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NUMERICAL SOLUTION OF PERIODIC BURGERS EQUATION

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A spectral method using the finitely reproducing property of the nonlinear operator is given to solve the periodic Burgers equation. This method gives a system of ordinary differential equations which is solved by the *s*-stage Runge-Kutta-Chebyshev Method. The numerical results show that the proposed method produces very accurate results.

1. INTRODUCTION

The one dimensional Burgers equation

$$u_t + uu_x - \varepsilon u_{xx} = 0, \ a < x < L, \ \ t > 0 \qquad \ldots (1.1)$$

with initial condition

$$u(x, 0) = u_0(x) \qquad \ldots (1.2)$$

where ε is a parameter, related to the Reynolds number $R(= 1/\varepsilon)$, was first introduced by Bateman[1] who found its steady solutions, descriptive of certain viscous flows. It was later proposed by Burgers[2] as one of a class of equations describing mathematical models of turbulence. In the context of gas dynamics, it was discussed by Hopf[3] and Cole[4]. They also illustrated independently that the Burgers equation can be solved exactly for an arbitrary initial condition. Benton and Platzman[5] have surveyed the analytical solutions of the one dimensional Burgers equation. It can be considered as a simplified form of the Navier-Stokes equation[6, 7] due to the form of nonlinear convection term and the occurrence of the viscosity term.

In this paper the Burgers equation (1.1) with periodic boundary conditions

$$u(x, t) = u(x + L, t) \qquad \ldots (1.3)$$

is considered. In contrast to Burgers equation with nonperiodic boundary conditions, there is relatively quite less work available in the literature dealing numerically with the Burgers equation with periodic boundary conditions. Aref and Daripa[8], and Hsia and Jeng[9] have applied finite difference methods to solve the Burgers equation with periodic boundary conditions. Schult and Wyld[10] have obtained results by using wavelets and compared them with finite difference results. A comparison of a number of different approaches is also done by Mulholland and Sloan[11].

In the present paper, a spectral method is presented to solve the Burgers equation with periodic boundary conditions. This method exploits the finitely reproducing property of nonlinear term in the equation. The concept of finitely reproducing nonlinearity is successfully used by Mittal and Singhal[12] to find the solution of the Burgers equation with Dirichlet's boundary conditions. In section 2, the nonlinear term uu_x in the Burgers equation is shown to be finitely reproducing with respect to the basis functions $\left[\frac{1}{\sqrt{2\pi}} e^{ijx}\right]_{j=-\infty}^{\infty}$. In section 3, a method is developed which gives a system of nonlinear ordinary differential equations to solved. This system of ODE's is solved by the s-stage Runge-Kutta-Chebyshev method given by Houwen *et al.*[13]. In section 4, results of three test problems are given. The results show that the method produces very accurate solutions.

2. Finitely Reproducing Nonlinearity

The concept of finitely reproducing nonlinearity was first introduced by Bazely[14]. Consider a nonlinear partial differential equation

$$u_t + Lu + M(u) = 0 \qquad \text{... (2.1)}$$

with suitable initial and boundary conditions. L is a linear and M is a nonlinear operator.

Definition

The operator M is called finitely reproducing with respect to a complete orthogonal set of functions $[\phi_k]_{k=1}^{\infty}$, if for every $n \in \mathbb{N}$, there exists $m(n) \in \mathbb{N}$, such that

$$M\left[\sum_{k=1}^{n} \alpha_k(t)\,\phi_k(x)\right] = \sum_{j=1}^{m(n)} \eta_j(\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))\,\phi_j(x) \qquad \text{... (2.2)}$$

where η_j,'s, $j = 1, 2, \ldots, m(n)$ are explicitly known functions of α_k's, $k = 1, 2, \ldots, n$.

