Indian J. pure appl. Math., 27(7) : 633-637, July 1996

CONVERGENCE AND PERFECT MAPS IN METRIC SPACES

G. L. GARG[1] AND ASHA GOEL[2]

[1]*Department of Mathematics, Punjabi University, Patiala* 147 002
[2]*Department of Mathematics, SLIET, Longowal* 148 106

(*Received* 15 *November* 1994; *after revision* 6 *September* 1995; *accepted* 19 *February* 1996)

We give characterizations of perfect maps (as well as of continuous maps and compact maps) in metric spaces in terms of convergence of subsequences. As corollaries, we obtain characterizations of compact-preserving injections, closed injections, closed embeddings and homeomorphisms.

Various properties of perfect maps in metric spaces have been studied by Vainstien[2]. A characterization of perfect maps in metric spaces in terms of compact sets has been given by the authors (see Garg and Goel[1]), namely; for arbitrary metric spaces X and Y, a map $f : X \to Y$ is perfect if and only if it preserves compactness of sets both ways. Also it is well known that in metric spaces a map $f : X \to Y$ is continuous if and only if $x_n \to x$ implies $f(x_n) \to f(x)$ for each point x of X. In this note, we give characterizations of perfect maps (as well as of continuous maps and compact maps) in terms of convergence of subsequences.

Throughout, X and Y are arbitrary metric spaces and $f : X \to Y$ is any map, not necessarily continuous or onto unless mentioned explicitly. For a subset A of X, cl (A) denotes the closure of A in X. A map $f : X \to Y$ is said to be compact (compact-preserving) if inverse image (image) of each compact set is compact; f is said to be 'perfect' if it is continuous, closed, and has compact fibers $f^{-1}(y), y \in Y$. f is called a 'local injection' if every $x \in X$ has a neighbourhood N_x such that $(y, z \in N_x$ and $y \neq z) \Rightarrow f(y) \neq f(z)$; i.e., f is one-to-one on N_x.

Theorem 1 — For any f, the following are equivalent :

(i) f is continuous i.e. if $x_n \to x$ in X, then $f(x_n) \to f(x)$.

(ii) If $x_n \to x$ in X, then the sequence $\{f(x_n)\}$ has a subsequence converging to $f(x)$.

(iii) If the sequence $\{x_n\}$ has a subsequence converging to x in X, then the sequence $\{f(x_n)\}$ has a subsequence converging to $f(x)$.

PROOF : (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious. To prove (iii) $\Rightarrow$ (i), suppose f is not continuous at some x. Then there exists $\varepsilon > 0$ and a sequence $x_n \to x$ such that $d(f(x_n), f(x)) \geq \varepsilon$ for each n, and so no subsequence of $\{f(x_n)\}$ converges to $f(x)$, a contradiction to (iii).

Corollary 1 — A local injection f is continuous if and only if it is compact-preserving.

PROOF : Suppose f is compact-preserving and $x_n \to x \in X$. Since f is a local injection, we can choose $n_0 \in N$ such that f is one-to-one on $H = \{x_n : n \geq n_0\} \cup \{x\}$. But H is compact and therefore $f(H)$ is compact by hypothesis. Consequently, the sequence $\{f(x_n)\}$ has a subsequence which converges to some $y \in f(H)$. Suppose $y \neq f(x)$. Then $y = f(z)$ for some $z \in H$, where $z \neq x$, and $d(x, z) = \varepsilon$ for some $\varepsilon > 0$. Since $x_n \to x$, there exists a positive integer $n_1 > n_0$ such that $d(x_n, x) < \varepsilon$ for all $n \geq n_1$. Let $K = \{x_n : n \geq n_1\} \cup \{x\}$. Clearly, $y \in \text{cl}\, f(K)$. Since K is compact, $f(K)$ is compact and therefore closed. But this is a contradiction, since f is an injection on H, $K \subset H$, $z \in H - K$, and therefore $f(z) = y \in \text{cl}\, f(K) - f(K)$. Consequently, $y = f(x)$ and hence f is continuous by condition (ii) of Theorem 1.

Theorem 2 — For any f, the following are equivalent :

(i) f is compact.

(ii) If $f(x_n) \to y$ in Y, then the sequence $\{x_n\}$ has a subsequence converging to a point in $f^{-1}(y)$. In particular, $f^{-1}(y) \neq \phi$.

(iii) If the sequence $\{f(x_n)\}$ has a subsequence converging to y in Y, then the sequence $\{x_n\}$ has a subsequence converging to a point in $f^{-1}(y)$. In particular, $f^{-1}(y) \neq \phi$.

