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EXPLICIT FORMULAS FOR BERNOULLI POLYNOMIALS OF ORDER n

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We reduce the n-ple Hurwitz zeta function to a finite series of generalized zeta functions by using Stirling numbers. By means of this result we express Bernoulli polynomials of order n as a finite double series of Bernoulli polynomials and Stirling numbers.

1. INTRODUCTION

For any complex x we define the functions $B_l(x)$ by the equation

$$\frac{ze^{xz}}{e^z-1}=\sum_{l=0}^{\infty} B_l(x)\frac{z^l}{l\,!}\ , \text{ where } |z|<2\pi.$$

The functions $B_l(x)$ are called l-th Bernoulli polynomials and the numbers $B_l(0)$ are called Bernoulli numbers and denoted by B_l. Thus,

$$\frac{z}{e^z-1}=\sum_{l=0}^{\infty} B_l\frac{z^l}{l\,!}\ , \text{ where } |z|<2\pi.$$

The Bernoulli polynomials and numbers of order n are defined respectively by, for any complex number x,

$$\frac{z^n e^{xz}}{(e^z-1)^n}=\sum_{l=0}^{\infty} B_l^{(n)}(x)\frac{z^l}{l\,!}\ , \text{ where } |z|<2\pi,$$

$$\frac{z^n}{(e^z-1)^n}=\sum_{l=0}^{\infty} B_l^{(n)}\frac{z^l}{l\,!}\ , \text{ where } |z|<2\pi.$$

Note that $B_l^{(1)}(x)=B_l(x)$, $B_l^{(1)}=B_l$ and $B_l^{(n)}(0)=B_l^{(n)}$ (Whittaker and Watson[15], pp. 25-29). Clearly

$$B_l^{(n)}(n-x) = (-1)^l B_l^{(n)}(x) \qquad \text{... (1.1)}$$

for every integer $l \geq 0$, so that

$$B_l^{(n)}(n) = (-1)^l B_l^{(n)}(0) = (-1)^l B_l^{(n)}.$$

The Stirling numbers of the first kind $(-1)^{n+k} s(n,k)$, denoted by $|s|\,(n, k)$ and called the unsigned or absolute Stirling numbers of the first kind $|s(n, k)|$, is the number of permutations of n symbols which have exactly k cycles. The $s(n, k)$ have for 'horizontal' generating function (this is taken as definition of the $s(n, k)$) (Abramowitz and Stegun[1], p. 824; Comtet[5], p. 213) :

$$x(x+1)\ldots(x+n-1) = \sum_{k=0}^{n} |s|\,(n,k)\, x^k. \qquad \text{... (1.2)}$$

The Stirling numbers of the first kind $s(n, k)$ satisfy the triangular recurrence relation (Comtet[5], p. 214) :

$$s(n, k) = s(n-1, k-1) - (n-1)\, s(n-1, k),\ n,\ k \geq 1; \qquad \text{... (1.3)}$$

$$s(n, 0) = s(0, k) = 0, \text{ except } s(0, 0) = 1.$$

According to the study of determinants of Laplacians the theory of multiple Gamma functions, which had been studied by Barnes[2] and others, has recently been revived[4, 9, 14]. There are two known ways to define multiple Gamma functions. Vignéras[13] gave the Weierstrass canonical product forms of them by recurrence formula. Barnes[2] defined them by using the r-ple Hurwitz zeta functions, which are given by, for $\mathrm{Re}(s) > r$,

$$\zeta_r(s, a \mid w_1, w_2, \ldots, w_r) = \sum_{m_1, m_2, \ldots, m_r = 0}^{\infty} \frac{1}{(a+\Omega)^s},$$

where $\Omega = m_1 w_1 + m_2 w_2 + \ldots + m_r w_r$ (Barnes[2], p. 387). He also represents the r-ple Hurwitz zeta functions by the contour integral

$$\zeta_r(s, a \mid w_1, w_2, \ldots, w_r) = \frac{i\Gamma(1-s)}{2\pi} \int_L \frac{e^{-az}(-z)^{s-1}}{\prod_{k=1}^{r} (1-e^{-w_k z})}\, dz$$

where the conditions for a and $w_1, w_2, \ldots, w_r$, the possible contour L are given in Barnes[2] (p. 388), and $\Gamma(z)$ is the well-known Gamma function which has simple poles at $z = 0, -1, -2, \ldots$; and $\operatorname{Res}_{z=-n} \Gamma(z) = (-1)^n/n!$, $n = 0, 1, 2, \ldots$ (Whittaker and Watson[15], pp. 235-264).