Consider the orthonormal set of eigen function $[\phi_j(x)]_{j=-\infty}^{\infty}$ of the linear operator d^2/dx^2 in the domain $[0, 2\pi)$ satisfying the periodic boundary conditions

$$\phi_j(x) = \frac{1}{\sqrt{2\pi}} e^{ijx}, \quad x \in [0, 2\pi), \; i^2 = -1. \qquad \text{... (2.3)}$$

The spatial domain of the Burgers equation ((1.1)-(1.3)) is transformed from $[0, L)$ to $[0, 2\pi)$ by using the transformation

$$\xi = \frac{x}{L} 2\pi. \qquad \text{... (2.4)}$$

The system thus becomes

$$u_t + A\, uu_\xi - \varepsilon A^2 u_{\xi\xi} = 0, \qquad 0 \le \xi < 2\pi \qquad \text{... (2.5)}$$

with initial condition

$$u(\xi, 0) = \bar{u}_0(\xi), \quad 0 \le \xi < 2\pi \qquad \text{... (2.6)}$$

where $A = 2\pi/L$.

Suppose that

$$u(\xi, t) = \sum_{j=-n}^{n} \alpha_j(t)\, \phi_j(\xi) \qquad \text{... (2.7)}$$

where ϕ_j's are given by (2.3), is an approximate solution of (2.5) satisfying the periodic boundary conditions. Now, it remains to be shown that the nonlinear term $M(u) = uu_\xi$ is finitely reproducing with respect to basis functions (2.3). Consider

$$M(u) = uu_\xi$$

$$= \left[\sum_{j=-n}^{n} \alpha_j(t)\, \phi_j(\xi)\right]\left[\sum_{k=-n}^{n} \alpha_k(t)\, \phi_k'(\xi)\right]$$

$$= \left[\sum_{j=-n}^{n} \alpha_j(t)\, \phi_j(\xi)\right]\left[\sum_{k=-n}^{n} \beta_k(t)\, \phi_k(\xi)\right]$$

where $\beta_k(t) = ik\alpha_k(t)$.

$$M(u) = \left[\sum_{j=0}^{n} \alpha_j \phi_j + \sum_{j=1}^{n} \alpha_{-j} \phi_{-j}\right]\left[\sum_{k=0}^{n} \beta_k \phi_k + \sum_{k=1}^{n} \beta_{-k} \phi_{-k}\right]$$

$$= \left[\sum_{j=0}^{n} \alpha_j \phi_j\right]\left[\sum_{k=0}^{n} \beta_k \phi_k\right] + \left[\sum_{j=1}^{n} \alpha_{-j} \phi_{-j}\right]\left[\sum_{k=1}^{n} \beta_{-k} \phi_{-k}\right]$$

$$+ \left[\sum_{j=0}^{n} \alpha_j \phi_j\right]\left[\sum_{k=1}^{n} \beta_{-k} \phi_{-k}\right] + \left[\sum_{j=1}^{n} \alpha_{-j} \phi_{-j}\right]\left[\sum_{k=0}^{n} \beta_k \phi_k\right].$$

$$\text{... (2.8)}$$

For further simplification of (2.8) the following formulae have been used

$$\left[\sum_{p=0}^{n} a_p y^p\right]\left[\sum_{q=0}^{n} b_q y^q\right]$$

$$= \sum_{p=0}^{n}\left[\sum_{s=0}^{p} a_s b_{p-s}\right] y^p + \sum_{p=0}^{n-1}\left[\sum_{s=p+1}^{n} a_s b_{n+p+1-s}\right] y^{n+p+1} \quad \dots (2.9)$$

$$\left[\sum_{p=1}^{n} a_p y^p\right]\left[\sum_{q=1}^{n} b_q y^{-q}\right]$$

$$= \sum_{p=0}^{n-1}\left[\sum_{s=1}^{n-p} a_{s+p} b_s\right] y^p + \sum_{p=1}^{n-1}\left[\sum_{s=1}^{n-p} a_s b_{p+s}\right] y^{-p}. \quad \dots (2.10)$$

