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PUTNAM-FUGLEDE TYPE INEQUALITIES FOR THE VON NEUMANN-SCHATTEN, AND UNITARILY INVARIANT, NORMS

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A Putnam-Fuglede type inequality is an inequality $|||A_1XB_1+A_2XB_2|||\geq|||A_1^*XB_1^*+A_2^*XB_2^*|||$ where $|||\cdot|||$ is some norm and the operators A_i, B_i^* are normal, hypernormal or k-quasihyponormal. Previous work has applied to the Hilbert-Schmidt norm $\|\cdot\|_2$: in this note we obtain a Putnam-Fuglede inequality for the von Neumann-Schatten norm $\|\cdot\|_p$, $1\leq p<\infty$, for the operator norm $\|\cdot\|$ on $\mathcal{L}(H)$ and for unitarily invariant norms $|||\cdot|||$ on operators of finite rank.

1. INTRODUCTION

A Putnam-Fuglede type inequality is an inequality

$$|||A_1XB_1+A_2XB_2|||\geq|||A_1^*XB_1^*+A_2^*XB_2^*|||. \qquad \ldots (1)$$

Here, $|||\cdot|||$ is some norm and the operators A_i and B_i^* are, for example, normal or hyponormal and satisfy some commutativity condition. The term "Putnam-Fuglede" is used because as Furuta[3] (Theorem 1(ii)), Gupta and Patel[4] (Theorem 1) and Duggal[1] (Corollary) have pointed out, there is a connection between the inequality (1) and the Putnam-Fuglede commutativity theorem[5] (Problem 192).

In this work we obtain a Putnam-Fuglede type inequality for the von Neumann-Schatten norms $\|\cdot\|_p$ on the class C_p, for the operator norm $\|\cdot\|$ on $\mathcal{L}(H)$ and for every unitarily invariant norm $|||\cdot|||$ on the space $\mathcal{F}$ of operators of finite rank; and for operators of finite rank we also get a corresponding singular values inequality (Theorem 3.2).

Previous Putnam-Fuglede type inequalities have involved the Hilbert-Schmidt norm $\|\cdot\|_2$ and class C_2. Thus, in now discussing such work we take $|||\cdot|||=\|\cdot\|_2$ and suppose $X\in C_2$. Weiss[11] (Corollary 2) showed that equality holds in (1) provided A_i and B_i are normal and $A_1A_2=A_2A_1$ and $B_1B_2=B_2B_1$. Furuta[3] extended this to hyponormal operators (Recall : an operator A is hyponormal if $A^*A - AA^* \geq 0$). Furuta[3] (Theorem 1(i)) showed that if A_i and B_i^*, $i = 1, 2$ are hyponormal then the inequality (1) holds provided that the operators A_i and B_i satisfy the commutativity condition

$$A_1^* A_2 = A_2 A_1^*, \; B_1 B_2^* = B_2^* B_1$$

which will be referred to in this note as the standard condition (Observe that the sum $A_1 + A_2$ of two hyponormal operators A_1 and A_2 satisfying the standard condition is itself hyponormal). Duggal[1] (Theorem 1) gave an alternative, elegant proof of Furuta's result and extended it to the more inclusive context of k-quasihyponormal operators (Related results had been obtained by Gupta and Patel[4]). An operator A is said to be k-quasihyponormal if

$$(A^*)^k (A^*A - AA^*) A^k \geq 0.$$

Thus, if $Q(k)$ denotes the set of all k-quasihyponormal operators then $Q(0)$ consists of the hyponormal operators and $Q(1)$ contains, as a subset, all quasinormal operators (Recall : an operator A is quasinormal if $A(A^*A) = (A^*A)A$).

In our extension of the Putnam-Fuglede type inequality to the $\|\cdot\|$, $\|\cdot\|_p$ and $|||\cdot|||$ norms we deal, like Duggal[1], with hyponormal, and k-quasihyponormal, operators A_i and B_i^*, $i = 1, 2$. We adopt conditions which, however, are stronger than (Theorem 3.2), or at least differ from (see Note 3 below), the standard condition. This might be because C_p, $1 \leq p < \infty$, and $\mathcal{L}(H)$, are not Hilbert spaces unlike C_2 on whose Hilbert space character Furuta and Duggal's proofs crucially depend.