PROOF : (i) $\Rightarrow$ (ii). Suppose f is compact and $f(x_n) \to y$. Then the set $K = \{f(x_n)\} \cup \{y\}$ is compact implies $f^{-1}(K)$ is compact and contains the sequence $\{x_n\}$. Thus the sequence $\{x_n\}$ has a subsequence $\{y_n\}$ converging to some $x \in f^{-1}(K)$. In fact, $y = f(x)$. Otherwise, $d(y, f(x)) = \varepsilon > 0$ and so $f(y_n) \to y$ implies there exists a positive integer n_0 such that $d(f(y_n), y) < \varepsilon$ for all $n \geq n_0$. Now the set $H = \{f(y_n) : n \geq n_0\} \cup \{y\}$ is compact, and so f is compact implies $f^{-1}(H)$ is closed, a contradiction, since it is easy to see that $x \in \text{cl}\, f^{-1}(H) - f^{-1}(H)$. Therefore, $x \in f^{-1}(y)$.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). Let B be a compact subset of Y and let $\{x_n\}$ be any sequence in $f^{-1}(B)$. Then $\{f(x_n)\}$, being a sequence in the compact set B, has a subsequence converging to some point, say y in B, and so by hypothesis, there exists a subsequence of $\{x_n\}$ converging to a point in $f^{-1}(y)$ and so in $f^{-1}(B)$. Thus $f^{-1}(B)$ is compact and hence f is compact.

Corollary 2 — For any injection (bijection) f, the following are equivalent :

(i) f is compact.

(ii) f is closed.

(iii) If $\{f(x_n)\}$ has a subsequence converging to y (respectively $f(x)$) in Y, then $\{x_n\}$ has a subsequence converging to $f^{-1}(y)$ (respectively x).

PROOF : (i) $\Rightarrow$ (ii). Suppose f is compact and let F be any closed subset of X such that $y \in \text{cl} f(F) - f(F)$. Then there exists a sequence $\{x_n\}$ of points in F such that $f(x_n) \to y$. Since f is a compact injection, by condition (ii) of Theorem 2, there exists a subsequence of $\{x_n\}$, converging to the point $x = f^{-1}(y)$. Then $x \in F$ and therefore, $y \in f(F)$ — a contradiction. Hence f must be closed.

(ii) $\Rightarrow$ (iii). Suppose f is closed and $\{f(x_n)\}$ has a subsequence $\{f(y_n)\}$ converging to y. If $f(y_n) = y$ for arbitrarily large n, then there exists a subsequence $\{z_n\}$ of $\{y_n\}$ such that $f(z_n) = y$ for all n. Since f is injective, $z_1 = z_n$ for all n and thus the constant sequence $\{z_n\}$ converges to $z_1 = f^{-1}(y)$, the desired conclusion. On the other hand, if there exists $n_0 \in N$ such that $f(y_n) \neq y$ for $n \geq n_0$, consider the subsequence $\{y_n\}$ of $\{x_n\}$. If $\{y_n\}$ has no convergent subsequence, then the set $H = \{y_n : n \geq n_0\}$ is closed. Therefore, the set $f(H)$ is closed — a contradiction, since $y \in \text{cl} f(H) - f(H)$. Thus, $\{y_n\}$ has a subsequence $\{z_n\}$ which converges to some x. If $y \neq f(x)$, then the set $H = \{z_n\} \cup \{x\}$ is compact and so closed implies $f(H)$ is closed, which is a contradiction, since it is easy to see that $y \in \text{cl} f(H) - f(H)$. Therefore, $y = f(x)$ and hence (iii) holds.

(iii) $\Rightarrow$ (i). Follows from (iii) implies (i) in Theorem 2.

Note 1 : The following Example 1 shows that an arbitrary compact-preserving map may not be continuous and an arbitrary closed map may not be compact, even in compact metric spaces.

Example 1 — Let $f : [0, 1] \to \{0, 1\}$ be defined by $f([0, 1/2)) = \{0\}$ and $f([1/2, 1]) = \{1\}$. Then f is compact-preserving and closed but is neither continuous, nor compact.

Note 2 : The following Example 2 shows that for an injection, the condition $f(x_n) \to f(x)$ implies $\{x_n\}$ has a subsequence converging to x may not imply f is compact even in compact metric spaces. However, for any onto map f, this condition implies compactness of f, by Theorem 2 above.

Example 2 — The injection $f : [0, 1] \to [0, 2]$ defined by $f(x) = x$ for $x \in [0, 1/2]$ and $f(x) = 2x$ for $x \in (1/2, 1]$ satisfies even the stronger condition $f(x_n) \to f(x)$ implies $x_n \to x$, but f is not compact.

Note 3 : Later, we will see in Example 3 below that for a compact map, the condition $f(x_n) \to f(x)$ implies $\{x_n\}$ has a subsequence converging to x may not hold, even in compact metric spaces.

Theorem 3 — For any $f : X \to Y$, where X, Y are metric (complete metric) spaces, the following are equivalent :

(i) f is perfect.

(ii) If $x_n \to x$ in X, then $f(x_n) \to f(x)$; and if $f(x_n) \to y$ in Y, then $\{x_n\}$ has a subsequence converging to a point in $f^{-1}(y)$. In particular, $f^{-1}(y) \neq \phi$.

(iii) If $\{x_n\}$ has a subsequence converging to x in X, then $\{f(x_n)\}$ has a subsequence converging to $f(x)$; and if $\{f(x_n)\}$ has a convergent (Cauchy) subsequence in Y, then $\{x_n\}$ has a convergent (Cauchy) subsequence.