Using (2.9) and (2.10), eqn. (2.8) can be written as

$$M(u) = \frac{1}{\sqrt{2\pi}}\left[\sum_{j=0}^{n}\left(\sum_{k=0}^{j} \alpha_k \beta_{j-k}\right)\phi_j + \sum_{j=0}^{n-1}\left(\sum_{k=j+1}^{n} \alpha_k \beta_{n+j+1-k}\right)\phi_{j+n+1}\right.$$

$$+ \sum_{j=2}^{n}\left(\sum_{k=1}^{j-1} \alpha_{-k} \beta_{k-j}\right)\phi_{-j} + \sum_{j=1}^{n}\left(\sum_{k=j}^{n} \alpha_{-k} \beta_{k-j-n}\right)\phi_{-(j+n)}$$

$$+ \sum_{j=0}^{n-1}\left(\sum_{k=1}^{n-j} (\alpha_{k+j}\beta_{-k} + \beta_{k+j}\alpha_{-k})\right)\phi_j$$

$$\left. + \sum_{j=1}^{n}\left(\sum_{k=0}^{n-j} (\alpha_k \beta_{-(j+k)} + \beta_k \alpha_{-(j+k)})\right)\phi_{-j}\right].$$

It can be further simplified as

$$M(u) = \frac{1}{\sqrt{2\pi}} \sum_{j=-n}^{n} \eta_j (\alpha_{-n}(t), \dots, \alpha_{-1}(t), \alpha_0(t), \alpha_1(t), \dots, \alpha_n(t))\, \phi_j(\xi)$$

$$+ \sum_{j=0}^{n-1}\left(\sum_{k=j+1}^{n} \alpha_k \beta_{n+j+1-k}\right)\phi_{j+n+1}(\xi)$$

$$+ \sum_{j=1}^{n}\left(\sum_{k=j}^{n} \alpha_{-k} \beta_{k-j-n}\right)\phi_{-(j+n)}(\xi) \quad \dots (2.11)$$

where η_j's are defined as follows :

$$\eta_j = \begin{cases} \sum_{k=0}^{n} \alpha_k \beta_{n-k}, & j = n \\ \sum_{k=0}^{j} \alpha_k \beta_{j-k} + \sum_{k=1}^{n-j} (\alpha_{k+j} \beta_{-k} + \beta_{k+j} \alpha_{-k}), & 0 < j < n \\ \alpha_0 \beta_0 + \sum_{k=1}^{n} (\alpha_k \beta_{-k} + \beta_k \alpha_{-k}), & j = 0 \\ \sum_{\substack{k=1 \\ j \neq -1}}^{-(j+1)} \alpha_{-k} \beta_{k+j} + \sum_{k=0}^{n+j} (\alpha_k \beta_{j-k} + \beta_k \alpha_{j-k}), & -n < j < 0 \\ \sum_{k=0}^{n} \alpha_{-k} \beta_{k-n}, & j = -n. \end{cases} \quad \text{... (2.12)}$$

Thus, it is shown that for every $n \in \mathbb{N}$ ($n \geq 1$), there exists $m(n)$ such that $M(u) = uu_\xi$ is finitely reproducing with respect to the basis functions $\left[\frac{1}{\sqrt{2\pi}} e^{ijx}\right]_{j=-n}^{n}$

3. Method of Solution

We substitute an approximate solution of the form

$$u(\xi, t) = \sum_{j=-N}^{N} \alpha_j(t)\, \phi_j(\xi) \quad \text{... (3.1)}$$

into eqn. (2.5). We get the error $E[\xi, \vec{\alpha}(t)]$ in the solution as

$$E[\xi, \vec{\alpha}(t)] = \sum_{j=-N}^{N} \dot{\alpha}_j(t)\, \phi_j(\xi) + \varepsilon A^2 \sum_{j=-N}^{N} \alpha_j(t)\, j^2 \phi_j(\xi)$$

