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PROVING FIXED POINT THEOREMS USING GENERAL PRINCIPLES

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The research literature contains a number of fixed point papers dealing with contractive conditions with rational expressions. It is the purpose of this paper to point that all of these results involving rational expressions are special cases, either of the general principles that appear in Park[32] and Hicks and Rhoades[17], or of the modest extensions of them that are proved in this paper.

The first general principle to be used is a modest extension of the theorem of Hicks and Rhoades[17] to the case in which the metric space is not complete. Let $O(x) := \{x, Tx, T^2x, ...\}$.

Proposition 1 — Let T be a selfmap of a metric space (X, d). Let $0 \leq h < 1$. Suppose there exists a point $x \in X$ such that

(A) $x_n := T^n x$ has a convergent subsequence with limit $z \in X$, and for this x,

(B) $d(Ty, T^2y) \leq hd(y, Ty)$ for each $y \in O(x)$. ... (1)

Then, for this x

(i) $\lim T^n x = z$,

(ii) $d(T^n x, z) \leq \dfrac{h^n}{1-h} d(x, Tx)$,

and

(iii) z is a fixed point of T in X if and only if $G(x) := d(x, Tx)$ is lower semicontinuous at z.

PROOF : Set $y = Tx$ and then $y = x$ in (1) to get

$$d(T^2x, T^3x) \leq hd(Tx, T^2x) \leq h^2 d(x, Tx).$$

By induction, $d(T^n x, T^{n+1} x) \leq h^n d(x, Tx)$, which implies that $\{x_n\}$ is Cauchy. Conclusion (i) now follows from (A). The rest of the proof is the same as that in Hicks and Rhoades[17].

Corollary 1 (Theorem 1 of Dass and Gupta[10]) — Let T be a selfmap of a complete metric space (X, d) such that

(i) $d(Tx, Ty) \le \frac{\alpha d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y)$

for all x, y in X, $\alpha, \beta > 0$, $\alpha + \beta < 1$, and

(ii) for some $x_0 \in X, x_n := T^n x_0$ has a convergent subsequence with limit z in X. Then z is the unique fixed point of T in X.

PROOF : Dass and Gupta[10] do not hypothesize that X is complete, but they use that fact in their proof. Set $y = Tx$ in (i) to get

$$d(Tx, T^2x) \le \alpha d(Tx, T^2x) + \beta d(x, Tx),$$

which implies that

$$d(Tx, T^2x) \le \frac{\beta}{1 - \alpha} d(x, Tx).$$

By Proposition 1, $\{x_n\}$ converges. Call the limit z. Now set $x = x_n, y = z$ in (i) to get

$$d(Tx_n, Tz) \le \frac{\alpha d(z, Tz)[1 + d(x_n, Tx_n)]}{1 + d(x_n, z)} + \beta d(x_n, z).$$

Taking the limit as $n \to \infty$ yields $d(z, Tz) \le \alpha d(z, Tz)$, which implies that z is a fixed point of T. As in Dass and Gupta[10], z is unique.

Note that condition (ii) is not needed. In fact, one could have used the Theorem in Hicks and Rhoades[17] to obtain their result.

If, in Corollary 1, one sets $x = T^p x, y = T^{p+1} x, p$ a non-negative integer, then one obtains Theorem 2 of Chatterjee[5].

Corollary 2 (Sharma and Yuel[41], Theorem 1) — Let T be a selfmap of a complete metric space (X, d) satisfying (i) of Corollary 1, with T asymptotically regular at some point in X. Then T has a unique fixed point in X.

PROOF : Since X is complete, conclusion (i) of Proposition 1 is true without hypothesizing condition (A). The balance of the proof is the same as that of Corollary 1.

Note that the assumption of asymptotic regularity is not needed.

Corollary 3 — (Jaggi and Dass[20]) — Let T be a selfmap of a metric space (X, d) satisfying :

(i) for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$,

$$d(Tx, Ty) \le \frac{\alpha d(x, Tx)\, d(y, Ty)}{d(x, Ty) + d(y, Tx) + d(x, y)} + \beta d(x, y) \text{ for all } x, y \in X,\ x \ne y.$$

(ii) There exists a point $x_0 \in X$ such that $\{T^n x_0\}$ has a convergent subsequence with limit z in X.

Then T has a unique fixed point z (in X).

PROOF : Set $x_0 = x$ and $y = Tx$ in (i). Assume that $x \neq Tx$. For, otherwise, x is a fixed point of T. Then, from (i),

$$d(Tx, T^2x) \leq \frac{\beta}{1-\alpha} d(x, Tx).$$

We may assume that $T^n x \neq T^{n+1} x$ for each $n \geq 0$. For, otherwise, T has a fixed point. It then follows that the above inequality is true for each $y \in O(x)$, and (1) is satisfied. By proposition 1, $\lim T^n x = z$.

Suppose that $z \neq Tz$. Now set $x = T^{n-1} x, y = z$ in (i) to get

$$d(T^n x, Tz) \leq \frac{\alpha d(T^{n-1}x, T^n x)\, d(z, Tz)}{d(T^{n-1}x, Tz) + d(z, T^n x) + d(T^{n-1}x, z)} + \beta d(T^{n-1}x, z).$$

Taking the limit as $n \to \infty$ yields $d(z, Tz) \leq 0$, a contradiction. Therefore z is a fixed point of T. Uniqueness follows from (i).

In proving each of the Corollaries 4-11 we shall assume that $T^n x \neq T^{n+1} x$ for each $n \geq 0$, since, otherwise, T has a fixed point. Then, as in the proof of Corollary 3, to show that $\{T^n x\}$ converges, it will be sufficient to show that the contractive condition of the specific corollary implies (1) with $y = Tx$.

Corollary 4 (Gupta and Saxena[16], Theorem 1) — Let T be a continuous selfmap of a metric space (X, d) such that

$$d(Tx, Ty) \leq \frac{a[1 + d(x, Tx)]\, d(y, Ty)}{1 + d(x, y)} + \frac{b d(x, Tx)\, d(y, Ty)}{d(x, y)} + c d(x, y)$$

for all x, y in X, $a, b, c \geq 0$, $a + b + c < 1$. If for some $x_0 \in X$, the sequence $\{x_n\}$, defined by $x_n = Tx_{n-1}$, has a convergent subsequence with limit $z \in X$, then T has a unique fixed point (in X).

PROOF : Set $y = Tx$ in the above inequality to obtain

$$d(Tx, T^2x) \leq (a + b)\, a d(Tx, T^2x) + c d(x, Tx),$$

which implies that

$$d(Tx, T^2x) \leq \frac{c}{1-a-b} d(x, Tx),$$

and (1) is satisfied. Therefore $\{x_n\}$ converges to z, and z is a fixed point of T, since T is continuous.

Now use the argument in Gupta and Saxena[16] to show that z is the unique fixed point of T in X.

As noted by Gupta and Saxena[16], their result includes theorems of Edelstein[11] and Dass and Gupta[10].

Corollary 5 (Jaggi[19], Theorem 1) — Let T be a selfmap of a metric space (X, d) satisfying

$$d(Tx, Ty) \le \frac{\alpha d(x, Tx)\, d(y, Ty)}{d(x, y)} + \beta d(x, y)$$

for all $x, y \in X$, $x \ne y$, and for some $\alpha, \beta \ge 0$ with $\alpha + \beta < 1$. If there exists an $x_0 \in X$ such that X is (x_0, T)-orbitally complete and T is x_0 orbitally continuous, then T has a unique fixed point.

PROOF : Set $y = Tx$, and assume that $x \ne Tx$, to get

$$d(Tx, T^2x) \le \alpha d(Tx, T^2x) + \beta d(x, Tx),$$

which implies that

$$d(Tx, T^2x) \le \frac{\beta}{1-\alpha}\, d(x, Tx).$$

Therefore the hypotheses of Proposition 1 are satisfied and T has a fixed point. As in Jaggi[19], the fixed point is unique.

Corollary 6 (Khan[23], Theorem 1) — Let (X, d) be a complete metric space, T a selfmap of X satisfying

$$d(Tx, Ty) \le k\, \frac{d(x, Tx)\, d(x, Ty) + d(y, Ty)\, d(y, Tx)}{d(x, Ty) + d(y + Tx)} \qquad \text{... (2)}$$

for all $x, y \in X$, where $0 \le k < 1$. Then T has a unique fixed point.

PROOF : One needs to add the hypothesis that (2) is true for all x, y such that $d(x, Ty) + d(y, Tx) \ne 0$.

Substituting $y = Tx$ into (2) yields

$$d(Tx, T^2x) \le k\, \frac{d(x, Tx)\, d(x, T^2x)}{d(x, T^2x)} = k\, d(x, Tx)$$

and (1) is satisfied.

Therefore $\{T^nx\}$ converges to a point z in X. Assume $z \ne Tz$. Using (2) with $x = T^{n-1}x, y = z$,

$$d(T^nx, Tz) \le k\, \frac{d(T^{n-1}x, T^nx)\, d(T^{n-1}x, Tz) + d(z, Tz)\, d(z, T^nx)}{d(T^{n-1}x, Tz) + d(z, T^nx)}.$$

Taking the limit as $n \to \infty$ yields $d(z, Tz) \le 0$, a contradiction. Therefore z is a fixed point of T. Uniqueness follows from (2).

Corollary 7 (Jain and Dixit[21], Theorem 1) — Let f be a selfmap of a metric space satisfying

$$\begin{aligned} d(fx, fy) \le{} & \frac{\alpha_1 d(x, fx)\, d(y, fy)}{d(x, y)} + \frac{\alpha_2 d(x, fy)\, d(y, fx)}{d(x, y)} + \frac{\alpha_3 d(y, fx)\, d(y, fy)}{d(x, y)} \\ & + \frac{\alpha_4 d(x, fx)\, d(y, fx)}{d(x, y)} + \beta_1\, d(x, y) + \beta_2\, d(x, fx) + \beta_3\, d(y, fy) \\ & + \beta_4\, d(x, fy) + \beta_5\, d(y, fx) \end{aligned} \qquad \text{... (3)}$$

for all $x, y \in X$, $x \neq y$, $fx \neq fy$, $\alpha_i, \beta_j \geq 0$, $\alpha_1 + 2\alpha_3 + 2\alpha_4 + \beta_1 + \beta_2 + \beta_3 + 2\beta_5 < 1$, $\alpha_2 + \beta_1 + \beta_4 + \beta_5 < 1$. If there exist an $x_0 \in X$ such that X is (x_0, f)-orbitally complete and f is x_0-orbitally continuous, then f has a unique fixed point.

