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CHARGED ANISOTROPIC FLUID SPHERES IN GENERAL RELATIVITY

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This is a pedagogical study of the general relativistic field equations which are solved for a charged anisotropic fluid sphere in analogy with the anisotropic uncharged models existing in literature. The solutions obtained are in a closed form without assuming any particular radial distribution for the 'mass' of the sphere.

1. INTRODUCTION

Anisotropic neutral self-gravitating fluid spheres have been extensively studied in general relativity[1-3]. On the other hand an attempt to model a charged anisotropic fluid sphere with certain simplifying assumptions like neglecting effect of anisotropy on the non-gravitational energy density distribution is also found in literature[4]. In this paper we attempt to develop a model of a charged anisotropic fluid sphere following the method of the neutral sphere as developed by Singh *et al.*[3] and the theory of charged spherically symmetric fluid and its distribution function as given by Maharaj and Maartens[5]. The general relativistic field equations along with the energy-momentum conservation equations are only used without any assumptions like the vanishing of the Weyl tensor. The form of the radial distribution of charges is kept fixed and the mass density and pressure distribution which follows along with the form of the metric is studied.

2. FIELD EQUATIONS OF GENERAL RELATIVITY FOR ANISOTROPIC CHARGED SPHERES

The usual spherically symmetric interior metric is

$$ds^2 = e^{\lambda}\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 - c^2 e^{\nu}\, dt^2. \qquad \dots (1)$$

The energy-momentum tensor given by eqn. (2.2) of Singh *et al.*[3] has to be modified to take into account the electromagnetic field energy and momentum which is given in the presence of a static spherically symmetric distribution of charges by Herrera and Leon[6] as

$$T_0^0 = T_1^1 = -T_2^2 = -T_3^3 = (1/8\pi)\frac{1}{r^4}\left[\int_0^r 4\pi\, e^{\lambda/2}\,\rho_c(R)\,R^2\,dR\right]^2 \qquad \text{... (2)}$$

where $\rho_c(r)$ is the charge density. The distribution function given by eqn. (4.5) of Maharaj and Maartens[5] for a spherically symmetric charged fluid shows that it is an invariant similar to the one for neutral sphere. So the dynamical quantitites of the fluid the density, pressure and the anisotropic pressure tensor takes the same form as those defined by eqns. (2.3)-(2.5) of Singh *et al.*[3]. In addition we have already assumed a charge distribution ρ_c which is a function of the radial co-ordinate only. With this additional source term that is the energy-momentum tensor given by eqn. (2) above and maintaining the same notations of Singh *et al.*[3] (without setting c = G = 1 to maintain the distinction with electromagnetic quantities expressed in e.s.u.) we have the general relativistic field equations

$$\frac{1}{r^2} + e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) = \frac{8\pi G}{c^4}\left(c^2\rho + \frac{1}{8\pi r^4}\left[\int_0^r 4\pi e^{\lambda/2}\,\rho_c(R)\,R^2\,dR\right]^2\right) \qquad \text{... (3)}$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = \frac{8\pi G}{c^4}\left(p + \frac{2s}{\sqrt{3}} - \frac{1}{8\pi r^4}\left[\int_0^r 4\pi e^{\lambda/2}\,\rho_c(R)\,R^2\,dR\right]^2\right) \qquad \text{... (4)}$$

$$e^{-\lambda}\left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} + \frac{\nu'-\lambda'}{2r}\right]$$

$$= \frac{8\pi G}{c^4}\left(p - \frac{s}{\sqrt{3}} + \frac{1}{8\pi r^4}\left[\int_0^r 4\pi e^{\lambda/2}\,\rho_c(R)\,R^2\,dR\right]^2\right). \qquad \text{... (5)}$$

We write $p_r = p + 2s/\sqrt{3}$ and $p_\perp = p - s/\sqrt{3}$ as the radial and tangential pressures respectively and ρ is the mass density of the fluid. The energy-momentum conservation equation gives

$$(\rho c^2 + p_r)\nu' + 2p_r' + \frac{4\sqrt{3}}{r}s - \frac{8\pi}{r^2}e^{\lambda/2}\rho_c\int_0^r e^{\lambda/2}\rho_c(R)\,R^2\,dR = 0. \qquad \text{... (6)}$$

Equations (3)-(6) consist of only three independent equations since the left-hand side of Einstein's field equations are related through the Bianchi identity. Thus for any assumed charge and mass distribution inside the anisotropic sphere we have four independent quantities $p_r, p_\perp$, e^ν and e^λ to determine from three equations. Thus we are free to choose any one quantity provided it satisfies the boundary conditions to be discussed in the next section.

For the charge distribution we assume a simple distribution to simplify the solution of our field equations and is taken as

$$e^{\lambda/2}\,\rho_c(R) = \rho_{0c}/R^2 \qquad \dots (7)$$

where ρ_{0c} is a constant. The total charge inside a body of 'radius' a is thus $4\pi\rho_{0c}\,a$. Although this functional form has a singularity at $r = 0$ the total charge in a finite radius sphere is free of any divergence and hence with slight modification can be used as any residual unneutralized charge density distribution in a realistic model. In this paper we accept the relation (7) throughout for our calculations. On the other hand we make no assumptions regarding the mass density distribution leaving the solution in terms of a general mass function $m(r)$ defined as

$$m(r) = 4\pi \int_0^r R^2\,\rho(R)\,dR \qquad \dots (8)$$

which in other words is $\rho(r) = m'(r)/(4\pi r^2)$. The $\rho(r)$ can be assumed as is done for the various realistic models and can be substituted in the general solution obtained here.

