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AN OSCILLATION THEOREM FOR SECOND ORDER SUPERLINEAR DIFFERENTIAL EQUATIONS*

FANWEI MENG

Mathematics Department, Qufu Normal University, Qufu, Shandong, P. R. China

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An oscillation criterion is given for the second order superlinear differential equation

$$x'' + a(t)\,|x|^{\gamma}\,\mathrm{sgn}\,x = 0,\ \gamma > 1,$$

where $a(t)$ is not assumed to be nonnegative for all large values of t. The result improves and extends a condition recently discovered by Butler *et al.*[2] for linear equation.

Consider the second order superlinear differential equation

$$x'' + a(t)\,|x|^{\gamma}\,\mathrm{sgn}\,x = 0,\ \gamma > 1, \qquad \text{... (1)}$$

where $a(t) \in C[0, +\infty)$. We restrict our attention to solutions of (1) which exist on some ray $[t_0, +\infty)$, where $t_0 \geq 0$ may depend on the particular solution. Such a solution is said to be oscillatory if it has arbitrary large zeros. Equation (1) is called oscillatory if all such solutions are oscillatory. We are here concerned with sufficient conditions on $a(t)$ for the oscillation of (1) when $a(t)$ is allowed to assume negative values for arbitrarily large values of t. In the linear case, the well-known Wintner's oscillation theorem states that if $a(t)$ satisfies

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t\int_0^s a(r)\,dr\,ds = +\infty, \qquad \text{... (2)}$$

then eqn. (1) is oscillatory for $\gamma = 1$, see Wintner[6]. Butler[1] proved that Wintner's theorem remains valid for eqn. (1) where $\gamma > 1$. Wintner's condition (2) was later improved by Hartman[3] (see also Hartman[4]) who proved that the following conditions i.e.,

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t A(s)\,ds > -\infty, \qquad \text{... (3)}$$

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where $A(t) = \int_0^t a(s)\,ds$ and that the limit in (2) does not exist, namely,

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t A(s)\,ds < \limsup_{t\to+\infty} \frac{1}{t}\int_0^t A(s)\,ds \qquad \text{... (4)}$$

are sufficient for oscillation of the linear equation. In the superlinear case, Wong[7] showed that conditions

$$\liminf_{t\to+\infty} \int_0^t a(s)\,ds > -\infty \qquad \text{... (5)}$$

and

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t \int_0^s a(r)\,dr\,ds = +\infty, \qquad \text{... (6)}$$

are sufficient for the oscillation of (1). Recently, Wong[9] proved that conditions (3) and (4) are sufficient for oscillation of (1).

Recently, in connection with the study of oscillation theory for linear systems, Butler *et al.*[2] showed that condition (3) together with

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t A^2(s)\,ds = +\infty, \qquad \text{... (7)}$$

are sufficient for oscillation of eqn. (1) in the linear case. Since (6) implies (7) by an application of Schwarz's inequality, their result extends part of Hartman's theorem. Recently, Wong[10] proved that this new oscillation criterion remains valid for the sublinear eqn. (1) with $0 < \gamma < 1$. It is therefore natural to ask whether this new oscillation criterion remains valid for the superlinear eqn. (1). The purpose of this note is to answer this question in the affirmative by proving.

Theorem — Suppose that (7) holds and

$$\liminf_{t\to+\infty} \frac{1}{t^m}\int_0^t (t-s)^m a(s)\,ds > -\infty, \text{ for some integer } m \geq 0. \qquad \text{... (8)}$$

Then eqn. (1) is oscillatory for $\gamma > 1$.

It is easy to see that the condition (8) is weaker than each of the conditions (3) and (5).

Proof of the theorem — Assume to the contrary that there exists a nonoscillatory solution $x(t)$ which may be assumed to be positive on $[t_0, +\infty)$. Define $y(t) = x^{1-\gamma}(t)$. It is easy to verify from (1) that $y(t)$ satisfies the second order nonlinear differential equation

$$y'' = (\gamma-1)\,a(t) + \gamma(\gamma-1)^{-1}y^{-1}(y')^2, \qquad \text{... (9)}$$

on $[t_0, +\infty)$. For convenience, denote $\alpha = \gamma - 1$, and $\beta = \gamma(\gamma-1)^{-1}$ then $\alpha > 0$, $\beta > 1$. Integrating (8) we obtain

$$y'(t) - y'(t_0) = \alpha \int_{t_0}^{t} a(s)\,ds + \beta \int_{t_0}^{t} y^{-1}(s)\,(y'(s))^2\,ds. \qquad \text{... (10)}$$

We consider two cases where $\int_{t_0}^{+\infty} y^{-1}(s)\,(y'(s))^2\,ds$ is finite or infinite.