PROOF : Assume that $x_n := f^n x_0$ is such that $x_n \neq x_{n+1}$ for each n. For, otherwise, f has a fixed point. With $y = fx$,

$$d(fx, f^2x) \leq \frac{\alpha_1\, d(x, fx)\, d(fx, f^2x)}{d(x, fx)}$$

$$+ 0 + 0 + 0 + (\beta_1 + \beta_2)\, d(x, fx)$$

$$+ \beta_3 d(fx, f^2x) + \beta_4\, d(x, f^2x) + 0$$

$$\leq \frac{\beta_1 + \beta_2 + \beta_4}{1 - \alpha_1 - \beta_3 - \beta_4}\, d(x, fx)$$

and f satisfies (1). The condition of orbital completeness implies that $\{f^n x_0\}$ is convergent. Call the limit z. Since f is x_0-orbitally continuous, $z = \lim f^{n+1} x_0 = f(\lim f^n x_0) = fz$. Uniqueness follows as in Jain and Dixit[21].

Corollary 8 (Sharma and Bajaj[40], Theorem 1) — Let T be a selfmap of a complete metric space (X, d) satisfying

$$d(Tx, Ty) \leq \frac{\beta d(x, Tx)\, d(x, Ty)}{d(x, Tx) + d(x, Ty)} \qquad \ldots (4)$$

for all $x, y \in X$ such that $d(x, Tx) + d(x, Ty) \neq 0$, $0 < \beta < 1/2$. Then T has a unique fixed point.

PROOF : Set $y = Tx$ in (4) to obtain

$$d(Tx, T^2x) \leq \frac{\beta d(x, Tx)\, d(x, T^2x)}{d(x, Tx) + d(x, T^2x)}$$

$$\leq \beta d(x, Tx).$$

Thus (1) is satisfied. Since X is complete, $\{T^n x\}$ converges to a point z. Assume that $z \neq Tz$.

Setting $x = T^{n-1}x, y = z$ in (4) we have

$$d(T^n x, Tz) \leq \frac{\beta d(T^{n-1}x, T^n x)\, d(T^{n-1}x, Tz)}{d(T^{n-1}x, T^n x) + d(T^{n-1}x, Tz)}.$$

Taking the limit as $n \to \infty$ yields $d(z, Tz) \leq 0$, a contradiction. Therefore $z = Tz$. Uniqueness follows from (4).

Corollary 9 (Das[9], Theorem 1) — Let T be a selfmap of a complete metric space (X, ρ) such that, for some positive integer m and, for some $\alpha_i, \beta_j > 0$ with

$\alpha_1 + \alpha_2 + \alpha_3 + \sum_{i=1}^{5} \beta_i < 1$,

$$\rho(T^m x, T^m y) \le \frac{\alpha_1 \rho(x, T^m x)\, \rho(y, T^m y)}{\rho(x, y)} + \frac{\alpha_2 \rho(x, T^m x)\, \rho(y, T^m x)}{\rho(T^m x, T^m y)}$$

$$+ \frac{\alpha_3 \rho(x, T^m y)\, \rho(y, T^m y)}{\rho(T^m x, T^m y)} + \beta\rho(x, y) + \beta_2 \rho(x, T^m x)$$

$$+ \beta_3 \rho(y, T^m y) + \beta_4 \sigma(x, T^m y) + \beta_5 \rho(y, T^m x) \qquad \ldots (5)$$

for all $x, y \in X$ such that $x \neq y$, $T^m x \neq T^m y$. If, for some positive integer p, T^{mp} is continuous, then T has a unique fixed point.

PROOF : Define $S = T^m$. Then (5) takes the form

$$\rho(Sx, Sy) \le \frac{\alpha_1 \rho(x, Sx)\, \rho(y, Sy)}{\rho(x, y)} + \frac{\alpha_2 \rho(x, Sx)\, \rho(y, Sx)}{\rho(Sx, Sy)}$$

$$+ \frac{\alpha_3 \rho(x, Sy)\, \rho(y, Sy)}{\rho(Sx, Sy)} + \beta_1 \rho(x, y) + \beta_2 \rho(x, Sx) + \beta_3 \rho(y, Sy)$$

$$+ \beta_4\, \rho(x, Sy) + \beta_5\, \rho(y, Sx). \qquad \ldots (6)$$

Interchanging x and y in (6) and then adding the result to (6) yields

$$\rho(Sx, Sy) \le \frac{\gamma_1\, \rho(x, Sx)\, \rho(y, Sy)}{\rho(x, y)} + \frac{\gamma_2\, [\rho(x, Sx)\, \rho(y, Sx) + \rho(x, Sy)\, \rho(y, Sy)]}{\rho(Sx, Sy)}$$

$$+ \gamma_3\, \rho(x, y) + \gamma_4\, [\rho(x, Sx) + \rho(y, Sy)]$$

$$+ \gamma_5\, [\rho(x, Sy) + \rho(y, Sx)], \qquad \ldots (7)$$

where $\gamma_1 = \alpha_1$, $\gamma_2 = (\alpha_2 + \alpha_3)/2$, $\gamma_3 = \beta_1$, $\gamma_4 = (\beta_2 + \beta_3)/2$, and $\gamma_5 = (\beta_4 + \beta_5)/2$. Note that $\gamma_1 + 2\gamma_2 + \gamma_3 + 2\gamma_4 + 2\gamma_5 = \alpha_1 + \alpha_2 + \alpha_3 + \sum_{i=1}^{5} \beta_i < 1.$

Assume that $S^{n+1} x \neq S^n x$ for each $n \ge 0$. Set $y = Sx$ in (7) to obtain

$$\rho(Sx, S^2x) \le \frac{\gamma_1\, \rho(x, Sx)\, \rho(Sx, S^2x)}{\rho(x, Sx)} + \frac{\gamma_2\, \rho(x, S^2x)\, \rho(Sx, S^2x)}{\rho(Sx, S^2x)} + \gamma_3\, \rho(x, Sx)$$

$$+ \gamma_4\, [\rho(x, Sx) + \rho(Sx, S^2x)] + \gamma_5\, (x, S^2x),$$

which implies that

$$\rho(Sx, S^2x) \le \frac{\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5}{1 - \gamma_1 - \gamma_2 - \gamma_4 - \gamma_5}\, \rho(x, S^2x),$$

and (1) is satisfied. Since the space is complete, $\{S^n x\}$ converges. Call the limit z. Since T^{mp} is continuous, S^p is continuous. Therefore $z = \lim S^{n+p} x = S^p (\lim S^n x) = S^p z$ and z is a fixed point of S^p. Let u denote the smallest positive integer such that $S^u z = z$. We wish to show that $u = 1$. Suppose that $u > 1$. Then

$$0 < \rho(z, Sz) = \rho(S^u z, S^{u+1} z) = \rho(S(S^{u-1} z), S(S^u z))$$

$$\leq \frac{\gamma_1 \rho(S^{u-1} z, S^u z)\, \rho(S^u z, S^{u+1} z)}{\rho(S^{u-1} z, S^u z)} + \frac{\gamma_2 \rho(S^{u-1} z, S^{u+1} z)\, \rho(S^u z, S^{u+1} z)}{\rho(S^u z, S^{u+1} z)}$$

$$+ \gamma_3 \rho(S^{u-1} z, S^u z) + \gamma_4 [\rho(S^{u-1} z, S^u z) + \rho(S^u z, S^{u+1} z)]$$

$$+ \gamma_5 (S^{u-1} z, S^{u+1} z),$$

which implies that

$$\rho(z, Sz) = \rho(S^n z, S^{n+1} z) \leq \frac{\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5}{1 - \gamma_1 - \gamma_2 - \gamma_4 - \gamma_5} \rho(S^{u-1} z, S^u z)$$

$$\leq r^2 \rho(S^{u-2} z, S^{u-1} z) \leq \ldots \leq r^u \rho(z, Sz),$$

a contradiction, where $r = (\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5)/(1 - \gamma_1 - \gamma_2 - \gamma_4 - \gamma_5)$. Therefore z is a fixed point of S.

It follows from (7) that z is unique. Thus z is the unique fixed point of T^m. But $z = T^m z$ implies that $Tz = T^{m+1} z$ and Tz is also a fixed point of T^m. By uniqueness, $z = Tz$. Again (7) forces z to be unique.

Corollary 10 (Pachpatte[27], Theorem 1) — Let T be an orbitally continuous selfmap of an orbitally complete space X satisfying

$$d(Tx, Ty) \leq q \max \left\{ d(x, y), \frac{d(x, Tx)\, d(y, Ty)}{d(x, y)}, \frac{d(x, Ty)\, d(y, Tx)}{d(x, y)}, \frac{d(x, Tx)\, d(x, Ty)}{2d(x, y)} \right\}$$

for all distinct $x, y \in X$ and $q \in (0, 1)$. Then, for each $x \in X$, $\{T^n x\}$ converges to the unique fixed point of T.

PROOF : Setting $y = Tx$ gives

$$d(Tx, T^2 x) \leq q \max \left\{ d(x, Tx), \frac{d(x, Tx)\, d(Tx, T^2 x)}{d(x, Tx)}, 0, \frac{d(x, Tx)\, d(x, T^2 x)}{2d(x, Tx)} \right\}$$

which implies that $d(Tx, T^2 x) \leq q d(x, Tx)$, and T satisfies (1). Thus $\lim T^n x = z$, say. Since T is orbitally continuous, $z = \lim T^{n+1} x = T(\lim T^n x) = Tz$. The contractive definition implies uniqueness of z.