Here we restrict these when $w_k = 1$, $k = 1, 2, \ldots, n$, $x > 0$ and the contour C given as in Whittaker and Watson[15] (p. 245). That is to say, we get

$$\zeta_n(s, x) = \sum_{k_1, k_2, \ldots, k_n = 0}^{\infty} (x + k_1 + k_2 + \ldots + k_n)^{-s}, \qquad \text{... (1.4)}$$

which can be shown to be an analytic function for Re $(s) > n$ by the Eisenstein's Theorem (Forsyth[6], p. 99). Furthermore $\zeta_n(s,x)$ can be continued to a meromorphic function with simple poles only at $s = 1, 2, ..., n$. For by the contour integral representation

$$\zeta_n(s,x) = \frac{i\Gamma(1-s)}{2\pi} \int_C \frac{e^{-xz}(-z)^{s-1}}{(1-e^{-z})^n} dz \qquad \text{... (1.5)}$$

the integral is valid for $x > 0$ and all s so $\zeta_n(s,x)$ has possible poles only at the poles of $\Gamma(1-s)$ i.e., $s = 1, 2, 3, ...$ But by the series definition $\zeta_n(s,x)$ is analytic for Re $(s) > n$ (Choi and Quine[3]). In particular, when $n = 1$, $\zeta_1(s,x) = \sum_{k=0}^{\infty} (x+k)^{-s} = \zeta(s,x)$ is the well-known Hurwitz ζ-function.

Among several explicit formulas for the Bernoulli numbers, the double series representation

$$B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n \qquad (n \geq 0) \qquad \text{... (1.6)}$$

is fairly well-known[7, 10]. Recently, Srivastava and Todorov[11], [p. 510, eqn. (3)] proved the following explicit formula for the generalized Bernoulli polynomials : For an arbitrary (real or complex) parameter α,

$$B_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{\alpha+k-1}{k} \frac{k!}{(2k)!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^{2k} (x+j)^{n-k}$$

$$\times F[k-n, k-\alpha;\ 2k+1;\ j/(x+j)], \qquad \text{... (1.7)}$$

where $F[a, b; c; z]$ denotes the Gaussian hypergeometric function (cf., e.g., Whittaker and Watson[15], Chap. 14). Some interesting special cases considered earlier by Gould[7] and Todorov[12] may be derived by applying the expression (1.7).

The object of the present note is to reduce the n-ple Hurwitz zeta function (1.4) to a finite sum of generalized Hurwitz zeta functions, which may be used to get another analytic continuation of $\zeta_n(s,x)$ different from the contour integral representation (1.5), and to prove a new explicit formula for the Bernoulli polynomials of order n :

$$B_{n+k}^{(n)}(x) = n \binom{n+k}{n} \sum_{j=0}^{n-1} (-1)^j \frac{B_{k+j+1}(x)}{k+j+1} \sum_{l=j}^{n-1} \binom{l}{j} s(n, l+1)\, x^{l-j},$$

$$\text{... (1.8)}$$

where $s(n, l+1)$ are the Stirling numbers of the first kind.

We can also give another explicit formula for $B_{n-k}^{(n)}(x)$ by comparing residues of the representation (1.5) with those of our reduced form for $\zeta_n(s,x)$ at $s = k$, $k = 1, 2, ..., n$.

2. Proof of the Explicit Formula (1.8)

We rewrite eqn. (1.5) as

$$\zeta_n(s,x) = \Gamma(1-s)\, I_n(s,x), \qquad \dots (2.1)$$

where

$$I_n(s,x) = \frac{i}{2\pi} \int_C \frac{e^{-xz}(-z)^{s-1}}{(1-e^{-z})^n}\, dz.$$