Case I — The integral $\int_{t_0}^{+\infty} y^{-1}(s)\,(y'(s))^2\,ds$ is finite. We will show that

$$\lim_{t \to +\infty} \frac{y(t)}{t} = 0. \qquad \text{... (11)}$$

Let ε be an arbitrary positive number. We can choose a $t_1 \geq t_0$ such that

$$\int_{t_1}^{+\infty} y^{-1}(s)\,(y'(s))^2\,ds < \varepsilon/4.$$

By Schwarz's inequality,

$$y(t) - y(t_1) \leq \left| \int_{t_1}^{t} y'(s)\,ds \right| \leq \left(\int_{t_1}^{t} y^{-1}(s)\,(y'(s))^2\,ds \right)^{1/2} \left(\int_{t_1}^{t} y(s)\,ds \right)^{1/2}$$

$$\leq \frac{\sqrt{\varepsilon}}{2} \left(\int_{t_1}^{t} y(s)\,ds \right)^{1/2}, \quad t \geq t_1. \qquad \text{... (12)}$$

If $\int_{t_1}^{+\infty} y(s)\,ds < +\infty$, then from (12) it follows that $y(t)$ is bounded on $[t_1, +\infty)$ and hence (11) is satisfied. So, we assume that $\int_{t_1}^{+\infty} y(s)\,ds = +\infty$. Then there exists a positive $t_2 > t_1$ such that $y(t_1) \leq \frac{\sqrt{\varepsilon}}{2} \left(\int_{t_1}^{t} y(s)\,ds \right)^{1/2}$, for $t > t_2$. Using this fact in (12),

$$y(t) \leq y(t_1) + \frac{\sqrt{\varepsilon}}{2} \left(\int_{t_1}^{t} y(s)\,ds \right)^{1/2} \leq \left(\frac{\sqrt{\varepsilon}}{2} + \frac{\sqrt{\varepsilon}}{2} \right) \left(\int_{t_1}^{t} y(s)\,ds \right)^{1/2}. \qquad \text{... (13)}$$

Dividing (13) through by $\left(\int_{t_1}^{t} y(s)\,ds \right)^{1/2}$ and integrating from t_2 to t we obtain

$$2\left(\int_{t_1}^{t} y(s)\,ds\right)^{1/2} - 2\left(\int_{t_1}^{t_2} y(s)\,ds\right)^{1/2} \le \sqrt{\varepsilon}\,(t-t_2) < \sqrt{\varepsilon}\,t. \qquad \ldots (14)$$

Now choose $t_3 \ge t_2$ such that $\int_{t_1}^{t_2} y(s)\,ds \le \frac{\varepsilon}{4} t_3^2$, hence for $t \ge t_3$, we have from (14)

$$\left(\int_{t_1}^{t} y(s)\,ds\right)^{1/2} < \frac{\sqrt{\varepsilon}}{2}\,t + \left(\int_{t_1}^{t_2} y(s)\,ds\right)^{1/2} \le \sqrt{\varepsilon}\,t. \qquad \ldots (15)$$

Combining (13) and (14), we conclude that $y(t) < \varepsilon t$, for $t \ge t_3$, proving (11).