Corollary 11 (Pachpatte[27], Theorem 2) — Let T be an orbitally continuous selfmap of an orbitally complete space X satisfying

$$\min \left\{ d(Tx, Ty), d(x, Tx), d(y, Ty), \frac{d(x, Tx)\, d(y, Ty)}{d(x, y)} \right\}$$

$$- \min \left\{ \frac{d(x, Ty)\, d(y, Tx)}{d(x, y)}, \frac{d(x, Tx)\, d(x, Ty)}{d(x, y)} \right\} \leq q d(x, y)$$

for all distinct $x, y \in X$ and $q \in (0, 1)$. Then, for each $x \in X$, $\{T^n x\}$ converges to a fixed point of T.

PROOF : Substituting into the contractive definition, with $y = Tx$, we obtain

$$\min\left\{ d(Tx, T^2x), d(x, Tx), d(Tx, T^2x), \frac{d(x, Tx)\, d(Tx, T^2x)}{d(x, Tx)} \right\} - \min\left\{ 0, \frac{d(x, Tx)\, d(x, T^2x)}{d(x, Tx)} \right\} \le q\, d(x, Tx),$$

or $\qquad \min\{d(Tx, T^2x), d(x, Tx)\} \le q\, d(x, Tx),$

which implies that $d(Tx, T^2x) \le qd(x, Tx)$, and (1) is satisfied. Call the limit z. Since T is orbitally continuous, $z = \lim T^{n+1}x = T(\lim T^n x) = Tz$.

The following result is an application of Rhoades[38].

Theorem 2 of Khan[22] — Let f be a surjective map of a complete metric space (X, d) satisfying

$$d(x, y) \le k \left\{ \frac{d(fx, x)\, d(fx, y) + d(fy, y)\, d(fy, x)}{d(fx, y) + d(fy, x)} \right\}$$

for all $x, y \in X$ and $0 \le k < 1$. Then f has a unique fixed point in X.

PROOF : The hypothesis that x and y are such that $d(fx, y) + d(fy, x) \ne 0$ needs to be added. With this additional assumption, set $y = fx$ to obtain

$$d(x, fx) \le k \left\{ \frac{0 + d(f^2x, fx)\, d(f^2x, x)}{0 + d(f^2x, x)} \right\}$$

$$= kd(fx, f^2x).$$

Now apply the Theorem of Rhoades[38] to obtain the result that f has a unique fixed point.

We now provide an alternative proof to

Theorem 2 of Basu[2] — Let T be a selfmap of a compact metric space X such that

(a) $$d(Tx, Ty) < \frac{a_1 d(x, Tx)\, d(y, Ty)}{d(x, Ty) + d(y, Tx) + d(x, y)} + \frac{a_2 d(x, Tx)\, d(x, Ty)}{d(x, Tx) + d(y, Ty)} + \frac{a_3 d(x, Tx)\, d(y, Ty)}{d(y, Tx) + d(x, Ty)} + \frac{a_4 d(y, Ty)\, d(y, Tx)}{d(y, Ty) + d(x, Tx)} + a_5\, d(x, y) \qquad \text{... (8)}$$

for all $x, y \in X, x \ne y$, where each $a_i > 0$ and $2a_1 + a_2 + a_3 + a_4 + 2a_5 = 2$.

(b) There exist a u in X such that $f(u) := \inf\{f(x) := d(x, Tx), x \in X\}$ exists.

Then T has a unique fixed point in X.

PROOF : It is necessary to add the hypotheses that x and y are chosen so that none of the denominators in (8) vanish. Interchange x and y in (8). Then add that result to (8) to obtain

$$d(Tx, Ty) < \frac{a_1 d(x, Tx)\, d(y, Ty)}{d(x, Ty) + d(y, Tx) + d(x, y)}$$

$$+ \left(\frac{a_2 + a_4}{2}\right)\left[\frac{d(x, Tx)\, d(x, Ty) + d(y, Ty)\, d(y, Tx)}{d(x, Tx) + d(y, Ty)}\right]$$

$$+ \frac{a_3\,[d(x, Tx)\, d(y, Tx) + d(y, Ty)\, d(x, Ty)]}{2[d(y, Tx) + d(x, Ty)]} + a_5\, d(x, y). \quad \ldots (9)$$

Assume that $T^{n+1}x \neq T^n x$ for each $n \geq 0$. Set $y = Tx$ in (9) to obtain

$$d(Tx, T^2x) < \frac{a_1 d(x, Tx)\, d(Tx, T^2x)}{d(x, T^2x) + d(x, Tx)} + \left(\frac{a_2 + a_4}{2}\right)\left[\frac{d(x, Tx)\, d(x, T^2x)}{d(x, Tx) + d(Tx, T^2x)}\right]$$

$$+ \frac{a_3\, d(Tx, T^2x)\, d(x, T^2x)}{2\, d(x, T^2x)} + a_5\, d(x, Tx),$$

which implies that

$$d(Tx, T^2x) < \frac{a_1 + (a_2 + a_4)/2 + a_5}{1 - a_3/2}\, d(x, Tx).$$

Note that $a_1 + (a_2 + a_3 + a_4)/2 + a_5 = 1$. Therefore $d(Tx, T^2x) < d(x, Tx)$, which implies that $\{d(T^n x, T^{n+1}x)\}$ is a strictly decreasing positive sequence. Condition (b) guarantees that the limit exists and is $f(u)$.

Since X is compact, if $f(u) = 0$, then T has a fixed point. Suppose $f(u) > 0$. Then $u \neq Tu$ and $Tu \neq T^2u$, and hence $1/d(u, T^2u) > 0$.

From (9),

$$d(Tu, T^2u) < \frac{a_1 d(u, Tu)\, d(Tu, T^2u)}{d(u, T^2u) + d(u, Tu)} + \left(\frac{a_2 + a_4}{2}\right)\left[\frac{d(u, Tu)\, d(u, T^2u)}{d(u, Tu) + d(Tu, T^2u)}\right]$$

$$+ \frac{a_3\, d(Tu, T^2u)\, d(u, T^2u)}{2d(u, T^2u)} + a_5\, d(u, Tu),$$

which implies that

$$d(Tu, T^2u) < \frac{a_1 + (a_2 + a_4)/2 + a_5}{1 - a_3/2}\, d(u, Tu),$$

a contradiction to $f(u)$. Therefore u is a fixed point of T. Condition (9) forces uniqueness. Note that the conditions that each $a_i > 0$ can be weakened to each $a_i \geq 0$ and $a_5 > 0$.

Using these same ideas we now provide an alternative proof of :

Theorem 1 of Das[8] — Let T be a selfmap of a compact metric space (X, ρ) such that, for some positive integer m, T^m is continuous and, for each $x, y \in X$, $x \neq y$, $T^m x \neq T^m y$,

$$\rho(T^m x, T^m y) < \frac{\alpha_1 \rho(x, T^m x)\, \rho(y, T^m y)}{\rho(x, y)} + \frac{\alpha_2 \rho(x, T^m x)\, \rho(y, T^m x)}{\rho(T^m x, T^m y)}$$

$$+ \frac{\alpha_3 \rho(x, T^m x)\, \rho(y, T^m y)}{\rho(T^m x, T^m y)} + \beta_1 \rho(x, y) + \beta_2 \rho(x, T^m x)$$

$$+ \beta_3 \rho(y, T^m y) + \beta_4 \rho(x, T^m y) + \beta_5 \rho(y, T^m x), \qquad \ldots (10)$$

where $\alpha_1 + \alpha_3 + \beta_3 + \beta_4 < 1$, $\alpha_3 \geq 0$, $\alpha_1 + 2\alpha_3 + \beta_1 + \beta_2 + \beta_3 + 2\beta_4 = 1$, $\beta_1 + \beta_4 + \beta_5 < 1$. Then T has a unique fixed point.

PROOF : For simplicity of notation, define $S = T^m$. Without loss of generality we may assume that $S^{n-1} x \neq S^n x$ for each $n > 0$, since otherwise S has a fixed point. Set $y = Sx$ in (10) to obtain

$$\rho(Sx, Sy) < \frac{\alpha_1 \rho(x, Sx)\, \rho(Sx, S^2x)}{\rho(x, Sx)} + \frac{\alpha_3 \rho(x, S^2x)\, \rho(Sx, S^2x)}{\rho(Sx, S^2x)}$$

$$+ \beta_1 \rho(x, Sx) + \beta_2 \rho(x, Sx) + \beta_3 \rho(Sx, S^2x) + \beta_4 \rho(x, S^2x),$$

which implies that

$$\rho(Sx, Sy) < \frac{\alpha_3 + \beta_1 + \beta_2 + \beta_4}{1 - \alpha_1 - \alpha_3 - \beta_3 - \beta_4} \rho(x, Sx).$$

Therefore $\{\rho(S^n x, S^{n+1} x)\}$ is a strictly decreasing positive sequence. Since X is compact $f(u) := \inf \{\rho(x, Sx) : x \in X\}$ exists and is the limit of $\rho(S^n x, S^{n+1} x)$.

If $f(u) = 0$, since X is compact, S has a fixed point. Suppose $f(u) > 0$. Then $u \neq Su$ and $Su \neq S^2u$. Then, from (10),

$$\rho(Su, S^2u) < \alpha_1 \rho(Su, S^2u) + \alpha_3 \rho(u, S^2u) + \beta_1 \rho(u, Su) + \beta_2 \rho(u, Su)$$

$$+ \beta_3 \rho(Su, S^2u) + \beta_4 \rho(u, S^2u),$$

which implies that

$$\rho(Su, S^2u) < \frac{\alpha_3 + \beta_1 + \beta_2 + \beta_4}{1 - \alpha_1 - \alpha_3 - \beta_3 - \beta_4} \rho(u, Su) < \rho(u, Su)$$

contradicting the definition of $f(u)$.

Condition (10) forces u to be unique. Thus u is the unique fixed point of T^m. But $T^m u = u$ implies that $T^{m+1} u = Tu$ and Tu is also a fixed point of T^m. Therefore $u = Tu$. Again condition (10) forces u to be unique. The continuity of T^m is not needed.

For completeness we point out an error in the following result.