(iv) If $\{x_n\}$ has a convergent (Cauchy) subsequence in X, then $\{f(x_n)\}$ has a convergent (Cauchy) subsequence; and if $\{f(x_n)\}$ has a subsequence converging to y in Y, then $\{x_n\}$ has a subsequence converging to a point in $f^{-1}(y)$. In particular, $f^{-1}(y) \neq \phi$.

PROOF : Firstly, suppose (i) holds. Then f is continuous and compact and therefore, by Theorems 1 and 2, condition (ii) and hence conditions (iii) and (iv) follow.

For the converse, assume X, Y are arbitrary metric spaces. It is sufficient to prove that each of the conditions (iii) and (iv) implies (i). Suppose (iii) holds. Then continuity of f follows from Theorem 1. In view of our Theorem in Garg and Goel[1], to prove f is perfect, it is sufficient to prove that f is compact. Let B be a compact subset of Y and $\{x_n\}$ be any sequence in $f^{-1}(B)$. Then $\{f(x_n)\}$, being a sequence in the compact set B, has a convergent subsequence and therefore, by condition (iii), $\{x_n\}$ has a convergent subsequence converging to some point (in $f^{-1}(B)$, since the set $f^{-1}(B)$ is closed). Therefore, $f^{-1}(B)$ is compact and hence f is compact.

Next suppose (iv) holds. Then the compactness of f follows from Theorem 2. To prove the continuity of f, suppose $x_n \to x$. Then, by condition (iv), $\{f(x_n)\}$ has a convergent subsequence converging to y, say, and so $\{x_n\}$ has a subsequence converging to a point in $f^{-1}(y)$. Since every subsequence of $\{x_n\}$ converges to x, it follows that $x \in f^{-1}(y)$. Thus $\{f(x_n)\}$ has a subsequence converging to $f(x)$ and hence f is continuous by the above Theorem 1.

The parenthesis part is now obvious.

Corollary 3 — For any injection (bijection) $f : X \to Y$, where X, Y are arbitrary metric spaces, the following are equivalent :

(i) f is a closed embedding (homeomorphism).

(ii) If $\{x_n\}$ has a subsequence converging to x in X, then $\{f(x_n)\}$ has a subsequence converging to $f(x)$; and if $\{f(x_n)\}$ has a convergent subsequence in Y, then $\{x_n\}$ has a convergent subsequence.

(iii) If $\{x_n\}$ has a convergent subsequence in X, then $\{f(x_n)\}$ has a convergent subsequence; and if $\{f(x_n)\}$ has a subsequence converging to y (respectively $f(x)$) in Y, then $\{x_n\}$ has a subsequence converging to $f^{-1}(y)$ (respectively x).

Corollary 4 — For any injection (bijection) $f : X \to Y$ where X, Y are complete metric spaces, the following are equivalent :

(i) f is a closed embedding (homeomorphism).

(ii) If $\{x_n\}$ has a subsequence converging to x in X, then $\{f(x_n)\}$ has a subsequence converging to $f(x)$; and if $\{f(x_n)\}$ has a Cauchy subsequence in Y, then $\{x_n\}$ has a Cauchy subsequence.

(iii) If $\{x_n\}$ has a Cauchy subsequence in X, then $\{f(x_n)\}$ has a Cauchy subsequence; and if $\{f(x_n)\}$ has a subsequence converging to y (respectively $f(x)$) in Y, then $\{x_n\}$ has a subsequence converging to $f^{-1}(y)$ (respectively x).

Note 4 : The following example 3 shows that even for a perfect surjection, the condition $f(x_n) \to f(x)$ implies $\{x_n\}$ has a subsequence converging to x may not hold even in compact metric spaces.

Example 3 — For the constant map $f : [0, 1] \to \{0\}$, the sequence $f(1/n) \to f(x) = 0$, for each x in $[0, 1]$, but the sequence $\{1/n\}$ has no subsequence converging to x for $x \neq 0$.

Note 5 : It is obvious from Theorem 3 that a perfect map satisfies each of the following two conditions.

(i) $\{x_n\}$ has a convergent subsequence implies $\{f(x_n)\}\}$ has a convergent subsequence.

(ii) $\{f(x_n)\}$ has a convergent subsequence implies $\{x_n\}$ has a convergent subsequence.

The following Example 4 shows that both these conditions together may not even imply that the map is compact-preserving or compact even in compact metric spaces.

Example 4 — Let $f : [0, 1] \to [0, 1]$ be defined by $f(x) = x$ for $x \in (0, 1)$, $f(0) = 1$, and $f(1) = 0$. Then f is a bijection satisfying the conditions (i) and (ii) of Note 5, but f is neither compact nor compact-preserving.

ACKNOWLEDGEMENT

The authors are grateful to the referee for useful suggestions.

REFERENCES

1. G. L. Garg and Asha Goel, *Soochow J. Math.* **20** (1994), 277-78.
2. I. A. Vainstein, *Doklady Akademii Nauk SSSR (N.S.)*, **57** (1947), 319-21.