Returning to (10), one can express $A(t)$ in the form

$$\alpha A(t) = y'(t) - y'(t_0) - \beta \int_{t_0}^{t} y^{-1}(s)\,(y'(s))^2\,ds + \alpha A(t_0). \qquad \ldots (16)$$

From (16), we obtain

$$\alpha^2 A^2(t) \le 3C_0^2 + 3(y'(t))^2 + 3\beta^2\left[\int_{t_0}^{t} y^{-1}(s)\,(y'(s))^2\,ds\right]^2$$

$$\le 3C_1^2 + 3(y'(t))^2, \qquad \ldots (17)$$

where $C_0 = \alpha A(t_0) - y'(t_0)$, $C_1^2 = C_0^2 + \beta^2\left[\int_{t_0}^{+\infty} y^{-1}(s)\,(y'(s))^2\,ds\right]^2$. Integrating (17) from t_0 to t and dividing through by t, we get

$$\frac{\alpha^2}{t}\int_{t_0}^{t} A^2(s)\,ds \le 3\,C_1^2\left(1 - \frac{t_0}{t}\right) + \frac{3}{t}\int_{t_0}^{t} (y'(s))^2\,ds. \qquad \ldots (18)$$

We note

$$\frac{1}{t}\int_{t_0}^{t} y^{-1}(s)\,(y'(s))^2\,y(s)\,ds \le \frac{1}{t}\max_{\{t_0 \le s \le t\}} |y(s)| \int_{t_0}^{t} y^{-1}(s)\,(y'(s))^2\,ds.$$

$$\ldots (19)$$

By (11) we can choose $T_0 > t_0$ such that $|y(t)| \le t$ for $t \ge T_0$. Using this

$$\max_{\{t_0 \le s \le t\}} |y(s)| \le \max_{\{t_0 \le s \le T_0\}} |y(s)| + t \le M_0 + t \qquad \ldots (20)$$

where M_0 is a constant. Substituting (19) and (20) in (18), we find the right-hand side is bounded as t tends to infinity, which is incompatible with condition (7).

Case II — The integral $\int_{t_0}^{+\infty} y^{-1}(s)\,(y'(s))^2\,ds$ is infinite. We choose an integer $k \geq \max_{\{m,\ 2\}}$ such that $\frac{k}{k-1} < \beta$. As $k \geq m$, it is easy to verify that condition (8) implies that

$$\liminf_{t \to +\infty} \frac{1}{t^k} \int_0^t (t-s)^k a(s)\,ds > -\infty. \qquad \text{... (21)}$$

Now, for each $t \geq t_0$ from (9) we obtain

$$\alpha \int_{t_0}^t (t-s)^k a(s)\,ds + \int_{t_0}^t \beta(t-s)^k y^{-1}(s)\,(y'(s))^2\,ds$$

$$= \int_{t_0}^t (t-s)^k y''(s)\,ds$$

$$= -(t-t_0)^k y'(t_0) + k \int_{t_0}^t (t-s)^{k-1} y'(s)\,ds.$$

So, we have

$$\frac{1}{t^k}\alpha \int_{t_0}^t (t-s)^k a(s)\,ds = -\left(1 - \frac{t_0}{t}\right)^k y'(t_0)$$

$$+ \frac{1}{t^k}\left\{k \int_{t_0}^t (t-s)^{k-1} y'(s)\,ds - \beta \int_{t_0}^t (t-s)^k y^{-1}(s)\,(y'(s))^2\,ds\right\},\ t \geq t_0. \qquad \text{... (22)}$$

Choose a constant μ with

$$\frac{k}{k-1} < \mu < \beta \qquad \text{... (23)}$$

we claim that

$$\limsup_{t \to +\infty}\left(\mu \int_{t_0}^t (t-s)^k y^{-1}(s)\,(y'(s))^2\,ds - k \int_{t_0}^t (t-s)^{k-1} y'(s)\,ds\right) > 0. \qquad \text{... (24)}$$

Assume that (24) does not hold, then there exists $t_1 \geq t_0$ such that

$$\int_{t_0}^{t} (t-s)^k y^{-1}(s)(y'(s))^2\, ds \leq \frac{k}{\mu} \int_{t_0}^{t} (t-s)^{k-1} y'(s)\, ds, \quad t \geq t_1. \qquad \ldots (25)$$