Theorem 6 of Sharma and Yuel[42] — Let T be a selfmap of a complete metric space (X, d) satisfying

$$d(Tx, Ty) \leq \alpha \left\{ \frac{d(x, Tx)\, d(y, Ty)}{d(x, y)} \right\} + \beta \{d(x, Tx) + d(y, Ty)\}$$

$$+ \gamma\{d(x, Ty) + d(y, Tx)\} + \delta d(x, y) \quad \ldots (11)$$

for all $x, y \in X$, where $0 \leq (\beta+\gamma+\delta)/(1-\alpha-\beta-\gamma) < 1$, $\beta+\gamma<1$, $2\gamma+\delta<1$, $\delta \geq 0$. Then T has a unique fixed point.

Note that the hypothesis needs to be amended to state that (11) is true for all $x \neq y$.

In the proof of the theorem the authors show that $\{T^n x\}$ is Cauchy, hence convergent to a point z.

Using the triangular inequality and (11), they obtain

$$\begin{aligned} d(z, Tz) &\leq d(z, T^n x) + d(T^n x, Tz) \\ &\leq d(z, T^n x) + \alpha \left\{ \frac{d(T^{n-1}x, T^n x)\, d(z, Tz)}{d(T^{n-1}x, z)} \right\} \\ &\quad + \beta\{d(T^{n-1}x, T^n x) + d(z, Tz)\} \\ &\quad + \gamma\{d(T^{n-1}x, Tz) + d(z, T^n x)\} + \delta d(T^{n-1}x, z), \end{aligned}$$

which is not valid, if $T^{n-1}x = z$ for any n. They then take the limit of the above inequality as $n \to \infty$. Unfortunately,

$$\lim \frac{d(T^{n-1}x, T^n x)}{d(T^{n-1}x, z)}$$

is an indeterminate form.

However, if one adds the hypothesis that T is orbitally continuous, then T has a unique fixed point, without assuming the condition $\beta+\gamma<1$.

We now establish the analog of Proposition 1 for Hausdorff spaces. Let $F : X \times X \to \mathbb{R}^+$, F continuous and such that $F(x, y) = 0$ implies that $x = y$.

Proposition 2 — Let $0 \leq h < 1$. Suppose there exists a point x in X such that

(A) $x_n := T^n x$ has a convergent subsequence with limit $z \in X$, and for this x,

(B) $F(Ty, T^2y) \leq hF(y, Ty)$ for each $y \in O(x)$, and

(C) T is orbitally continuous.

Then, for this x, $\lim T^n x = z$.

PROOF : From Condition (B), with $y = T^{n-1}x$,

$$F(T^n x, T^{n+1}x) \leq hF(T^{n-1}x, T^n x) \leq h^n F(x, Tx).$$

Therefore $\{F(T^n x, T^{n+1}x)\}$ is a monotone decreasing non-negative sequence with limit zero.

Using (A) and (C),

$$\begin{aligned} F(z, Tz) &= F(\lim_k T^{n_k} x, T(\lim_k T^{n_k} x)) = F(\lim_k T^{n_k} x, \lim_k T^{n_k+1} x) \\ &= \lim_k F(T^{n_k} x, T^{n_k} x) = 0, \end{aligned}$$

and $z = Tz$.

Corollary 12 — Let T a selfmap of a Hausdorff space X, $F : X \times X \to \mathbb{R}^+$, F continuous and such that $F(x, y) = 0$ implies that $x = y$ and such that, for each pair of points x, y in X

$$F(Tx, Ty) \leq \frac{\alpha F(y, Ty)\,[1 + F(x, Tx)]}{1 + F(x, y)} + \beta[F(x, Tx) + F(y, Ty)] + \gamma F(x, y), \quad \ldots (12)$$

where $\alpha, \beta, \gamma \geq 0$ and such are that $\alpha + 2\beta + \gamma < 1$. If, for some $x_0 \in X$, $\{T^n x_0\}$ has a convergent subsequence with limit $z \in X$, and T is x_0 orbitally continuous, then z is a fixed point of T in X.

PROOF : For notational simplicity set $x = x_0$, $y = Tx$ in (12) to obtain

$$F(Tx, T^2x) \leq \alpha F(Tx, T^2x) + \beta[F(x, Tx) + F(Tx, T^2x)] + \gamma F(x, Tx),$$

which implies that

$$F(Tx, T^2x) \leq \frac{\beta + \gamma}{1 - \alpha - \beta} F(x, Tx),$$

and condition (B) of Proposition 2 is satisfied. Since conditions (A) and (C) are also true, $\{T^n x_0\}$ converges to z.

Now set $x = T^{n-1}x$, $y = T^n x$ in (12) to get

$$F(T^n x, T^{n+1}x) \leq \frac{\alpha F(T^n x, T^{n+1}x)\,[1 + F(T^{n-1}x, T^n x)]}{1 + F(T^{n-1}x, T^n x)} + \beta[F(T^{n-1}x, T^n x) + F(T^n x, T^{n+1}x)] + \gamma F(T^{n-1}x, T^n x).$$

Taking the limit as $n \to \infty$ yields

$$F(z, z) \leq \frac{\alpha F(z, z)\,[1 + F(z, z)]}{1 + F(z, z)} + 2\beta F(z, z) + \gamma F(z, z) = (\alpha + 2\beta + \gamma)\, F(z, z),$$

which implies that $F(z, z) = 0$.

Now set $x = T^{n-1}x, y = z$ in (12) to obtain

$$F(T^n x, Tz) \leq \frac{\alpha F(z, Tz)\,[1 + F(T^{n-1}x, T^n x)]}{1 + F(T^{n-1}x, z)} + \beta[F(T^{n-1}x, T^n x) + F(z, Tz)] + \gamma F(T^{n-1}x, z).$$

Taking the limit as $n \to \infty$, and using the fact that $F(z, z) = 0$, yields $F(z, Tz) \leq (\alpha + \beta)\, F(z, Tz)$, which implies that z is a fixed point of T.

To prove uniqueness, suppose that w is also a fixed point of T. Then, with $x = z$, $y = w$ in (12),

$$F(z, w) = F(Tz, Tw) \leq \frac{\alpha F(w, Tw)\,[1 + F(z, Tz)]}{1 + F(z, w)} + \beta[F(z, Tz) + F(w, Tw)] + \gamma F(z, w),$$

or $F(z, w) \leq \gamma F(z, w)$, which implies that $z = w$.

Corollary 13 (Chatterjee[5], Theorem 1) — Let T be a continuous selfmap of a Hausdorff space X, $F : X \times X \to \mathbb{R}^+$, F continuous and such that $F(x, y) = 0$ implies that $x = y$ and such that, for each pair of distinct points x, y in X,

$$F(Tx, Ty) \leq \frac{\alpha F(y, Ty)\,[1 + F(x, Tx)]}{1 + F(x, y)} + \beta F(x, y),$$

where $\alpha, \beta \geq 0$ are constants such that $\alpha + \beta < 1$. If, for some $x_0 \in X$ $\{T^n x_0\}$ has a convergent subsequence with limit z in X, then z is a fixed point of T.

PROOF : In Corollary 12, set $\beta = 0$, $\gamma = \beta$.

Note that x and y need not be distinct, and that the fixed point is unique.

Corollary 14 (Chugh and Rani[7], Theorem 3) — Let T be a continuous selfmap of a Hausdorff space X, $F : X \times X \to \mathbb{R}^+$, F continuous and such that

(a) $F(x, y) = 0$ implies that $x = y$,

(b) $F(Tx, Ty) \leq \dfrac{\alpha F(y, Ty)\,[1 + F(x, Tx)]}{1 + F(x, y)} + \beta F(x, y)$

for all $x \neq y$, $\alpha, \beta > 0$, $\alpha + \beta < 1$,
and

(c) $F(x, y) \geq \dfrac{1 + F(x, x)}{1 + F(x, y)} F(y, y)$, for each x, y in X.

If, for some $x_0 \in X$, the sequence $\{T^n x_0\}$ has a convergent subsequence with limit z in X, then T has a unique fixed point.

PROOF : The result follows from Corollary 12, without the assumption that $x \neq y$, or condition (c).

Theorem 2 of Popa[34] is similar to Corollary 14 only with (b) replaced by

(b′) $F(Tx, Ty) \leq \dfrac{aF(x, Tx)\,F(y, Ty)}{F(x, y)} + bF(x, y)$

for all $x \neq y$, where a, $b \geq 0$ with $a + b < 1$.

This result is also a special case of Proposition 2, and does not require condition (c).

We now obtain a generalization and extension of the Theorem of Iséki *et al.*[18] for two maps. The proof uses one of the general principles developed by Park[32], which is quoted here for completeness.

Let ω denote the set of non-negative integers.

Theorem P1 (Theorem 3.1 of Park[32]) — Let S and T be selfmaps of a metric space (X, d). If there exists a sequence $\{x_i \mid i \in \omega\} \subset X$, where $x_{2n+1} := Sx_{2n}$, $x_{2n+2} := Tx_{2n+1}$, such that $\overline{\{x_i\}}$ is complete, and if there exists a $\lambda \in [0, 1)$ such that

$$d(Sx, Ty) \leq \lambda d(x, y) \qquad \text{... (13)}$$

for each distinct x, $y \in \overline{\{x_i\}}$ satisfying either $x = Ty$ or $y = Sx$, then either :

(i) S or T has a fixed point in $\{x_i\}$, or

(ii) $\{x_i\}$ converges to some $z \in X$ and $d(x_i, z) \leq \lambda^i d(x_0, x_1)/(1-\lambda)$ for $i > 0$.

Further, if either S or T is continuous at z and (13) holds for any distinct x, $y \in \overline{\{x_i\}}$, then z is a common fixed point of S and T.

Corollary 15 — Let S and T be two selfmaps of a complete metric space satisfying

$$d(Sx, Ty) \leq \frac{\alpha d(y, Ty)\,[1 + d(x, Sx)]}{[1 + d(x, y)]} + \beta[d(x, Sx) + d(y, Ty)]$$

$$+ \gamma[d(x, Ty) + d(y, Sx)] + \delta d(x, y) \quad \ldots (14)$$

for all x, y in X, where $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + 2\beta + 2\gamma + \delta < 1$. Then S and T have a unique common fixed point in X.