The value of $\zeta_n(-l,x)$ can be calculated explicitly if l is a nonnegative integer. Taking $s = -l$ in the relation $\zeta_n(s,x) = \Gamma(1-s)\, I_n(s,x)$ we find

$$\zeta_n(s,-l) = \Gamma(1+l)\, I_n(-l,x) = l!\, I_n(-l,x).$$

We also have

$$\begin{aligned} I_n(-l,x) &= -\frac{1}{2\pi i} \int_C \frac{(-z)^{-l-1} e^{-xz}}{(1-e^{-z})^n}\, dz \\ &= -\operatorname*{Res}_{z=0} \frac{(-z)^{-l-1} e^{-xz}}{(1-e^{-z})^n} \\ &= (-1)^l \operatorname*{Res}_{z=0} z^{-n-l-1} \frac{z^n e^{(n-x)z}}{(e^z-1)^n} \\ &= (-1)^l \operatorname*{Res}_{z=0} z^{-n-l-1} \sum_{k=0}^{\infty} B_k^{(n)}(n-x) \frac{z^k}{k!} \\ &= (-1)^l \frac{B_{n+l}^{(n)}(n-x)}{(n+l)!}, \end{aligned}$$

from which we have the following relation with eqn. (1.1) : For every integer $l \geq 0$,

$$\zeta_n(-l,x) = (-1)^n \frac{l!}{(n+l)!} B_{n+l}^{(n)}(x). \qquad \dots (2.2)$$

Replacing n by 1 in (2.2), we have the well-known result :

$$\zeta(-l,x) = -\frac{B_{l+1}(x)}{l+1}. \qquad \dots (2.3)$$

Since the number of solutions of $k_1 + k_2 + \dots + k_n = k$, $k = 0, 1, 2, \dots$, $(k_1, k_2, \dots, k_n) \in N^n$, N the set of nonnegative integers, is equal to the coefficient of x^k in the expansion of the Maclaurin series of $(1-x)^{-n}$, i.e., $\binom{-n}{k} = (-1)^k \binom{k+n-1}{n-1}$. So the Maclaurin series expansion of $(1-x)^{-n}$ is

$$\sum_{k=0}^{\infty}\binom{-n}{k}(-x)^k=\sum_{k=0}^{\infty}\binom{k+n-1}{n-1}x^k.$$

The multiple Hurwitz zeta function in (1.4) can thus be represented as a simple series :

$$\zeta_n(s,x)=\sum_{k=0}^{\infty}\binom{k+n-1}{n-1}\Big/(x+k)^s. \qquad \ldots (2.4)$$

From (1.2) we obtain

$$\binom{k+n-1}{n-1}=\frac{1}{(n-1)!}\sum_{i=0}^{n-1}|s|(n,i+1)k^i.$$

We therefore have

$$\zeta_n(s,x)=\frac{1}{(n-1)!}\sum_{k=0}^{\infty}\left(\sum_{i=0}^{n-1}|s|(n,i+1)k^i\right)\Big/(x+k)^s.$$

Considering, by the binomial theorem,

$$k^i=\{(-x)+(x+k)\}^i=\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}(x+k)^j,$$

we have

$$\zeta_n(s,x)=\frac{1}{(n-1)!}\left[\sum_{i=0}^{n-1}|s|(n,i+1)\left\{\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\right\}\right]\left(\sum_{k=0}^{\infty}\frac{1}{(x+k)^{s-j}}\right)$$

$$=\frac{1}{(n-1)!}\sum_{i=0}^{n-1}|s|(n,i+1)\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\zeta(s-j,x).$$

It is also easy to show that $\zeta_n(s,x)$ can be expressed as the following :

$$\zeta_n(s,x)=\sum_{j=0}^{n-1}p_{n,j}(x)\,\zeta(s-j,x), \qquad \ldots (2.5)$$

where

$$p_{n,j}(x)=\frac{1}{(n-1)!}\sum_{l=j}^{n-1}(-1)^{n+1-j}\binom{l}{j}s(n,l+1)\,x^{l-j}$$

and so we find $p_{n,j}(x)$ a polynomial in x of degree $n-1-j$ with rational coefficients.

