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ALL 5-STARS ARE SKOLEM GRACEFUL

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A graph $G = (V, E)$ is Skolem graceful if its vertices can be labelled 1, 2, ... $|V|$ so that the edges are labelled 1, 2, ... $|E|$ where each edge label is the absolute difference of the labeLs of the two end vertices. A k-star is a graph with k components, with atmost one vertex of degree exceeding 1 in each component. It is known that all 1-stars, 4-stars are Skolem graceful and that not all 2-stars and 3-stars are Skolem graceful. In this paper we prove that all 5-stars are Skolem graceful, and obtain a necessary condition for a k-star to be Skolem graceful.

INTRODUCTION

A simple graph $G = (V, E)$ is said to be graceful if there exists an injection $g : V \to \{0, 1, ..., |E|\}$ such that the induced function $g' : E(G) \to \{1, 2, ..., |E|\}$ defined by $g'(u, v) = |g(u) - g(v)|$ is a bijection. Recently, there have been several variations of this interesting concept. See Gallian[3] for more details.

In Lee and Wui[4], a Skolem graceful labelling of G is defined as a bijection $f : V \to \{1, 2, .., |V|\}$ such that the induced labelling $f^* : E \to \{1, 2, ..., |E|\}$ defined by $f^*(u, v) = |f(u) - f(v)|$, $(u, v) \in E$, is also a bijection. Such an f is called an S-labelling of G and the graph G is said to be Skolem graceful.

It is obvious that if G is Skolem graceful then $|E| \leq |V| - 1$ and moreover, a tree (more generally any graph with vertices and $n - 1$ edges) is Skolem graceful if and only if it is graceful.

In Lee and Wui[6], an interesting observation was made connecting the existence of Skolem sequences (Nickerson[7] and Skolem[8]) and Skolem gracefulness of the graph kK_2 and deduced that kK_2 is Skolem graceful if and only if $k \equiv 0$ or 1 (mod 4). As a natural generalization of the graph kK_2, Lee and Wui[4] defined a k-star $St(\alpha_1, \alpha_2, ..., \alpha_k)$ to be a disconnected graph with k components, $K_{1,\alpha_1}, K_{1,\alpha_2}, ..., K_{1,\alpha_k}$ where $K_{1,n}$ denotes a star with $n + 1$ vertices. Clearly, all 1-stars are Skolem graceful. Lee and Wui[4] proved that a 2-star $St(\alpha_1, \alpha_2)$ is Skolem graceful if and only if $\alpha_1 \alpha_2$ is even and a 3-star $St(\alpha_1, \alpha_2, \alpha_3)$ is Skolem graceful if and only if $\alpha_1 \alpha_2 \alpha_3$ is even. Denham *et al.*[2] proved that all 4-stars are Skolem graceful thus settling a conjecture of Lee *et al.*[5] and remarked that their *S*-labelling of 4-stars is not promising enough to obtain *S*-labelling of *k*-stars even for the case $k = 5$. This remark and the following problem motivated us to do this research.

Problem (Lee and Wui[4]) — Give a characterization of Skolem graceful *k*-stars.

In this paper, we first obtain a necessary condition for a *k*-star to be Skolem graceful and then show that every 5-star is Skolem graceful.

RESULTS

Theorem 1 — If a *k*-star $St(\alpha_1, \alpha_2, ..., \alpha_k)$ is Skolem graceful then either some α_i is even or $k \equiv 0$ or $1 \pmod 4$.

PROOF : Assume that $St(\alpha_1, \alpha_2, ..., \alpha_k)$ is Skolem graceful through a Skolem graceful labelling f and that every α_i is odd. We show that $k \equiv 0$ or $1 \pmod 4$.

Let $c_1; c_2, ..., c_k$ denote the labels of the central vertices of the *k*-stars.

For $1 \le i \le k$, let $a_{i1}, a_{i2}, ..., a_{i\beta_i}$ be the vertex labels of the vertices adjacent with c_i such that $a_{ij} > c_i$ for $1 \le j \le \beta_i$; and let $b_{i1}, b_{i2}, ..., b_{i\gamma_i}$ be the vertex labels of the vertices adjacent with c_i such that $b_{ij} \le c_i$ for $1 \le j \le \gamma_i$.