PROOF : In (14) set $y = Sx$ to obtain

$$d(Sx, TSx) \leq \alpha d(Sx, TSx) + \beta[d(x, Sx) + d(Sx, TSx)]$$

$$+ \gamma d(x, TSx) + \delta d(x, Sx)$$

and hence

$$d(Sx, TSx) \leq \frac{(\beta + \gamma + \delta)}{(1 - \alpha - \beta - \gamma)}\, d(x, Sx),$$

and (13) is satisfied. By Theorem P1, either condition (i) or (ii) holds.

Lemma 1 — Under the conditions of Corollary 15, if either S or T has a fixed point p, then p is the unique common fixed point of S and T.

PROOF : Suppose that p is a fixed point of S. Setting $x = y = p$ in (14), yields $d(p, Tp) = d(Sp, Tp) \leq (\alpha + \beta + \gamma)\, d(p, Tp)$, which implies that $p = Tp$. The proof for p a fixed point of T is similar.

Suppose that p and q are fixed points of S and T. Then, substituting into (14), $d(p, q) = d(Sp, Tq) \leq [2\gamma + \delta]\, d(p, q)$, which implies that $p = q$.

Returning to the proof of Corollary 15, suppose that condition (ii) of Theorem P1 is satisfied. Using (14) with $x = x_n, y = z$,

$$d(Sx_{2n}, Tz) \leq \frac{\alpha d(z, Tz)\,[1 + d(x_{2n}, Sx_{2n})]}{1 + d(x_{2n}, z)} + \beta[d(x_{2n}, Sx_{2n}) + d(z, Tz)]$$

$$+ \gamma[d(x_{2n}, Tz) + d(z, Sx_{2n})] + \delta d(x_{2n}, z).$$

Taking the limit as $n \to \infty$ yields $d(z, Tz) \leq (\alpha + \beta + \gamma)\, d(z, Tz)$, and p is a fixed point of T. The result now follows from Lemma 1.

Corollary 16 (Iséki *et al.*[18], Theorem) — Let T be a continuous selfmap of a complete metric space X satisfying (14) with $S = T$. Then T has a unique fixed point in X.

PROOF : In Corollary 15, set $S = T$. Note that the hypothesis that T be continuous is not needed.

Corollary 17 (Bhola and Sharma[4], Theorem) — Let S, T, P be three selfmaps of a complete metric space (X, d) satisfying

$$d(SPx, TPy) \leq \frac{\alpha d(y, TPy)\,[1 + d(x, SPx)]}{1 + d(x, y)} + \beta[d(x, SPx) + d(y, TPy)$$

$$+ \gamma[d(x, TPy) + d(y, SPx)] + \delta d(x, y)$$

for all $x, y \in X$, where $0 \leq (\beta + \gamma + \delta)/(1 - \alpha - \beta - \gamma) < 1$, $\beta + \gamma < 1$, $2\gamma + \delta < 1$, $\gamma \geq 0$. Further, assume that $SP = PS$ and $TP = PT$. Then S, P and T have a unique common fixed point in X.

PROOF : If one defines $A = SP$, $B = TP$, then A and B have a unique common fixed point z by Corollary 15; i.e., SP and TP have a unique common fixed point. Since S commutes with P, $z = SPz$ implies that $Pz = PSPz = SPPz$, and Pz is also a fixed point of SP. By Lemma 1, Pz is also a fixed point of TP. By uniqueness, $z = Pz$. A similar argument shows that $Tz = z$. Since $z = SPz = Sz$, z is also a fixed point of S, without using the condition that $\beta + \gamma < 1$. Uniqueness follows from the contractive condition.

Corollary 18 (Chatterjee[5], Theorem 2) — Let (X, d) be a complete metric space, T a selfmap of X satisfying

$$d(T^{p+1}x, T^{p+2}y) \leq \frac{\alpha d(T^{p+1}y, T^{p+2}y)\,[1 + d(T^{p}x, T^{p+1}x)]}{1 + d(T^{p}x, T^{p+1}y)}$$

$$+ \beta d(T^{p}x, T^{p+1}y)$$

for each x, y in X and for nonnegative constants α, β with $\alpha + \beta < 1$, and for any nonnegative integer p. Then T has a unique fixed point.

PROOF : The above definition is a special case of (14) with $\beta = \gamma = 0$, $\delta = \beta$, $S = T$, evaluated at $x = T^{p}x$, $y = T^{p+1}y$.

Corollary 19 (Sharma and Bajaj[40], Theorem 2) — Let S, T be selfmaps of a complete metric space (X, d) satisfying

$$d(Sx, Ty) \leq \beta \frac{d(x, Sx)\, d(x, Ty) + d(y, Sx)\, d(y, Ty)}{d(x, Sx) + d(x, Ty)} \quad \ldots (15)$$

for all $x, y \in X$ such that $d(x, Sx) + d(x, Ty) \neq 0$, $0 < \beta < 1/2$. Then S and T have a unique common fixed point.

Lemma 2 — Let S, T satisfy the hypotheses of Corollary 19. Then p is a fixed point of S iff it is a fixed point of T.

PROOF : Suppose that $p = Sp$ and $p \neq Tp$. Then, from (15),

$$d(p, Tp) = d(Sp, Tp) \leq \beta \frac{0 + 0}{d(p, Tp)} = 0,$$

a contradiction. Therefore p is also a fixed point of T.

Similarly, any fixed point of T is also a fixed point of S.

PROOF OF COROLLARY 19 : Applying Theorem P1, either (i) S or T has a fixed point, or (ii) $\{x_n\}$ converges to a point z. If (i) is true, then we have a common fixed point by Lemma 2. Assume that (ii) is true. Without loss of generality we may assume that $x_n \neq x_{n+1}$ for each $n \geq 0$, since, otherwise, there is a common fixed point of S and T.

$$d(Sx_{2n}, Tz) \leq \beta \frac{d(x_{2n}, Sx_{2n})\, d(x_{2n}, Tz) + d(z, Sx_{2n})\, d(z, Tz)}{d(x_{2n}, Sx_{2n}) + d(x_{2n}, Tz)}.$$

If $z = Tz$, then we are finished by Lemma 2. Assume $z \neq Tz$. Then, taking the limit as $n \to \infty$ yields

$$d(z, Tz) \leq \beta \frac{0 + 0}{d(z, Tz)} = 0,$$

a contradiction. Therefore $z = Tz$. From Lemma 2, $z = Sz$. Uniqueness follows from (15).

Corollary 20 (Fisher[12], Theorem 3) — Let S, T be selfmaps of a complete metric space (X, d) such that

$$d(Sx, Ty) \leq \frac{cd(x, Sx)\, d(y, Ty) + bd(x, Ty)\, d(y, Sx)}{d(x, Sx) + d(y, Ty)} \qquad \text{... (16)}$$

for each $x, y \in X$ such that $d(x, Sx) + d(y, Ty) \neq 0$ and $d(Sx, Ty) = 0$, otherwise, where $b \geq 0$, $1 < c < 2$. Then each of S and T has a unique fixed point and these points coincide.

Lemma 3 — Let S, T satisfy the hypotheses of Corollary 20. Then p is a fixed point of S iff it is a fixed point of T.

PROOF : Suppose $p = Sp$ and $p \neq Tp$. Then, from (16) with $x = y = p$,

$$d(Sp, Tp) \leq \frac{0 + 0}{d(p, Tp)} = 0,$$

a contradiction.

Similarly, $p = Tp$ implies $p = Sp$.

PROOF OF COROLLARY 20 : Assume that $x \neq Sx$. Then, with $y = Sx$ in (16), we obtain

$$d(Sx, TSx) \leq \frac{cd(x, Sx)\, d(Sx, TSx)}{d(x, Sx) + d(Sx, TSx)}. \qquad \text{... (17)}$$

If $Sx = TSx$, then Sx is a fixed point of T, and hence of S by Lemma 3. If $Sx \neq TSx$, then we have, from (17),

$$d(x, Sx) + d(Sx, TSx) \leq cd(x, Sx),$$

or $d(Sx, TSx) \leq (c-1)\, d(x, Sx)$. By Theorem P1, either (i) S or T has a fixed point or (ii) $\{x_n\}$ converges.

Assumption (i) leads, by Lemma 3, to a common fixed point. Assume condition (ii). Without loss of generality we may assume that $x_n \neq x_{n+1}$ for all $n \geq 0$.

If $z = Tz$, then $z = Sz$ by Lemma 3. Suppose that $z \neq Tz$. From (16), with $x = x_{2n}, y = z$,

$$d(Sx_{2n}, Tz) \leq \frac{cd(x_{2n}, Sx_{2n})\, d(z, Tz) + bd(x_{2n}, Tz)\, d(z, Sx_{2n})}{d(x_{2n}, Sx_{2n}) + d(z, Tz)}.$$

Taking the limit as $n \to \infty$ yields

$$d(z, Tz) \leq \frac{0+0}{d(z, Tz)} = 0,$$

a contradiction.

Therefore $z = Tz$, which implies that $z = Sz$ by Lemma 3. The uniqueness of z follows from (16).

To prove that S and T have separate unique fixed points, suppose that z and w are fixed points of S. Then, by Lemma 3 they are also fixed points of T. From the uniqueness of common fixed points, $z = w$. A similar argument shows that the fixed point of T is unique.

Corollary 21 (Pachpatte[26], Theorem 1) — Let (X, d) be a complete metric space, T_1, T_2 selfmaps of X satisfying

$$d(T_1^p x, T_2^q y) \leq k \max \left\{ \frac{d(x, T_2^q y)\,[d(x, y) + d(x, T_1^p x) + d(y, T_1^p x)]}{2[d(x, T_2^q y) + d(y, T_1^p x)]}, \right.$$

$$\left. \frac{d(y, T_1^p x)\,[d(x, y) + d(y, T_2^q y) + d(x, T_2^q y)]}{2[d(x, T_2^q y) + d(y, T_1^p x)]} \right\} \quad \ldots (17a)$$

for all $x, y \in X$, $k \in (0, 1)$, p, q fixed natural numbers. Then T_1 and T_2 have a common unique fixed point in X.