2. Preliminaries

Let H denote a separable Hilbert space, let $\mathcal{L}(H)$ denote the space of all bounded, linear operators mapping H into H and let $\mathcal{F}$ denote the space of all operators in $\mathcal{L}(H)$ of finite rank. For details of the von Neumann-Schatten classes C_p and norms $\|\cdot\|_p$, $1 \leq p < \infty$ (see Dunford and Schwartz[2], Chap XI; Ringrose[10], Chap 2). The symbol $\|\cdot\|_\infty$, when used, denotes the operator norm $\|\cdot\|$ on $\mathcal{L}(H)$. A unitarily invariant norm $|||\cdot|||$ satisfies $|||UA||| = |||AV|||$ for all unitary operators U and V (provided $|||A||| < \infty$). A unitarily invariant norm $|||\cdot|||$ on $\mathcal{F}$ is self-adjoint, i.e. $|||A||| = |||A^*|||$ if $A \in \mathcal{F}$[9] (Chapter 10). For an operator A of rank n,

$$s_1(A), ..., s_n(A)$$

will denote the singular values of A (i.e. the eigenvalues of $|A|$) arranged in decreasing order and repeated according to multiplicity. Notice that $s_i(A) = s_i(A^*)$, $1 \leq i \leq n$, because the non-zero eigenvalues of A^*A $(= |A|^2)$ coincide with those of AA^* $(= |A^*|^2)$.

We require the following lemma.

Lemma 2.1 (Maher[7], Lemma 3.1) — (a) If $|A|^2 \geq |B|^2$ then

$$\|A\| \geq \|B\| \,;$$

(b) if, further, $A \in C_p$ for $1 \leq p < \infty$ then $B \in C_p$ and

$$\|A\|_p \geq \|B\|_p, \quad 1 \leq p < \infty;$$

(c) if, further, (Maher[8], Lemma 3.1) A is of finite rank n, say, then rank $B \leq n$ and

$$s_i(A) \geq s_i(B), \quad i = 1, \ldots, n;$$

(d) and for every unitarily invariant norm $|||\cdot|||$ on $\mathcal{F}$

$$|||A||| \geq |||B|||.$$

3. Inequalities Concerning k-quasihyponormal Operators

Lemma 3.1 is a first step towards the main result.

Lemma 3.1 — (a) If T is hyponormal then for every X in $\mathcal{L}(H)$

$$|TX|^2 \geq |T^*X|^2;$$

(b) the same result holds if $T \in Q(k)$ and if, for every X in $\mathcal{L}(H)$, there exists some Y in $\mathcal{L}(H)$ such that $T^k Y = X$.

Proof : (a) Consider the following identity, valid for an arbitrary operator T :

$$\langle |TX|^2 f, f \rangle = \langle |T^*X|^2 f, f \rangle + \langle (T^*T - TT^*)Xf, Xf \rangle. \qquad \ldots (1)$$

Hence, if T is hyponormal, $|TX|^2 \geq |T^*X|^2$.

(b) If $T \in Q(k)$ and $T^k Y = X$ then

$$\langle (T^*T - TT^*)Xf, Xf \rangle = \langle (T^*)^k (T^*T - TT^*) T^k Yf, Yf \rangle \geq 0$$

which, again from (1), gives $|TX|^2 \geq |T^*X|^2$. □

Observe that equality holds in Lemma 3.1 if T is normal or if T is quasi-normal.

Armed with Lemma 3.1 we now prove the main result.

Theorem 3.2 — (a) If A_i and B_i^*, for $i = 1, 2$, are hyponormal and if $A_2 A_1^* = 0 = B_2^* B_1$ then, provided $X \in C_p$ if $1 \leq p < \infty$,

$$\|A_1XB_1 + A_2XB_2\|_p \geq \|A_1^*XB_1^* + A_2^*XB_2^*\|_p, \; 1 \leq p \leq \infty;$$

(b) if further, rank $X = n$ $(< \infty)$ then $n \geq$ rank $(A_1XB_1 + A_2XB_2) \geq$ rank $(A_1^*XB_1^* + A_2^*XB_2^*)$,

$$s_i(A_1XB_1 + A_2XB_2) \geq s_i(A_1^*XB_1^* + A_2^*XB_2^*), \quad i = 1, \ldots, n$$

and for every unitarily invariant norm $|||\cdot|||$ on $\mathcal{F}$

$$|||A_1XB_1 + A_2XB_2||| \geq |||A_1^*XB_1^* + A_2^*XB_2^*|||\ .$$

(c) If $A_i \in Q(k)$ and $B_i^* \in Q(k)$ for $i = 1, 2$, if $A_2A_1^* = 0 = B_2^*B_1$ and $A_1^*A_2 = 0 = B_1B_2^*$ and if X is such that there exist operators $Y_i^{(1)}$ and $Y_i^{(2)}$ for $i = 1, 2$ satisfying $XB_i = A_i^k Y_i^{(1)}$ and $X^*A_1 = (B_i^*)^k Y_i^{(2)}$ then, provided $X \in C_p$ if $1 \leq p < \infty$, the same result as in (a) holds; and if, further, rank $X = n$ $(< \infty)$ then the same result as in (b) holds.