Let $x = \alpha_1 + \alpha_2 + ... + \alpha_k$. Then, by definition, vertices have received the labels $1, 2, ..., x + k$ and edges have received the labels $1, 2, ..., x$. Since these labels are graceful labels, we get :

$$\frac{x(x+1)}{2} = \sum_{e \in E(G)} f^*(e) = \sum_{i=1}^{k} \sum_{j=1}^{\beta_i} (a_{ij} - c_i) + \sum_{i=1}^{k} \sum_{j=1}^{\gamma_i} (c_i - b_{ij}) \quad ... (1)$$

$$\frac{(x+k)(x+k+1)}{2} = \sum_{v \in V(G)} f(v) = \sum_{i=1}^{k} c_i + \sum_{i=1}^{k} \sum_{j=1}^{\beta_i} a_{ij} + \sum_{i=1}^{k} \sum_{j=1}^{\gamma_i} b_{ij} . \quad ... (2)$$

Adding (1) and (2) we get,

$$x^2 + (k+1)x + \frac{k^2+k}{2}$$

$$= \sum_{i=1}^{k} c_i + \sum_{i=1}^{k} \left(\sum_{j=1}^{\beta_i} (a_{ij} - c_i + a_{ij}) + \sum_{j=1}^{\gamma_i} (c_i - b_{ij} + b_{ij}) \right)$$

$$= \sum_{i=1}^{k} c_i + \sum_{i=1}^{k} ((\beta_i + \gamma_i) c_i) + 2 \sum_{i=1}^{k} \left(\sum_{j=1}^{\beta_i} a_{ij} - \beta_i c_i \right)$$

$$= \sum_{i=1}^{k} (\beta_i + \gamma_i + 1) c_i + 2 \sum_{i=1}^{k} \left(\sum_{j=1}^{\beta_i} a_{ij} - \beta_i c_i \right)$$

$$= \sum_{i=1}^{k} (\alpha_i + 1) c_i + 2 \sum_{i=1}^{k} \left(\sum_{j=1}^{\beta_i} a_{ij} - \beta_i c_i \right), \text{ since } \alpha_i = \beta_i + \gamma_i.$$

Since every α_i is odd, the right-hand side of the above expression is even. Hence, $x^2 + (k+1)x + \frac{k^2+k}{2}$ is even. But if $k \equiv 2$ or 3 (mod 4), it is easy to see that $x^2 + (k+1)x + \frac{k^2+k}{2}$ is an odd integer. Hence $k \equiv 0$ or 1 (mod 4). ■

We believe that the converse of Theorem 1 is true, i.e.,

(i) if $k \equiv 0$ or 1 (mod 4), then every k-star is Skolem graceful;

(ii) if $k \equiv 2$ or 3 (mod 4), then a k-star is Skolem graceful if some α_i is even.

The results of Lee and Wui[4], Denham *et al.*[2] and our Theorem 2 show that (i) and (ii) are true for all $k \leq 5$. In Choudum and Kishore[1], we have shown that (i) and (ii) are true for several values of k.

To prove every 5-star is Skolem graceful, we adopt the following notation :

Let c_1, c_2, c_3, c_4, c_5 denote the central vertices of the 5-star $St(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$.

Let v_{ij}, $1 \leq j \leq \alpha_i$ be the vertices joined to c_i, $1 \leq i \leq 5$.

Lemma 1 — $St(2p, 2q, 2r, 2s, 2t)$ is Skolem graceful.

PROOF : We define the S-labelling f as follows :

$$f(c_1) = 3,\ f(c_2) = 2,\ f(c_3) = 1,$$

$$f(c_4) = 2p + 2q + 2r + 2s + 2t + 5,$$

$$f(c_5) = 2p + 2q + 2r + 2s + 2t + 4,$$

$$f(v_{1i}) = \begin{cases} 3 + i & \text{for } 1 \leq i \leq p, \\ 2q + 2r + 2s + 2t + 3 + i & \text{for } p + 1 \leq i \leq 2p, \end{cases}$$

$$f(v_{2i}) = \begin{cases} p + 3 + i & \text{for } 1 \leq i \leq q, \\ p + 2r + 2s + 2t + 2 + i & \text{for } q + 1 \leq i \leq 2q, \end{cases}$$

$$f(v_{3i}) = \begin{cases} p + q + 4 & \text{for } i = 1, \\ p + q + s + 3 + i & \text{for } 2 \leq i \leq r, \\ p + q + s + 2t + 1 + i & \text{for } r + 1 \leq i \leq 2r, \end{cases}$$

$$f(v_{4i}) = \begin{cases} p + q + 4 + i & \text{for } 1 \leq i \leq s, \\ p + q + 2r + 2t + 1 + i & \text{for } s + 1 \leq i \leq 2s, \end{cases}$$

$$f(v_{5i}) = \begin{cases} p+q+r+s+3+i & \text{for } 1 \le i \le 2t-2, \\ p+q+2r+2s+2t+2 & \text{for } i = 2t-1, \\ p+2q+2r+2s+2t+3 & \text{for } i = 2t. \end{cases}$$ ■

Lemma 2 — St($2p$, $2q$, $2r+1$, s, $2t$) is Skolem graceful.