For simplicity of notation define $S = T_1^p$ and $T = T_2^q$.

Lemma 4 — Let S and T satisfy the conditions of Corollary 21. Then z is a fixed point of S iff it is a fixed point of T.

PROOF : Suppose that $z = Sz$ and $z \neq Tz$. Then, from (17a),

$$d(z, Tz) \leq k \max \left\{ \frac{d(z, Tz)\,[0 + d(z, Tz) + 0]}{2[d(z, Tz) + 0]}, 0 \right\} = \frac{k}{2} d(z, Tz),$$

a contradiction. Therefore $z = Tz$. Similarly, $z = Tz$ implies $z = Sz$.

PROOF OF COROLLARY 21 : As pointed out by Fisher[15] it is necessary to add the hypothesis that $d(x, T_2^q y) + d(y, T_1^p x) \neq 0$. By adding the condition $d(x, T_2^q y) + d(y, T_1^p x) = 0$ implies $d(T_1^p s, T_2^q y) = 0$, Fisher[15] was able to prove the result. We now supply a different proof.

Assume that $x \neq Sx$ and $Sx \neq TSx$. For otherwise either S or T has a fixed point, and, by Lemma 4, it is a common fixed point. Therefore $1/d(x, TSx) > 0$. Setting $y = Sx$ in (17) we obtain

$$d(Sx, TSx) \leq k \max \left\{ \frac{d(x, TSx)\,[2d(x, Sx) + 0]}{2d(x, TSx)}, 0 \right\} = kd(x, Sx).$$

From Theorem P1, either (i) S or T has a fixed point or (ii) $\{x_n\}$ converges to a point z. Condition (i), by Lemma 4 gives a common fixed point. Assume (ii). Without loss of generality we may assume that $x_n \neq z$ for any n. Also assume that $z \neq Tz$. Using (17), since $z \neq Tz$, for all n sufficiently large, $d(x_n, Tz) \neq 0$, and we have

$$d(Sx_{2n}, Tz) \leq k \max \left\{ \frac{d(x_{2n}, Tz)\,[d(x_{2n}, z) + d(x_{2n}, Sx_{2n}) + d(z, Sx_{2n})]}{2[d(x_{2n}, Tz) + d(z, Sx_{2n})]}, \right.$$
$$\left. \frac{d(z, Sx_{2n})\,[d(x_{2n}, z) + d(z, Tz) + d(x_{2n}, Tz)]}{2[d(x, Tz) + d(z, Sx_{2n})]} \right\}.$$

Taking the limit as $n \to \infty$ we obtain

$$d(z, Tz) \leq k \max \left\{ \frac{d(z, Tz)}{2}, 0 \right\},$$

a contradiction. Therefore $z = Tz$. By Lemma 4, $z = Sz$. Uniqueness follows from (17). Thus T_1^p and T_2^q have a unique common fixed point. It is then easy to show that T_1 and T_2 have a unique common fixed point.

Corollary 22 (Pathak[33], Theorem 1) — Let (X, d) be a complete metric space, T_1, T_2 selfmaps of X satisfying

$$d(T_1x, T_2T_1y) \leq q \max \left\{ \frac{1}{2} d(T_1x, T_1y), \right.$$
$$\frac{d(T_1y, T_2T_1y)\,[1 + d(x, T_1x)\, d(x, T_2T_1y)]}{1 + d(x, T_1y)\, d(x, T_2T_1y)},$$
$$\left. \frac{d(x, T_2T_1y)\,[1 + d(x, T_1x) + d(T_1y, T_1x)]}{2[1 + d(x, T_1y)]} \right\} \quad \ldots (18)$$

for all $x, y \in X$, $0 \leq q < 1$. Then T_1 and T_2 have a unique common fixed point.

Lemma 5 — Let T_1, T_2 satisfy the conditions of Corollary 22. Then p is a fixed point of T_1 iff it is a fixed point of T_2.

PROOF : Suppose that $p = T_1p$. Then, from (18), with $x = y = p$,

$$d(p, T_2p) \leq q \max \left\{ 0, \frac{d(p, T_2p)\,[1 + 0]}{1 + 0}, \frac{d(p, T_2p)\,[1 + 0 + 0]}{2[1 + 0]} \right\},$$

which implies that $p = T_2p$. The proof for $p = T_2p$ is similar.

PROOF OF COROLLARY 22 : Set $y = x$ in (18) to get

$$d(T_1x, T_2T_1x) \leq q \max \left\{ 0, \frac{d(T_1x, T_2T_1x)\,[1 + d(x, T_1x)\, d(x, T_2\, T_1x)]}{1 + d(x, T_1x)\, d(x, T_2\, T_1x)}, \right.$$

$$\left. \frac{d(x, T_2T_1x)\,[1 + d(x, T_1x) + 0]}{2[1 + d(x, T_1x)]} \right\}$$

$$= q \max \left\{ d(T_1x, T_2T_1x), \frac{d(x, T_2T_1x)}{2} \right\},$$

which implies that $d(T_1x, T_2T_1x) \leq qd(x, T_1x)$.

From Theorem P1 either (i) T_1 or T_2 has a fixed point or (ii) $\{x_n\}$ converges to a point z. Condition (i) implies a common fixed point by Lemma 5. Assume (ii).

Using (18) with $x = z$, $y \cdot = x_{2n}$,

$$d(T_1z, T_2T_1x_{2n}) \leq q \max \left\{ \frac{d(T_1z, T_1x_{2n})}{2}, \right.$$

$$\frac{d(T_1x_{2n}, T_2T_1x_{2n})\,[1 + d(z, T_1z)\, d(x_{2n}, T_2T_1\, x_{2n})]}{1 + d(z, T_1x_{2n}\, d(z, T_2T_1x_{2n})},$$

$$\left. \frac{d(z, T_2T_1x_{2n}\,[1 + d(z, T_1z) + d(T_1x_{2n}, T_1z)]}{2[1 + d(z, T_1x_{2n})]} \right\}.$$

Taking the limit as $n \to \infty$ yields

$$d(T_1p, p) \leq q \max \left\{ \frac{d(T_1p, p)}{2}, 0 \right\}$$

which implies that $p = T_1p$. By Lemma 5, $p = T_2p$. Condition (18) implies uniqueness.

Corollary 23 (Pachpatte[27], Theorem 1) — Let X be orbitally complete, T_1, T_2 selfmaps of X, T_1 and T_2 orbitally continuous and satisfying

$$\min\, \{d(T_1x, T_2y), d(x, T_1x), d(y, T_2y)\} - \min\, \{d(x, T_2y), d(y, T_1x)\}$$

$$\leq q \max \left\{ \frac{d(x, T_2y)\,[d(x, y) + d(x, T_1x) + d(y, T_1x)]}{2[d(x, T_2y) + d(y, T_1x)]}, d(x, y) \right.$$

$$\left. \frac{d(y, T_1x)\,[d(x, y) + d(y, T_2y) + d(x, T_2y)]}{2[d(x, T_2y) + d(y, T_1x)]} \right\}$$

for all $x, y \in X$, $q \in (0, 1)$. Then T_1 and T_2 have a common fixed point.

PROOF : The hypotheses must be amended to state that the above inequality is true for all x and y such that $d(x, T_2y) + d(y, T_1x) \neq 0$.

If x is such that $x = T_1x$, then x is a fixed point of T_1. If $T_1x = T_2T_1x$, then T_1x is a fixed point of T_2. Assume that $x \neq T_1x$ and $T_1x \neq T_2T_1x$. Then $1/d(x, T_2T_1x) > 0$. Using the contractive definition with $y = T_1x$ we obtain

$$\min \{d(T_1x, T_2T_1x), d(x, T_1x), d(T_1x, T_2T_1x)\} - \min \{d(x, T_2T_1x), 0\}$$

$$\leq q \max \left\{ \frac{d(x, T_2T_1x)\,[d(x, T_1x) + d(x, T_1x) + 0]}{2[d(x, T_2T_1x) + 0]}, d(x, T_1x), 0 \right\}$$

or $$\min \{d(T_1x, T_2T_1x), d(x, T_1x) \leq qd(x, T_1x),$$

which implies that $d(T_1x, T_2T_1x) \leq qd(x, T_1x)$. By Theorem P1, $\{x_n\}$ converges to a point z. Since T_1 and T_2 are orbitally continuous, z is a common fixed point of T_1 and T_2.

Corollary 24 (Bajaj[1], Theorem 1) — Let S, T be selfmaps of a complete metric space (X, d) satisfying

$$d(Sx, Ty) \leq \alpha \frac{d(x, Sx)\, d(x, Ty) + [d(x, y)]^2 + d(x, Sx)\, d(x, y)}{d(x, Sx) + d(x, y) + d(x, Ty)} \qquad \ldots (19)$$

for all $x, y \in X$, $x \neq y$, where $0 < \alpha < 1$ and $d(x, Sx) + d(x, y) + d(x, Ty) \neq 0$. Then S and T have a common fixed point. Further, if $d(x, Sx) + d(x, y) + d(x, Ty) = 0$ implies $d(Sx, Ty) = 0$, then S and T have a unique common fixed point.

Lemma 6 — Let S, T satisfy the conditions of Corollary 24. Then p is a fixed point of S iff it is a fixed point of T.

PROOF : Suppose $p = Sp$ and $p \neq Tp$. Then, using (19) with $x = y = p$,

$$d(p, Tp) \leq \alpha \frac{0 + 0 + 0}{0 + 0 + d(p, Tp)},$$

a contradiction. The proof for $p = Tp$ is similar.

PROOF OF COROLLARY 24 : Assume $x \neq Sx$. Then, from (19), with $y = Sx$,

$$d(Sx, TSx) \leq \alpha \frac{d(x, Sx)\, d(x, TSx) + [d(x, Sx)]^2 + [d(x, Sx)]^2}{d(x, Sx) + d(x, Sx) + d(x, TSx)}$$

$$= \alpha \frac{d(x, Sx)\,\{ d(x, TSx) + 2d(x, Sx) \}}{2d(x, Sx) + d(x, TSx)} = \alpha d(x, Sx).$$

From Theorem P1, either (i) S or T has a fixed point or (ii) $\{x_n\}$ converges to a point z. Condition (i), from Lemma 6, leads to a common fixed point. Assume condition (ii).

