[3] has defined a topological space X to be locally s-closed if each point of X belongs to a regular open neighbourhood which is an s-closed subspace. A subset A of X is said to be regular open if $A = Int(Cl(A))$.

Theorem 3.1 — A space X is locally s-closed if and only if each point of X has an open neighbourhood which is s-closed relative to X.

Proof : In Theorem 1 of Basu[3] it is shown that a preopen set A of a space X is s-closed as a subspace if and only if it is s-closed relative to X. Since every open set is pre-open, the proof is obvious. A subset A is called preopen if $A \subset Int\ (Cl(A))$.

Definition 3.2 — A space X is said to be locally S-closed (Noiri[11]) (resp. locally quasi H-closed if each point of X has an open neighbourhood which is an S-closed (resp. quasi H-closed) subspace.

A space X is said to be extremally disconnected if the closure of each open set of X is open in X.

Corollary 3.1 — The following are equivalent for an extremely disconnected space X :

(a) X is locally s-closed;

(b) X is locally S-closed;

(c) X is locally quasi H-closed.

PROOF : This immediately follows from Lemma 3.8 of Noiri[12].

Theorem 3.2 — If X is a locally s-closed space and $A \in SO(X)$, then A is locally s-closed.

PROOF : Let $x \in A$. Since X is locally s-closed, by Theorem 3.1 there exists an open neighbourhood U of x such that U is s-closed relative to X. Since A is semi-open, by Theorem 4 of Khan *et al.*[8] $A \cap U$ is s-closed relative to A. Since $A \cap U$ is an open neighbourhood of x in the subspace A, by Theorem 3.1 A is locally s-closed.

Corollary 3.2 — Local s-closedness is an open hereditary property.

Theorem 3.3 — A space X is locally s-closed if and only if for each point x of X, there exists an open set A of X such that $x \in A$ and A is locally s-closed.

PROOF : *Necessity* — This is obvious.

Sufficiency — Let $x \in X$. There exists an open set A such that $x \in A$ and A is locally s-closed. Therefore, there exists an open neighbourhood U of x in A such that U is an s-closed subspace of A. Since A is open in X, U is open in X. By Theorem 11 of Khan *et al.*[8], U is an s-closed subspace of X. This proves that X is locally s-closed.

4. LOCALLY s-CLOSED SPACES AND SOME FUNCTIONS

Theorem 4.1 — If X is a locally s-closed space and $f : X \to Y$ is an open quasi-irresolute surjection, then Y is locally s-closed.

PROOF : Let $y \in Y$. There exists $x \in X$ such that $f(x) = y$. Since X is locally s-closed, by Theorem 3.1 there exists an open neighbourhood of x which is s-closed relative to X. Since f is open, $f(U)$ is an open neighbourhood of y. Since f is quasi-irresolute, by Corollary 5.1 of Di Maio and Noiri[7] $f(U)$ is s-closed relative to Y. Hence by Theorem 3.1 Y is locally s-closed.

Corollary 4.1 — Local s-closedness is preserved under continuous open surjections.

PROOF : This follows from the fact that every continuous open function is irresolute (Crossley and Hildebrand[6], Theorem 1.8) and irresoluteness implies quasi-irresoluteness (Di Maio and Noiri[7], Lemma 5.1).

Corollary 4.2 — Let $\{X_\alpha : \alpha \in \nabla\}$ be any family of topological spaces. If the product space $\Pi \{X_\alpha : \alpha \in \nabla\}$ is locally s-closed, then X_α is locally s-closed for each $\alpha \in \nabla$.

Definition 4.1 — A function $f: X \to Y$ is said to be strongly semi-continuous El-Monsef *et al.*[1] if for each semi-open set V of $Y, f^{-1}(V)$ is open in X.

Theorem 4.2 — If $f: X \to Y$ is an open strongly semi-continuous surjection and X is locally quasi H-closed, then Y is locally s-closed.