PROOF : We define the S-labelling f as follows :

$f(c_1) = 3$, $f(c_2) = 2$, $f(c_3) = 1$, $f(c_4) = 2p + 2q + 2r + s + 2t + 6$,

$f(c_5) = 2p + 2q + 2r + s + 2t + 5$,

$$f(v_{1i}) = \begin{cases} 3+i & \text{for } 1 \le i \le p, \\ 2q+2r+s+2t+4+i & \text{for } p+1 \le i \le 2p, \end{cases}$$

$$f(v_{2i}) = \begin{cases} p+3+i & \text{for } 1 \le i \le q, \\ p+2r+s+2t+3+i & \text{for } q+1 \le i \le 2q, \end{cases}$$

$$f(v_{3i}) = \begin{cases} p+q+t+2+i & \text{for } 1 \le i \le r+1, \\ p+q+s+t+2+i & \text{for } r+2 \le i \le 2r+1, \end{cases}$$

$$f(v_{4i}) = p + q + r + t + 3 + i \qquad \text{for } 1 \le i \le s,$$

$$f(v_{5i}) = \begin{cases} p+q+3+i & \text{for } 1 \le i \le t-1, \\ p+q+2r+s+4+i & \text{for } t \le i \le 2t-1, \\ p+2q+2r+s+2t+4 & \text{for } i = 2t. \end{cases}$$ ■

Lemma 3 — St($2p$, $2q+1$, $2r+1$, $2s+1$, t) is Skolem graceful.

PROOF : We define the S-labelling f as follows :

$f(c_1) = 3$, $f(c_2) = 2$, $f(c_3) = 1$, $f(c_4) = 2p + 2q + 2r + 2s + t + 8$,

$f(c_5) = 2p + 2q + 2r + 2s + t + 7$,

$$f(v_{1i}) = \begin{cases} 3+i & \text{for } 1 \le i \le p, \\ 2q+2r+2s+t+6+i & \text{for } p+1 \le i \le 2p, \end{cases}$$

$$f(v_{2i}) = \begin{cases} p+3+i & \text{for } 1 \le i \le q+1, \\ p+2r+2s+t+4+i & \text{for } q+2 \le i \le 2q+1, \end{cases}$$

$$f(v_{3i}) = \begin{cases} p+q+s+4+i & \text{for } 1 \le i \le r, \\ p+q+s+t+3+i & \text{for } r+1 \le i \le 2r+1, \end{cases}$$

$$f(v_{4i}) = \begin{cases} p+q+4+i & \text{for } 1 \le i \le s, \\ p+q+2r+t+4+i & \text{for } s+1 \le i \le 2s+1, \end{cases}$$

$$f(v_{5i}) = \begin{cases} p+q+r+s+4+i & \text{for } 1 \le i \le t-1, \\ p+2q+2r+2s+t+6 & \text{for } i = t. \end{cases}$$ ■

Lemma 4 — St($2p+1$, $2q+1$, $2r+1$, $2s+1$, $2t+1$) is Skolem graceful.

PROOF : We define the S-labelling f as follows :

$$f(c_1) = 3,\ f(c_2) = 2,\ f(c_3) = 1,$$

$$f(c_4) = 2p + 2q + 2r + 2s + 2t + 10,$$

$$f(c_5) = 2p + 2q + 2r + 2s + 2t + 9,$$

$$f(v_{1i}) = \begin{cases} 3+i & \text{for } 1 \le i \le p+1, \\ 2q+2r+2s+t+7+i & \text{for } p+2 \le i \le 2p+1, \end{cases}$$

$$f(v_{2i}) = \begin{cases} p+4+i & \text{for } 1 \le i \le q, \\ p+2r+2s+2t+6+i & \text{for } q+1 \le i \le 2q+1, \end{cases}$$

$$f(v_{3i}) = \begin{cases} p+q+t+4+i & \text{for } 1 \le i \le r+1, \\ p+q+2s+t+4+i & \text{for } r+2 \le i \le 2r+1, \end{cases}$$

$$f(v_{4i}) = \begin{cases} p+q+r+t+5+i & \text{for } 1 \le i \le 2s, \\ p+2q+2r+2s+2t+8 & \text{for } i = 2s+1, \end{cases}$$

$$f(v_{5i}) = \begin{cases} p+q+4+i & \text{for } 1 \le i \le t, \\ p+q+2r+2s+5+i & \text{for } t+1 \le i \le 2t+1. \end{cases}$$ ■

Lemmas 1 to 4 imply :

Theorem 2 — All 5-stars are Skolem graceful.

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