Assume that $Sz \neq z$. Using (19), with $x = z$, $y = x_{2n+1}$,

$$d(Sz, Tx_{2n+1}) \leq \frac{d(z, Sz)\, d(z, Tx_{2n+1}) + [d(z, x_{2n+1})]^2 + d(z, Sz)\, d(z, x_{2n+1})}{d(z, Sz) + d(z, x_{2n+1}) + d(z, Tx_{2n+1})}.$$

Taking the limit as $n \to \infty$, we obtain

$$d(Sz, z) \leq \alpha \frac{0 + 0 + 0}{d(z, Sz) + 0 + 0},$$

and $z = Sz$. By Lemma 6, $z = Tz$.

To prove uniqueness, suppose that w is also a common fixed point of S and T, and that $z \neq w$. Using (19), with $x = z$, $y = w$,

$$d(z, w) \le \alpha \frac{0 + [d(z, w)]^2 + 0}{0 + d(z, w) + 0} = \alpha\, d(z, w),$$

a contradiction. Therefore $z = w$. Note that the condition $d(x, Sx) + d(x, y) + d(x, Ty) = 0$ implies $d(Sx, Ty) = 0$ is not needed.

Corollary 25 (Mukherjee and Som[24], Theorem 1) — Let T_1, T_2 be selfmaps of a complete metric space (X, d) satisfying one of the following :

(a) $\dfrac{d(x, T_1x)\, d(y, T_2y)}{d(T_1x, T_2y)} + d(T_1x, T_2y) \le \alpha d(x, y),\ 1 \le \alpha < 2,$

(b) $\dfrac{d(x, T_1x)\, d(y, T_2y)}{d(T_1x, T_2y)} + d(T_1x, T_2y) \le \beta\{d(x, T_2y) + d(y, T_1x) + d(x, y)\},$

$1/2 \le \beta < 2/3$

for each $x, y \in X$ with $T_1x \neq T_2y$. Then T_1 and T_2 have a common fixed point.

Lemma 7 — Let T_1 and T_2 be as in Corollary 25. Then p is a fixed point of T_1 iff it is a fixed point of T_2.

PROOF : Suppose that $p = T_1p$ and $p \neq T_2p$. Then, from (a) and (b), with $x = y = p$, we have $0 + d(p, T_2p) \le 0$ and $0 + d(p, T_2p) \le \beta\{d(p, T_2p) + 0 + 0\}$, which imply that $p = T_2p$.

PROOF OF COROLLARY 25 : Suppose that $T_1x \neq T_2T_1x$. In (a) and (b) set $y = T_1x$ to get

$$d(x, T_1x) + d(T_1x, T_2T_1x) \le \alpha d(x, T_1x),$$

which implies that $d(T_1x, T_2T_1x) \le (\alpha - 1)\, d(x, T_1x)$,

and

$$d(x, T_1x) + d(T_1x, T_2T_1x) \le \beta\{d(x, T_2T_1x) + 0 + d(x, T_1x)\},$$

which implies that

$$d(T_1x, T_2T_1x) \le \frac{2\beta - 1}{1 - \beta}\, d(x, T_1x).$$

Thus, in either case

$$d(T_1x, T_2T_1x) \le \max\left\{\alpha - 1, \frac{2\beta - 1}{1 - \beta}\right\} d(x, T_1x),$$

and Theorem P1 is satisfied. Either (i) T_1 or T_2 has a fixed point, or (ii) $\{x_n\}$ converges. Assumption (i) leads to a common fixed point. Assume condition (ii), and call the limit z.

Suppose that $z \neq T_2z$. Using (a) with $x = x_{2n}$, $y = z$, we obtain, for all n sufficiently large,

$$\frac{d(x_{2n}, T_1x_{2n})\, d(z, T_2z)}{d(T_1x_{2n}, T_2z)} + d(T_1x_{2n}, T_2z) \le \alpha\, d(x_{2n}, z).$$

Using (b), with $x = x_{2n}$, $y = z$, we obtain, for all n sufficiently large,

$$\frac{d(x_{2n}, T_1x_{2n})\, d(z, T_2z)}{d(T_1x_{2n}, T_2z)} + d(T_1x_{2n}, T_2z) \le \beta$$

$$\{d(x_{2n}, T_2z) + d(z, T_1x_{2n}) + d(x_{2n}, z)\}.$$

In either case we have

$$\frac{d(x_{2n}, T_1x_{2n})\, d(z, T_2z)}{d(T_1x_{2n}, T_2z)} + d(T_1x_{2n}, T_2z) \le \max\ \{\alpha d(x_{2n}, z),$$

$$\beta\{d(x_{2n}, T_2z) + d(z, T_1x_{2n}) + d(x_{2n}, z)\}\}.$$

Taking the limit as $n \to \infty$ yields $d(z, T_2z) \le \max\ \{0,\ \beta(d(z, T_2z) + 0 + 0)\}$, and $z = T_2z$. By Lemma 7, $z = T_1z$.

Corollary 26 (Fisher[13], Theorem 2) — Let S and T be selfmaps of a complete metric space (X, d) satisfying

$$d(Sx, Ty) \le c\,\frac{\{d(x, Sx)\}^2 + \{d(y, Ty)\}^2}{d(x, Sx) + d(y, Ty)} \qquad \text{... (20)}$$

for each $x, y \in X$ for which $d(x, Sx) + d(y, Ty) \ne 0$, where $0 < c < 1$. Then S and T have a common fixed point z. Further, if $d(x, Sx) + d(y, Ty) = 0$ implies that $d(Sx, Ty) = 0$, then z is the unique common fixed point of S and T.

Lemma 8 — Let S and T satisfy the conditions of Corollary 26. Then p is a fixed point of S iff it is a fixed point of T.

PROOF : Suppose that $p = Sp$ and $p \ne Tp$. Using (20) with $x = y = p$,

$$d(p, Tp) \le c\,\frac{0 + \{d(p, Tp)\}^2}{0 + d(p, Tp)} = cd(p, Tp),$$

a contradiction. The proof for $p = Tp$ is similar.

PROOF OF COROLLARY 26 : Assume that $x \ne Sx$. Then, from (20) with $y = Sx$,

$$d(Sx,\ TSx) \le c\,\frac{\{d(x, Sx)\}^2 + \{d(Sx, TSx)\}^2}{d(x, Sx) + d(Sx, TSx)}.$$

Clearing of fractions we obtain

$$d(Sx,\ TSx)\ d(x,\ Sx) + \{d(Sx,\ TSx)\}^2 \le c[\{d(x,\ Sx)\}^2 + \{d(Sx,\ TSx)\}^2],$$

or

$$(1 - c)\ \{d(Sx,\ TSx\}^2 + d(Sx,\ TSx)\ d(x,\ Sx) \le c\{d(x,\ Sx)\}^2$$

which implies that

$$\{d(Sx,\ TSx)\}^2 + \frac{d(x,Sx)}{(1-c)}\, d(Sx, TSx) \le \frac{c}{1-c}\ \{d(x,\ Sx)\}^2$$

or

$$\left\{ d(Sx, TSx) + \frac{d(x,Sx)}{2(1-c)} \right\}^2 \le \left\{ \frac{c}{1-c} + \frac{1}{4(1-c)^2} \right\}\ \{d(x,\ Sx)\}^2.$$

Thus

$$d(Sx, TSx) \le \left[\left\{ \frac{4c(1-c)+1}{4(1-c)^2} \right\}^{1/2} - \frac{1}{2(1-c)} \right] \{d(x,\ Sx)\}.$$

It is easy to show that the quantity in brackets is positive and less than one. Therefore Theorem P1 applies. Hence either (i) S or T has a fixed point, or (ii) $\{x_n\}$ converges. Condition (i) implies a common fixed point. Assume that condition (ii) is satisfied, and call the limit z.

Assume that $z \ne Tz$. Then, using (20) with $x = x_{2n}$, $y = z$,

$$d(Sx_{2n}, Tz) \le c\,\frac{\{d(x_{2n}, Sx_{2n})\}^2 + \{d(z, Tz)\}^2}{d(x_{2n}, Sx_{2n}) + d(z, Tz)}.$$

Taking the limit as $n \to \infty$ yields $d(z,\ Tz) \le cd(z, Tz)$, a contradiction. Therefore $z = Tz$, and, by Lemma 8, $z = Sz$.

Corollary 27 (Fisher[14], Theorem 2) — Let S, T be selfmaps of a complete metric space $(X,\ d)$ such that, for each $x,\ y \in X$, either

$$d(Sx, Ty) \le \frac{b\{d(x, Ty)\}^2 + c\{d(y, Sx)\}^2}{d(x, Ty) + d(y, Sx)} \qquad \text{... (21)}$$

if $d(x,\ Ty) + d(y,\ Sx) > 0$, $0 \le b, c, b+c < 1$, or $d(Sx,\ Ty) = 0$ if $d(x,\ Ty) + d(y,\ Sx) = 0$. Then S and T have a unique common fixed point.

Lemma 9 — Let S and T satisfy the conditions of Corollary 27. Then p is a fixed point of S iff it is a fixed point of T.

PROOF : Suppose that $p = Sp$ and $p \ne Tp$. Then, setting $x = y = p$ in (21)

$$d(p, Tp) \le \frac{b\{d(p, Tp)\}^2 + 0}{d(p, Tp) + 0} = bd(p, Tp),$$

a contradiction. The proof for $p = Tp$ is similar.

PROOF OF COROLLARY 27 : Since $b + c < 1$, either $b < 1/2$ or $c < 1/2$. For, if $b,\ c \ge 1/2$, then $b + c \ge 1$, a contradiction. Assume that $b < 1/2$.