PROOF : Let $y \in Y$. There exists $x \in X$ such that $f(x) = y$. Since X is locally quasi H-closed, there exists an open neighbourhood U of x which is a quasi H-closed subspace. Every quasi H-closed subspace is quasi H-closed relative to X. Now, let $\{V_\alpha : \alpha \in \nabla\}$ be a cover of $f(U)$ and $V_\alpha \in SO(Y)$ for each $\alpha \in \nabla$. Then we have $U \subset \bigcup \{f^{-1}(V_\alpha) : \alpha \in \nabla\}$ and $f^{-1}(V_\alpha)$ is open in X for each $\alpha \in \nabla$. Since U is quasi H-closed relative to X, there exists a finite subfamily ∇_0 of ∇ such that $U \subset \bigcup \{Cl(f^{-1}(V_\alpha)) : \alpha \in \nabla_0\}$. The strong semi-continuity of f implies that

$$f(U) \subset \bigcup \{f(Cl(f^{-1}(V_\alpha))) : \alpha \in \nabla_0\} \subset \bigcup \{sCl(V_\alpha) : \alpha \in \nabla_0\}.$$

This shows that $f(U)$ is s-closed relative to Y. Since f is open, $f(U)$ is an open neighbourhood of y. Therefore, by Theorem 3.1, Y is locally s-closed.

Theorem 4.3 — If $f: X \to Y$ is a semi-continuous surjection and X is an s-closed space, then Y is quasi H-closed.

PROOF : Let $\{V_\alpha : \alpha \in \nabla\}$ be an open cover of Y. Then $\{f^{-1}(V_\alpha) : \alpha \in \nabla\}$ is a semi-open cover of X. Since X is s-closed, there exists a finite subfamily ∇_0 of ∇ such that $X = \bigcup \{sCl(f^{-1}(V_\alpha)) : \alpha \in \nabla_0\}$. It follows from Theorem 4 of Biswas[4] that

$$Y = f(X) = \bigcup \{f(sCl(f^{-1}(V_\alpha))) : \alpha \in \nabla_0\} \subset \bigcup \{Cl(V_\alpha) : \alpha \in \nabla_0\}.$$

This shows that Y is quasi H-closed.

Lemma 4.1 — The following are equivalent for a bijection $f: X \to Y$:

(a) f is pre-semi-open;

(b) f is semi-closed preserving;

(c) $f^{-1}: Y \to X$ is irresolute.

Lemma 4.2 — If $f: X \to Y$ is a semi-closed preserving bijection, then $f(U) \in SR(Y)$ for every $U \in SR(X)$.

Theorem 4.4 — If $f: X \to Y$ is a semi-closed preserving bijection and K is s-closed relative to Y, then $f^{-1}(K)$ is s-closed relative to X.

PROOF : Let $\{U_\alpha : \alpha \in \nabla\}$ be a cover of $f^{-1}(K)$ by semi-regular sets of X. Since f is a semi-closed preserving bijection, by Lemma 4.2 $\{f(U_\alpha) : \alpha \in \nabla\}$ is a cover of K by semi-regular sets of Y. Since K is s-closed relative to Y, by Proposition 4.1 Di Maio and Noiri[7] there exists a finite subfamily ∇_0 of ∇ such that $K \subset \bigcup \{f(U_\alpha) : \alpha \in \nabla_0\}$. Therefore, we obtain $f^{-1}(K) \subset \bigcup \{U_\alpha : \alpha \in \nabla_0\}$. This shows that $f^{-1}(K)$ is s-closed relative to X.

Corollary 4.3 — If $f: X \to Y$ is a semi-closed preserving bijection and Y is an s-closed space, then X is s-closed.

Theorem 4.5 — If Y is a locally s-closed space and $f: X \to Y$ is a continuous semi-closed preserving bijection, then X is locally s-closed.

PROOF : Let $x \in X$ and $y = f(x)$. Since Y is locally s-closed, there exists an open neighbourhood V of $f(x)$ which is s-closed relative to Y by Theorem 3.1. Since f is a continuous and semi-closed preserving bijection, $f^{-1}(V)$ is an open neighbourhood of x and s-closed relative to X by Theorem 4.4. Hence by Theorem 3.1, X is locally s-closed.