$$+ \frac{A}{\sqrt{2\pi}} \sum_{j=-N}^{N} \eta_j (\alpha_{-N}(t), \ldots, \alpha_{-1}(t), \alpha_0(t), \alpha_1(t), \ldots, \alpha_N(t))\, \phi_j(\xi)$$

$$+ \frac{A}{\sqrt{2\pi}} \sum_{j=0}^{N-1} \left(\sum_{k=j+1}^{N} \alpha_k(t)\, \beta_{N+j+1-k}(t) \right) \phi_{j+N+1}(\xi)$$

$$+ \frac{A}{\sqrt{2\pi}} \sum_{j=1}^{N} \left(\sum_{k=j}^{N} \alpha_{-k}(t)\, \beta_{k-j-N}(t) \right) \phi_{-(j+N)}(\xi) \quad \text{... (3.2)}$$

where $\vec{\alpha}(t) = (\alpha_{-N}(t), ..., \alpha_{-1}(t), \alpha_0(t), ..., \alpha_1(t), ..., \alpha_N(t))^T$ is the unknown column vector and η_j's are given by (2.12).

For minimizing the error, the orthogonality condition of the basis functions is used as follows

$$\int_0^{2\pi} E[\xi, \vec{\alpha}(t)]\, \phi_j(\xi)\, d\xi = 0, \quad j = -N, ..., -1, 0, 1, ..., N. \qquad ... (3.3)$$

This gives the following system of $2N + 1$ ordinary differential equations

$$\alpha_j(t) + \varepsilon j^2 A^2 \alpha_j(t) + \frac{A}{\sqrt{2\pi}} \eta_j(\alpha_{-N}(t), ..., \alpha_{-1}(t),$$

$$\alpha_0(t), ..., \alpha_1(t), ..., \alpha_N(t)) = 0, \qquad ... (3.4)$$

for $j = -N, ..., -1, 0, 1, ..., N$

The initial conditions for the system of ordinary differential equations are given by

$$\alpha_j(0) = \int_0^{2\pi} \bar{u}_0(\xi)\, \phi_j(\xi)\, d\xi, \text{ for } j = -N, ..., -1, 0, 1, ..., N. \qquad ... (3.5)$$

In the present case, it is observed that

$$\alpha_{-j}(t) = \alpha_j^*(t), \quad \text{for} \quad j = 1, 2, ..., N$$

where * denotes the complex conjugate, so that the system (3.4) of $(2N + 1)$ ordinary differential equations reduces to a system of $(N + 1)$ ordinary differential equations.

The exact solution of the nonlinear ordinary differential equations (3.4) with initial condition (3.5) is not known. However, it can be solved numerically. We have applied s-stage Runge-Kutta-Chebyshev method given by Houwen *et al.*[13] whose convergence is discussed by Verwer *et al.*[15]. The system (3.4) provides the coefficients $\alpha_j(t)$'s.

Thus, the numerical solution of Burgers equation (1.1) with initial condition (1.2) at any given x is computed by substituting $\xi = \frac{x}{L} 2\pi$ in (3.1).

4. Numerical Results and Discussion

The proposed method for periodic boundary conditions is applied on three test problems of different kinds.

Example 1 — Consider the Burgers equation (1.1) with the following 2π periodic exact solution[16]

$$u(x,t) = \frac{\frac{1}{t+1} \sum_{r=-\infty}^{\infty} [x-(2r+1)] \exp\left[-\left\{x-(2r+1)^2 \pi\right\}^2/(4\varepsilon\,(t+1))\right]}{\sum_{r=-\infty}^{\infty} \exp\left[-\left\{x-(2r+1)\pi\right\}^2/(4\varepsilon\,(t+1))\right]} . \quad \text{... (4.1)}$$

The computed maximum pointwise error L_∞ for $\varepsilon = 0.5$, 0.1 and 0.05 with $\tau = 0.01$ at different instants of time are given in Table I. Mulholland and Sloan[11] have reported maximum point wise error for $\varepsilon = 0.2$ at $t = 0.4$ with $\tau = 0.1$ and 0.01 using the Fourier pseudo-spectral method. In order to compare our results with those given in Mulholland and Sloan[11], the results are computed for $\varepsilon = 0.2$ with $\tau = 0.1$, 0.01 and 0.05 and are given in Table II. From the Table it is clear that the errors are less than those given in Mulholland and Sloan[11]. It is also clear from the table that the error in the solution decreases as N increases. It is also found decreasing with decrease in τ.