PROOF : (a) Observe that for a hyponormal operator A, Ran$A \subseteq$ RanA^* because Ker$A \subseteq$ KerA^* (for if $f \in$ Ker$A$ then $\langle (A^*A - AA^*)f, f\rangle = -\langle AA^*f, f\rangle = -\|A^*f\|^2 \geq 0$ which forces $A^*f = 0$). The condition $A_2A_1^* = 0$ means that Ran $A_1^* \perp$ Ran A_2^* and hence, because the A_i are hyponormal Ran $A_1 \perp$ Ran A_2 so that $A_1^*A_2 = 0$ $(= A_2A_1^*)$. Thus, the cross terms in the expansion of $|A_1XB_1 + A_2XB_2|^2$ (and of $|A_1^*XB_1 + A_2^*XB_2|^2$) vanish so that

$$|A_1XB_1 + A_2XB_2|^2 = |A_1XB_1|^2 + |A_2XB_2|^2$$

$$\geq |A_1^*XB_1|^2 + |A_2^*XB_2|^2 = |A_1^*XB_1 + A_2^*XB_2|^2 \quad \dots (1)$$

the above inequality following from Lemma 3.1(a) since the A_i are hyponormal. Hence, applying Lemma 2.1 (a), (b) and taking adjoints we obtain, with $1 \leq p \leq \infty$,

$$\|A_1XB_1 + A_2XB_2\|_p \geq \|A_1^*XB_1 + A_2^*XB_2\|_p = \|B_1^*X^*A_1 + B_2^*X^*A_2\|_p; \quad \dots (2)$$

and since, similarly,

$$|B_1^*X^*A_1 + B_2^*X^*A_2|^2 \geq |B_1X^*A_1 + B_2X^*A_2|^2 \quad \dots (3)$$

(because the B_i^*, $i = 1, 2$, are hyponormal and satisfy $B_2^*B_1 = 0$, and hence $B_1B_2^* = 0$) we find, from Lemma 2.1 (a), (b) on taking adjoints, that, with $1 \leq p \leq \infty$,

$$\|B_1^*X^*A_1 + B_2^*X^*A_2\|_p \geq \|A_1^*XB_1^* + A_2^*XB_2^*\|_p,$$

which, from (2), gives the desired $\|\cdot\|_p$ inequality.

(b) If rank $X = n$ then obviously $n \geq$ rank $(A_1XB_1 + A_2XB_2)$. The singular values inequality in (b), and hence the other rank inequality there, follow as in (a), by Lemma 2.1 (c), via taking adjoints from the modulus inequalities (1) and (3). The norm $|||\cdot|||$ inequality follows similarly.

(c) For operators A_i, B_i^* in $Q(k)$ the condition $A_1^*A_2 = 0 = B_1B_2^*$ is required because if, say, $A \in Q(k)$ it need not follow that Ran$A \subseteq$ RanA^*. The proof of the result in part (c) follows as in (a) from Lemma 2.1 and from Lemma 3.1(b). If rank $X = n < \infty$ then the same result as in (b) is proved identically to (b). □

4. NOTES

(1) Notice that equality holds in Theorem 3.2 if the operators A_i, B_i^* are normal and/or quasinormal.

(2) For hyponormal A_i, B_i^* the condition $A_2 A_1^* = 0 = B_2^* B_1$ is stronger than the standard one : for, as shown in the proof of Theorem 3.2(a), the condition $A_2 A_1^* = 0$ implies that $A_1^* A_2 = 0$ whilst, similarly, the condition $B_2^* B_1 = 0$ implies that $B_1 B_2^* = 0$; and thus $A_1^* A_2 = A_2 A_1^*$ and $B_1 B_2^* = B_2^* B_1$.

(3) (a) If, instead of the condition of Theorem 3.2, viz. $A_2 A_1^* = B_2^* B_1 = 0$ (= $A_1^* A_2 = B_1 B_2^*$), it is assumed that $A_1^* A_2 = A_1 A_2^*$ and $B_1 B_2^* = B_1^* B_2$ then the result of Theorem 3.2 follows because the cross- terms in the expansions of $|A_1XB_1 + A_2XB_2|^2$ and $|A_1^* XB_1 + A_2^* XB_2|^2$ are identical (as are the cross terms in the expansions of $|B_1^* X^*A_1 + B_2^* X^*A_2|^2$ and $|B_1 X^*A_1 + B_2 X^*A_2|^2$).

(b) The condition $A_1^* A_2 = A_1 A_2^*$ differs from the standard condition $A_1^* A_2 = A_2 A_1^*$; but each condition can be derived from the other provided the operator $A_i A_j^*$ (or $A_i^* A_j$), $i = 1, 2$, is self-adjoint; with similar comments about the B^s.

(4) Instead of expanding moduli, the result of Theorem 3.2 can be obtained by using the inequalities of Maher[6], (Theorem 1.7 (b), (d)) concerning operators having orthogonal ranges/co-ranges.

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