A point $x \in X$ is said to be in the semi θ-closure Di Maio and Noiri[7] of a subset A of X, denoted by $sCl_\theta(A)$, if $A \cap sCl(U) \neq \phi$ for every $U \in SO(X)$ conaining x. A subset A is said to be semi θ-closed (Di Maio and Noiri[7]) if $A = sCl_\theta(A)$. The complement of a semi θ-closed set is said to be semi θ-open.

Definition 4.2 — A function $f: X \to Y$ is said to be semi θ-closed (Di Maio and Noiri[7]) if $f(K)$ is semi θ-closed in Y for every semi θ-closed set K of X.

Theorem 4.6 — The following are equivalent for a function $f: X \to Y$:

(a) f is semi θ-closed;

(b) $sCl_\theta(f(A)) \subset f(sCl_\theta(A))$ for every subset A of X;

(c) for every subset B of Y and every semi θ-open set U of X containing $f^{-1}(B)$, there exists a semi θ-open set V of Y containing B such that $f^{-1}(V) \subset U$;

(d) for every point $y \in Y$ and every semi θ-open set U of X containing $f^{-1}(y)$, there exists a semi θ-open set V of Y containing y such that $f^{-1}(V) \subset U$.

PROOF : The proof is similar to that for analogous results in the case of a closed function and is thus omitted.

It is shown in Proposition 4.1 of Di Maio and Noiri[7] that a subset A of a space X is s-closed relative to X if and only if every cover of A by semi-regular sets of X has a finite subcover.

Lemma 4.3 (Basu[3]) — A subset A of a space X is s-closed relative to X if and only if every cover of A by semi θ-open sets of X has a finite subcover.

Theorem 4.7 — Let $f: X \to Y$ be a semi θ-closed function such that $f^{-1}(y)$ is s-closed relative to X for each point y of Y. If K is s-closed relative to Y, then $f^{-1}(K)$ is s-closed relative to X.

PROOF : Let $\{U_\alpha : \alpha \in \nabla\}$ be any cover of $f^{-1}(K)$ by semi θ-open sets of X. For each $y \in K, f^{-1}(y)$ is s-closed relative to X and by Lemma 4.3 there exists a finite subset $\nabla(y)$ of ∇ such t hat $f^{-1}(y) \subset \bigcup \{U_\alpha : \alpha \in \nabla(y)\}$. Let $U(y) = \bigcup \{U_\alpha : \alpha \in \nabla(y)\}$, then $U(y)$ is semi θ-open in X. Since f is semi θ-closed, by Theorem 4.6 there exists a semi θ-open set $V(y)$ containing y such that $f^{-1}(V(y)) \subset U(y)$. Since $\{V(y) : y \in K\}$ is a semi θ-open cover of K, by Lemma 4.3

there exists a finite subset K_0 of K such that $K \subset \bigcup \{V(y) : y \in K_0\}$. Therefore, we obtain

$$f^{-1}(K) \subset \bigcup \{f^{-1}(V(y)) : y \in K_0\} \subset \bigcup \{U_\alpha : \alpha \in \nabla(y),\ y \in K_0\}.$$

This shows that $f^{-1}(K)$ is s-closed relative to X.

Corollary 4.4 — Let $f : X \to Y$ be a semi θ-closed surjection such that $f^{-1}(y)$ is s-closed relative to X for each point $y \in Y$. If Y is s-closed, then X is s-closed.

Theorem 4.8 — Let $f : X \to Y$ be a continuous semi θ-closed function such that $f^{-1}(y)$ is s-closed relative to X for each $y \in Y$. If Y is locally s-closed, then X is locally s-closed.

PROOF : Let $x \in X$. Since Y is locally s-closed, by Theorem 3.1 there exists an open neighbourhood V of $f(x)$ such that V is s-closed relative to Y. Since f is continuous, $f^{-1}(V)$ is an open neighbourhood of x and is s-closed relative to X by Theorem 4.7. Therefore, it follows from Theorem 3.1 that X is locally s-closed.

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