Example 2 — In this example the initial condition for the Burgers equation is taken as

$$u(x,\ 0) = -R \sin x, \quad 0 < x < 2\pi$$

where R is the Reynolds number. The results are presented in Figs. 1(a) and 1(b) for $R = 1$ and $R = 100$ respectively. In Figure 1(a), an initial simple harmonic waveform and decays to zero. But for large Reynolds number $R = 100$, as time advances, the curves steepen and develops a shock like discontinuity. The figure shows a good agreement with the corresponding results given in Benton and Platzman[5].

Example 3 — Initial condition for this example is

$$u(x,\ 0) = \begin{cases} \sin x, & 0 \le x < \pi \\ 0, & \pi \le x < 2\pi. \end{cases}$$

Numerical solution of Burgers equation (1.1) with the above initial condition, are given in Figs. 2 and 3. It is clear from the figures that the wave front for $\varepsilon = 0.1$ and 0.05 have smooth slope which becomes steeper for $\varepsilon = 0.01$ and 0.005, eventually leading to a discontinuity for smaller values of ε. For a half sine wave initial condition, as time advances, the discontinuity propagates to the right and the curves away from the discontinuity become straight lines with slope gradually decreasing.

In Examples 2 and 3 the integral (3.5) is solved analytically and in Example 1 is solved numerically by using Simpson's 1/3 rule. The number N considered in the solutions series (3.1) is judged by approximating the initial condition $u_0(x)$ to a desired accuracy. For small values of ε, one must consider a large value of N to obtain proper solution. To achieve a better accuracy large values of N and small values of τ must be taken. The present method is straight forward and easy to apply.

TABLE I

Maximum pointwise error (L_∞) in the numerical solution of Example 1 with $N = 50$ and $\tau = 0.01$

t	$\varepsilon = 0.5$	$\varepsilon = 0.1$	$\varepsilon = 0.01$
0.2	1.43721E-05	4.59368E-03	9.40661E-02
0.4	1.77907E-05	1.40353E-03	4.81911E-02
0.6	1.76531E-05	4.70346E-04	2.48140E-02
0.8	1.62724E-05	1.66144E-04	1.35673E-02
1.0	1.45209E-05	6.55941E-05	7.51181E-03

TABLE II

Maximum pointwise error (L_∞) in numerical solution of Example 1 for $\varepsilon = 0.2$ for different values of N and τ

	$\tau = 0.1$				$\tau = 0.01$				$\tau = 0.005$	
t	$N = 20$	$N = 30$	$N' = 32$*	$N' = 64$*	$N = 25$	$N = 35$	$N' = 32$*	$N' = 64$*	$N = 35$	$N = 45$
0.2	2.86844E-02	3.60909E-03	--	--	5.46483E-03	5.15487E-04	--	--	5.03787E-04	2.66071E-05
0.4	1.62093E-02	2.80816E-03	0.036	3.2E-03	1.76680E-03	1.28062E-04	0.036	3.9E-04	1.14162E-04	7.72534E-06
0.6	1.38203E-02	2.35117E-03	--	--	5.71582E-04	4.10107E-05	--	--	2.79907E-05	5.11777E-06
0.8	1.37357E-02	2.01345E-03	--	--	1.83297E-04	2.08884E-05	--	--	8.61554E-06	4.56800E-06
1.0	1.32388E-02	1.79398E-03	--	--	6.47568E-05	1.76647E-05	--	--	4.66467E-06	4.06705E-06

*Results taken from Mulholland and Sloan[11], where N' denotes the number of equidistant points at which the interval is descritised in Fourier pseudo-spectral method.