3. Solution of the Field Equations and the Boundary Conditions

The various boundary conditions that any spherical body in equilibrium must satisfy are :

(1) The internal metric of the body must go over to the external Reissner-Nordstrom metric at $r = a$ for a charged sphere which is given as

$$ds^2 = -\left(1 - \frac{2\,Gm(a)}{c^2 r} + \frac{G(4\pi\rho_{0c})^2\,a^2}{c^4 r^2}\right)c^2\,dt^2$$

$$+\left(1 - \frac{2\,Gm(a)}{c^2 r} + \frac{G(4\pi\rho_{0c})^2\,a^2}{c^4 r^2}\right)^{-1} dr^2 + r^2\,(d\theta^2 + \sin^2\theta\,d\phi^2) \qquad \dots (9)$$

(2) The radial pressure p_r on the surface must vanish (Gokhroo and Mehra[7]) to prevent a massive radial pressure gradient there causing inequilibrium which will eject fluid radially outward (or inward). However one need not demand the vanishing of the tangential pressure over there since this will only cause superficial tangential streaming motion which is perfectly possible without disturbing the spherical symmetry of the geometry. As we had noted in the last section we have three equations in four unknowns so we make a judicious choice of p_r to simplify our solution without violating the boundary condition $p_r = 0$ at $r = a$.

We set,

$$e^{\lambda} r (c^2 \rho(r) + p_r) = A \qquad \text{... (10)}$$

where A is a constant to be determined from the boundary condition. The solutions to equations (3) and (4) yield

$$e^{-\lambda} = 1 - \frac{2Gm(r)}{c^2 r} + \frac{G}{c^4}(4\pi \rho_{0c})^2 \qquad \text{... (11)}$$

and

$$e^{\nu} = e^{-\lambda} e^{\frac{8\pi G A}{c^4}(r-a)}. \qquad \text{... (12)}$$

It can be easily verified that these are consistent with the external Reissner-Nordstorm field. We also have

$$p_r = \frac{A}{r}\left[1 - \frac{2Gm(r)}{c^2 r} + \frac{G}{c^4}(4\pi \rho_{0c})^2\right] - C^2 \rho(r). \qquad \text{... (13)}$$

Thus if $p_r = 0$ at $r = a$ we have

$$A = ac^2 \rho(a)\left[1 - \frac{2Gm(a)}{c^2 a} + \frac{G}{c^4}(4\pi \rho_{0c})^2\right]^{-1}. \qquad \text{... (14)}$$

The perpendicular pressure is easily evaluated now

$$p_{\perp} = p_r + \frac{r}{4}\left[(C^2 \rho(r) + p_r)\nu' + 2p_r' - \frac{(4\pi\rho_{0c})^2}{2\pi r^3}\right]. \qquad \text{... (15)}$$

A point that needs to be paid attention to is that e^{ν} might possibly have a singularity at $r = 0$. This depends on whether the mass distribution $m(r)$ has a term which is proportional to r^{α} where $\alpha < 1$. One can make some finer adjustments around this point in the distribution functions leaving the rest of it as it is. This will change the solutions obtained in this paper around the point $r = 0$ only without any alteration in the rest of the spherical body. Another objection to this 'exact' solution of Einstein's equation is that $p_{\perp}$ is not zero at $r = a$ so it will given rise to a streaming motion just at the surface. One can contend however that the motion does not go beyond the surface and hence does not affect the bulk energy-momentum tensor we assumed in eqns. (3)-(6). We can still make $p_{\perp} = 0$ at $r = a$ by choosing an appropriate value of ρ_{0c} without imposing any condition on $m(r)$.

4. Constraints on the Stellar Density Distribution $\rho(r)$.

As pointed out[8] that in models of realistic stars one should expect the pressure and density to be positive and their respective gradients to be negative. These conditions are to be applied in our opinion to the quantities p_r and ρ in the case of the anisotropic model of the hypothetical body we have analysed. The conditions that $p_r > 0$ and $dp_r/dr < 0$ gives the two inequalities which appear as constraints on $\rho(r)$

$$\frac{\rho(a)\, a}{1 - \frac{2Gm(a)}{c^2 a} + \frac{G}{c^4}(4\pi\, \rho_{oc})^2} > \frac{\rho(r)\, r}{1 - \frac{2Gm(r)}{c^2 r} + \frac{G}{c^4}(4\pi\, \rho_{0c})^2} \quad \text{... (16)}$$

and

$$\frac{\rho(a)\, a}{1 - \frac{2Gm(a)}{c^2 a} + \frac{G}{c^4}(4\pi\, \rho_{0c})^2} > \frac{-\frac{d\rho}{dr} r^2}{1 - \frac{4Gm}{c^2 r} + \frac{2Gm'}{c^2} + \frac{G}{c^4}(4\pi\, \rho_{0c})^2} \quad \text{... (17)}$$

For spheres which are not super-dense like neutron stars or white dwarfs for example one can show that the density distribution cannot contain for n greater than unity terms like K/r^n where n and K are constants. This will ensure that both inequalities (16) and (17) are satisfied simultaneously. However the requirement that the pressure and density must be finite at the origin cannot be obtained for this closed form solution. Models in which pressure and density diverge at the centre do exist in literature[9, 10]. The only remedy (Knutsen[8]) is possibly to seek a numerical solution and match it to the solution obtained in this paper at a finite radius as has been done in Durgapal and Gehlot[11] (analytically) to remove the singularity in Durgapal and Gehlot[10].

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