By using the Schwarz inequality and (25), for $t \geq t_1$, we obtain

$$0 \leq \int_{t_0}^{t} (t-s)^{k-1} y'(s)\, ds \leq \left(\int_{t_0}^{t} (t-s)^k y^{-1}(s)(y'(s))^2\, ds \right)^{1/2} \times \left(\int_{t_0}^{t} (t-s)^{k-2} y(s)\, ds \right)^{1/2}$$

$$\leq \left(\frac{k}{\mu} \int_{t_0}^{t} (t-s)^{k-1} y'(s)\, ds \right)^{1/2} \left(\int_{t_0}^{t} (t-s)^{k-2} y(s)\, ds \right)^{1/2}$$

and so we obtain

$$\int_{t_0}^{t} (t-s)^{k-1} y'(s)\, ds \leq \frac{k}{\mu} \int_{t_0}^{t} (t-s)^{k-2} y(s)\, ds, \quad t \geq t_1. \qquad \ldots (26)$$

But, for $t \geq t_1$, we have

$$\int_{t_0}^{t} (t-s)^{k-2} y(s)\, ds = \frac{1}{k-1} (t-t_0)^{k-1} y(t_0) + \frac{1}{k-1} \int_{t_0}^{t} (t-s)^{k-1} y'(s)\, ds.$$

So (26) gives

$$\left(\mu \frac{k-1}{k} - 1 \right) \int_{t_0}^{t} (t-s)^{k-1} y'(s)\, ds \leq (t-t_0)^{k-1} y(t_0).$$

Therefore, by using (25), we find

$$\left(\mu \frac{k-1}{k} - 1 \right) \frac{1}{t^{k-1}} \int_{t_0}^{t} (t-s)^k y^{-1}(s)(y'(s))^2\, ds \leq \frac{k}{\mu} \left(1 - \frac{t_0}{t} \right)^{k-1} y(t_0), \qquad \ldots (27)$$

But one has $\int_{t_0}^{+\infty} y^{-1}(s)\,(y'(s))^2\,ds = +\infty$ implies that

$$\lim_{t \to +\infty} \frac{1}{t^{k-1}} \int_{t_0}^{t} (t-s)^k y^{-1}(s)\,(y'(s))^2\,ds = +\infty.$$

On the other hand, (23) implies that $\mu \frac{k-1}{k} - 1 > 0$. Thus (27) leads to a contradiction. Hence our claim is proved, and there exists a sequence $\{t_n\}$ such that

$$\lim_{n \to +\infty} \left(\mu \int_{t_0}^{t_n} (t_n - s)^k y^{-1}(s)\,(y'(s))^2\,ds - k \int_{t_0}^{t_n} (t_n - s)^{k-1} y'(s)\,ds \right) > 0. \quad \text{... (28)}$$

Using (28) in (22), we find for sufficiently large t_n,

$$\frac{\alpha}{t_n^k} \int_{t_0}^{t_n} (t_n - s)^k a(s)\,ds < -\left(1 - \frac{t_0}{t_n}\right)^k y'(t_0)$$

$$+ (\mu - \beta) \frac{1}{t_n^k} \int_{t_0}^{t_n} (t_n - s)^k y^{-1}(s)\,(y'(s))^2\,ds. \quad \text{... (29)}$$

Since $\mu < \beta$, $\alpha > 0$, and

$$\lim_{t \to +\infty} \frac{1}{t^k} \int_{t_0}^{t} (t-s)^k y^{-1}(s)\,(y'(s))^2\,ds = \int_{t_0}^{+\infty} y^{-1}(s)\,(y'(s))^2\,ds = +\infty,$$

(29) implies

$$\liminf_{t \to +\infty} \frac{1}{t^k} \int_{0}^{t} (t-s)^k a(s)\,ds = -\infty,$$

which contradicts (21). The proof of the theorem is now complete.

Remark 1 : It is easy to given an example of $a(t)$ which satisfies conditions (7) and (8) but fails to satisfy conditions (6) and (8). Take $A(t) = t^{\mu} \sin t$, $1/2 < \mu < 1$. The example $a(t) = t^{\mu} \cos t + \mu t^{\mu-1} \sin t$, $1/2 < \mu < 1$ does not satisfy (4) or (6), but our theorem applies.

Remark 2 : Condition (6) has been extended by Kamenev[5] to the weaker condition for the linear equation, i.e., for some $\mu > 1$,

$$\limsup_{t \to +\infty} \frac{1}{t^{\mu}} \int_{0}^{t} (t-s)^{\mu} a(s)\,ds = +\infty. \quad \text{... (30)}$$

It is known that (3) and (30) are also sufficient for the superlinear equation, see Wong[9].

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