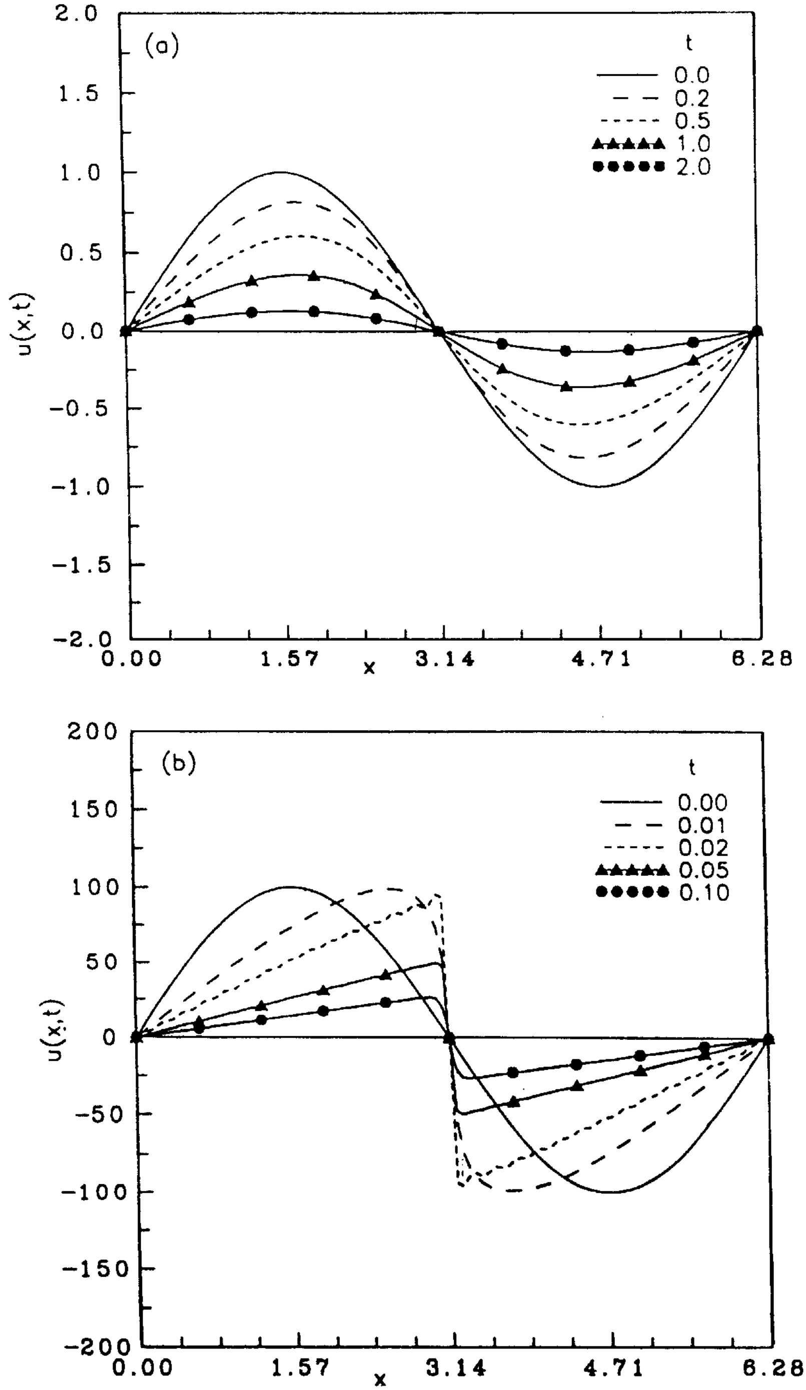


FIG. 1. Numerical solution of example 2 for $\varepsilon = 1.0$. (a) $R = 1$, $N = 10$ and $\tau = 0.05$ and (b) $R = 100$, $N = 70$ and $\tau = 0.0005$.