Since $\zeta(s,x)$ can be continued analytically to a meromorphic function having a simple pole at $s = 1$ with its residue 1, the representation (2.5) shows us that $\zeta_n(s,x)$ is analytic for all s except simple poles only at $s = 1, 2, \ldots, n$ with their residues : We find, for $k = 1, 2, \ldots, n$,

$$\operatorname*{Res}_{s=k} \zeta_n(s,x) = p_{n,k-1}(x). \qquad \ldots (2.6)$$

Using the recurrence relation (1.3) for $s(n, k)$, we express $\zeta_n(s,x)$ explicitly for small values n :

$$\zeta_2(s,x) = (1-x)\,\zeta(s,x) + \zeta(s-1,x),$$

$$\zeta_3(s,x) = \frac{1}{2}(x^2-3x+2)\,\zeta(s,x) + \left(\frac{3}{2}-x\right)\zeta(s-1,x) + \frac{1}{2}\zeta(s-2,x),$$

$$\zeta_4(s,x) = \frac{1}{6}\{(-x^3+6x^2-11x+6)\,\zeta(s,x) + (3x^2-12x+11)$$
$$\zeta(s-1,x) - (3x-6)\,\zeta(s-2,x) + \zeta(s-3,x)\}.$$

Letting $s = -l$ in (2.5) with (2.2), we obtain the desired formula (1.8). Replacing x by n with the formula (1.1), we obtain

$$B^{(n)}_{n+k} = (-1)^{n+k}\, n \binom{n+k}{n} \sum_{j=0}^{n-1} (-1)^j \frac{B_{k+j+1}(n)}{k+j+1}$$
$$\times \sum_{l=j}^{n-1} \binom{l}{j} s(n, l+1)\, n^{l-j}. \qquad \ldots (2.7)$$

Now we shall compute $\operatorname*{Res}_{s=k} \zeta_n(s,x)$ by using eqn. (2.1). If l is any integer, we find that

$$I_n(l,x) = -\frac{1}{2\pi i}\int_C \frac{(-z)^{l-1}e^{-xz}}{(1-e^{-z})^n}\,dz$$
$$= -\operatorname*{Res}_{z=0} \frac{(-z)^{l-1}e^{-xz}}{(1-e^{-z})^n}.$$

We can show that $(-z)^{k-1}e^{-xz}/(1-e^{-z})^n$ has a pole of order $n+1-k$ at $z = 0$, $1 \le k$ we therefore have

$$I_n(k,x) = \frac{(-1)^k}{(n-k)!} \lim_{z\to 0} \frac{d^{n-k}}{dz^{n-k}} \frac{z^n e^{-xz}}{(1-e^{-z})^n}.$$

To find the residue of $\zeta_n(s,x)$ at $s = k$, $1 \le k \le n$, by recalling the well-known formula $\Gamma(1-s) = \pi/(\Gamma(s)\sin \pi s)$ (Whittaker and Watson[15], p. 239) we have to compute the limit

$$\lim_{s \to k} (s-k)\,\zeta_n(s,x) = \lim_{s \to k} (s-k)\,\Gamma(1-s)\,I_n(s,x)$$

$$= I_n(k,x) \lim_{s \to k} (s-k) \frac{\pi}{\Gamma(s)\sin \pi s}$$

$$= \frac{1}{(n-k)!\,(k-1)!} \lim_{z \to 0} \frac{d^{n-k}}{dz^{n-k}} \frac{z^n e^{-xz}}{(1-e^{-z})^n}$$

$$= \frac{1}{(n-k)!\,(k-1)!} \lim_{z \to 0} \frac{d^{n-k}}{dz^{n-k}} \sum_{i=0}^{\infty} B_i^{(n)}(n-x) \frac{z^i}{i!}$$

$$= \frac{B_{n-k}^{(n)}(n-x)}{(n-k)!\,(k-1)!}.$$

We thus have another expression for $\operatorname*{Res}_{s=k} \zeta_n(s,x)$:

$$\operatorname*{Res}_{s=k} \zeta_n(s,x) = \frac{B_{n-k}^{(n)}(n-x)}{(n-k)!\,(k-1)!}. \qquad \text{... (2.8)}$$

Now combining formulas (2.6) and (2.8) with (1.1), we obtain

$$B_{n-k}^{(n)}(x) = \frac{1}{\binom{n-1}{k-1}} \sum_{l=k-1}^{n-1} \binom{l}{k-1} s(n,l+1)\,x^{l+1-k}. \qquad \text{... (2.9)}$$

Letting $x = n$ in (2.9), we have, for $n = 1, 2, \ldots, 1 \le k \le n$,

$$B_{n-k}^{(n)} = \frac{(-1)^{n-k}}{\binom{n-1}{k-1}} \sum_{l=k-1}^{n-1} \binom{l}{k-1} s(n,l+1)\,n^{l+1-k}. \qquad \text{... (2.10)}$$

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