The solution can be obtained at any point in the space domain directly and if the initial condition can be reproduced accurately enough, the corresponding results are quite accurate.

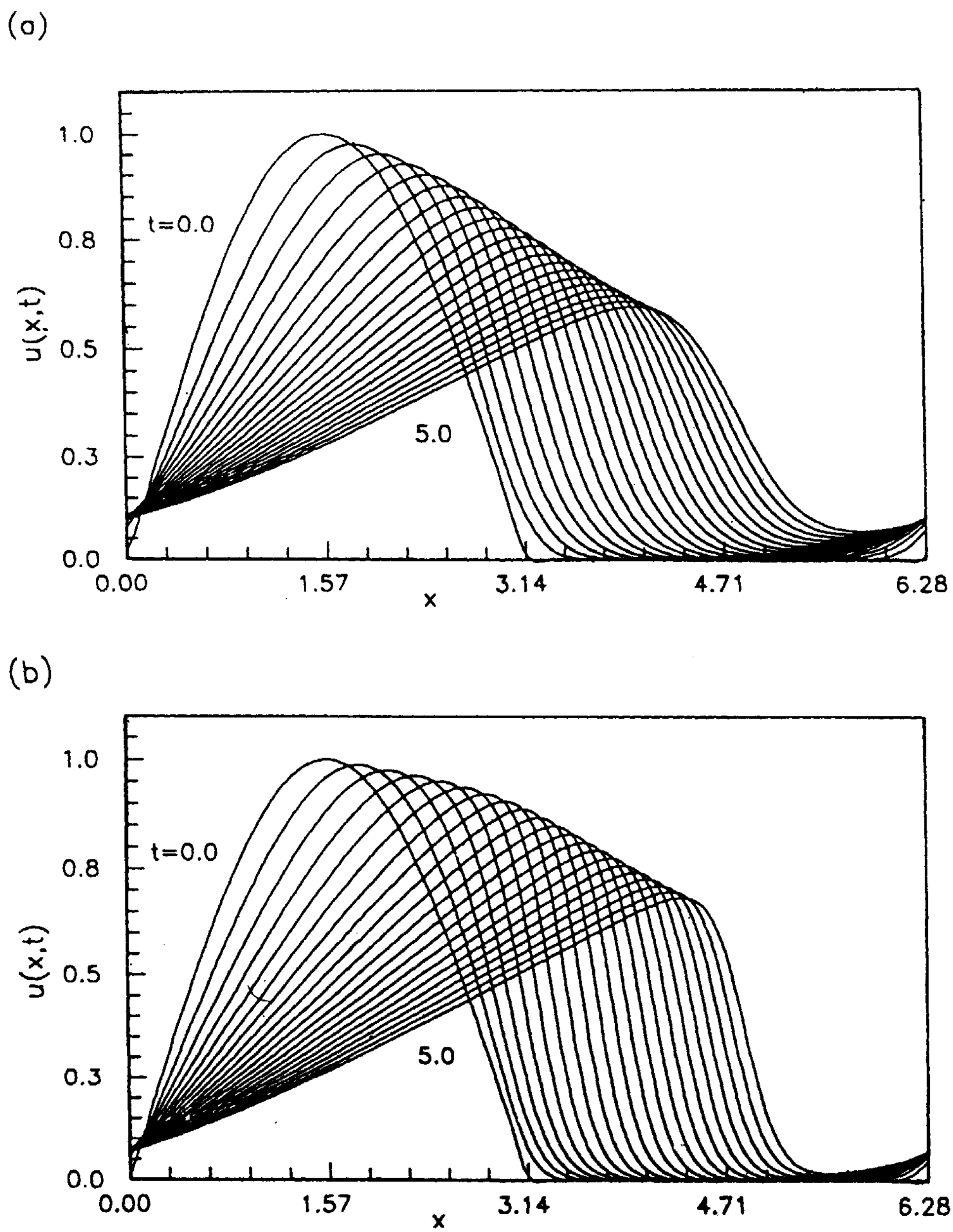
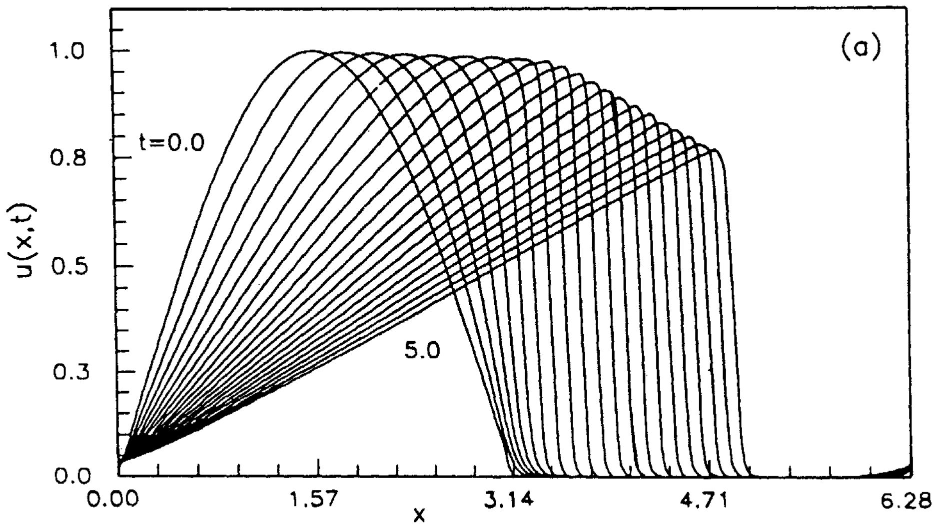


FIG. 2. Numerical solution of example 3 with $\tau = 0.01$. (a) $\varepsilon = 0.1$, $N = 20$ and (b) $\varepsilon = 0.05$, $N = 30$. The time interval between curves is 0.25.



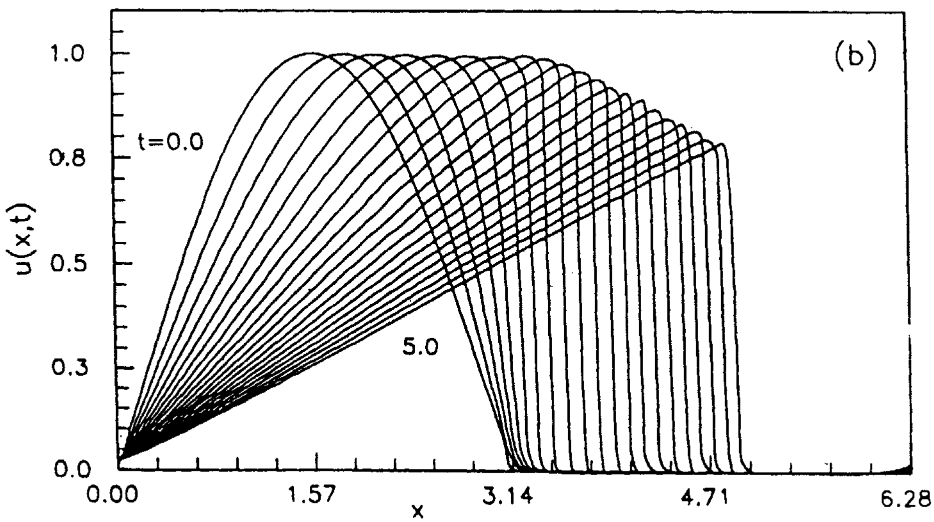


FIG. 3. Numerical solution of example 3. (a) $N = 70$, $\varepsilon = 0.01$ and $\tau = 0.01$ and (b) $N = 120$, $\varepsilon = 0.005$ and $\tau = 0.005$. The time interval between curves is 0.25.

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