Assume that $x \ne Sx$ and $Sx \ne TSx$. Then $1/d(x,\ TSx) > 0$. In (21) set $y = Sx$ to obtain

$$d(Sx, TSx) \le \frac{b\{d(x, TSx)\}^2 + 0}{d(x, TSx) + 0} \le b[d(x, Sx) + d(Sx, TSx)],$$

which implies that

$$d(Sx, TSx) \le \frac{b}{1-b}\, d(x, Sx),$$

and Theorem P1 is satisfied. Therefore, either (i) S or T has a fixed point or (ii) $\{x_n\}$ converges. Condition (i), by Lemma 9, leads to a common fixed point. Assume condition (ii), with z as the limit point. Suppose that $z \neq Tz$. Then, from (21) with $x = x_{2n}$, $y = z$,

$$d(Sx_{2n}, Tz) \le \frac{b\{d(x_{2n}, Tz)\}^2 + c\{d(z, Sx_{2n})\}^2}{d(x_{2n}, Tz) + d(z, Sx_{2n})}.$$

Now take the limit as $n \to \infty$ to obtain

$$d(z, Tz) \le \frac{b\{d(z, Tz)\}^2 + 0}{d(z, Tz) + 0} = bd(z, Tz),$$

a contradiction. Therefore $z = Tz$ and, from Lemma 9, $z = Sz$.

To show uniqueness, suppose that $w \neq z$ is another common fixed point of S and T. From (21), with $x = z$, $y = w$,

$$d(z, w) \le \frac{b\{d(z, w)\}^2 + \{d(w, z)\}^2}{2d(z, w)} = \frac{(b + c)}{2}\, d(z, w),$$

a contradiction. Therefore $z = w$.

Corollary 28 (Pachpatte[28], Theorem 1) — Let S, T be selfmaps of a complete metric space (X, d) satisfying

$$d(Sx, Ty) \le \frac{q \max\left\{ [d(x, y)]^2, [d(x, Sx)]^2, [d(y, Ty)]^2, \frac{1}{2}[d(x, Ty)]^2, \frac{1}{2}[d(y, Sx)]^2 \right\}}{d(x, Sx) + d(y, Ty)} \quad \ldots (22)$$

for all $x, y \in X$ for which $d(x, Sx) + d(y, Ty) \neq 0$, $q \in (0, 1)$. Then S and T have a common fixed point z. Further, if $d(x, Sx) + d(y, Ty) = 0$ implies that $d(Sx, Ty) = 0$, then z is the unique common fixed point of S and T.

Lemma 10 — Let S and T satisfy the conditions of Corollary 28. Then p is a fixed point of S iff it is a fixed point of T.

PROOF : Suppose that $p = Sp$ and $p \neq Tp$. Setting $x = y = p$ in (22) yields

$$d(p, Tp) \leq \frac{q \max\left\{0, 0, [d(p, Tp)]^2, \frac{1}{2}[d(p, Tp)]^2, \frac{1}{2}[d(p, Tp)]^2\right\}}{d(x, Sx) + d(p, Tp)}$$

$$\leq qd(p, Tp)$$

a contradiction. The assumption $p = Tp$ is proved similarly.

PROOF OF COROLLARY 28 : Assume that $x \neq Sx$. Then, from (22), with $y = Sx$,

$$d(Sx, TSx) \leq \frac{q \max\left\{[d(x, Sx)]^2, [d(Sx, TSx)]^2, \frac{1}{2}[d(x, TSx)]^2, 0\right\}}{d(x, Sx) + d(Sx, TSx)}$$

$$= qd(x, Sx),$$

and Theorem P1 applies; i.e., (i) either S or T has a fixed point, or (ii) $\{x_n\}$ converges. Condition (i) leads to a common fixed point for S and T by Lemma 10. Assume condition (ii) with limit z.

Suppose that $z \neq Tz$. Then, from (22) with $x = x_{2n}$, $y = z$,

$$d(Sx_{2n}, Tz)$$

$$\leq \frac{q \max\left\{[d(x_{2n}, Sx_{2n})]^2, [d(z, Tz)]^2, \frac{1}{2}[d(x_{2n}, Tz)]^2, \frac{1}{2}[d(z, Sx_{2n})]^2\right\}}{d(x_{2n}, Sx_{2n}) + d(z, Tz)}.$$

Taking the limit as $n \to \infty$ yields

$$d(z, Tz) \leq \frac{q \max\left\{0, [d(z, Tz)]^2, \frac{1}{2}[d(z, Tz)]^2, 0\right\}}{0 + d(z, Tz)} = qd(z, Tz),$$

a contradiction.

The following result of Pachpatte, although expressed in a rational format, is a special case of a fixed point theorem not in a rational form.

Theorem of Pachpatte[29] — Let S, T be selfmaps of a complete metric space satisfying

$$d(Sx, Ty) \leq q \max\left\{\frac{d(x, Sx)\, d(x, Ty)}{d(x, Ty) + d(y, Sx)}, \frac{d(y, Ty)\, d(y, Sx)}{d(x, Ty) + d(y, Sx)}\right\} \quad \text{... (23)}$$

for all $x, y \in X$ for which $d(x, Ty) + d(y, Sx) \neq 0$, $q \in (0, 1)$. Then S and T have a common fixed point z. Further, if $d(x, Ty) + d(y, Sx) = 0$ implies that $d(Sx, Ty) = 0$, then z is the unique common fixed point of S and T.

PROOF : Note that (23) implies that

$$d(Sx, Ty) \leq q \max \{d(x, Sx), d(y, Ty)\},$$

and the theorem is a special case of Theorem 14 of Rhoades[36].

We now make a modest extension of Theorem P1 to the situation in which the space is not complete.

Proposition 3 — Let S and T be selfmaps of a metric space (X, d). Suppose that there exists a point x in X such that the sequence $\{x_n\}$ defined by $x_0 = x$, $x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}$ has a convergent subsequence with limit z in X. Suppose there exists a $\lambda \in [0, 1)$ such that (13) is satisfied for each disticnt $x, y \in \{x_i\}$ satisfying either $x = Ty$ or $y = Sx$. Then either

(i) S or T has a fixed point in $\{x_i\}$ or

(ii) $\{x_i\}$ converges to z and $d(x_i, z) \le \lambda^i d(x_0, x_1)/(1-\lambda)$ for $i > 0$.

PROOF : As in the proof of Theorem P1, $\{x_i\}$ is Cauchy. Since it has a convergent subsequence with limit z, $\{x_i\}$ converges to z. The rest of the proof follows as in Park[32].

Corollary 29 (Murthy and Pathak[25], Theorem 1) — Let (X, d) be a metric space, T_1, T_2 selfmaps of X such that

$$d(T_1^r x, T_2^s y) \le \frac{\alpha d(x, T_1^r x)\, d(y, T_2^s y)}{d(x, T_2^s y) + d(y, T_1^r x) + d(x, y)} + \beta d(x, y) \qquad \text{... (24)}$$

for all x, y in $X, x \ne y$, where $r, s > 0$ are integers and $\alpha, \beta \ge 0$ are such that $\alpha + \beta < 1$. If for some x in X the sequence $\{x_n\}$ defined by $x_{2n+1} = T_1^r x_{2n}, x_{2n+2} = T_2^s x_{2n+2}$ has a convergent subsequence with limit point in X, then T_1 and T_2 have a unique common fixed point p in X.

PROOF : The condition $x \ne y$ in (24) is imposed in order to guarantee that the denominator does not vanish. A more appropriate condition to impose would be to have (24) exist for each $x, y \in X$ such that $d(x, T_2^s y) + d(y, T_1^r x) + d(x, y) \ne 0$.

For notational simplicity, set $S = T_1^r, T = T_2^s$ and evaluate (24) at $x_0 = x$, $y = Sx$.

$$d(Sx, TSx) \le \frac{\alpha d(x, Sx)\, d(Sx, TSx)}{d(x, TSx) + d(x, Sx)} + \beta d(x, Sx)$$

$$\le \alpha d(Sx, TSx) + \beta d(x, Sx),$$

which implies that

$$d(Sx, TSx) \le \frac{\beta}{1-\alpha} d(x, Sx),$$

and (13) is satisfied. Therefore either condition (i) or (ii) or Proposition 3 holds.

Lemma 11 — Let $S = T_1^r, T = T_2^s$ satisfy the conditions of Corollary 29. If S or T has a fixed point, then it is the unique common fixed point of T_1 and T_2.

PROOF : Let z be a fixed point of S, and assume that $z \ne Tz$. Then, from (24),

$$d(Sz, Tz) \le \frac{\alpha d(z, Sz)\, d(z, Tz)}{d(z, Tz) + d(z, Sz) + 0} = 0,$$

a contradiction. Therefore z is also a fixed point of T. Similarly, if z is a fixed point of T, then it is also a fixed point of S.

To show that z is unique, suppose that z and w are common fixed points of S and T, with $z \neq w$. Then, from, (24)

$$d(z, w) = d(Sz, Tw) \leq \frac{\alpha d(z, Sz)\, d(w, Tw)}{d(z, Tw) + d(w, Sz) + d(z, w)} + \beta d(z, w) = \beta d(z, w),$$

a contradiction.

As in Murthy and Pathak[25], it follows that z is the unique fixed point of T_1 and T_2.

Now suppose that condition (ii) is satisfied. Assume $z \neq Tx$. Substituting $x = x_{2n}, y = z$ in (24), we have, for all n sufficiently large,

$$d(Sx_{2n}, Tz) \leq \frac{\alpha d(x_{2n}, x_{2n+1})\, d(z, Tz)}{d(x_{2n}, Tz) + d(z, Sx_{2n}) + d(x_{2n}, z)} + \beta d(x_{2n}, z)$$

$$\leq \frac{\alpha d(x_{2n}, x_{2n+1})\, [d(z, x_{2n}) + d(x_{2n}, Tz)]}{d(z, Tz) + d(z, Sx_{2n}) + d(z, z)} + \beta d(x_{2n}, z)$$

$$\leq \alpha d(x_{2n}, x_{2n+1}) + \beta d(x_{2n}, z).$$

Taking the limit as $n \to \infty$ gives $z = Tz$. By Lemma 11 z is the unique common fixed point of T_1 and T_2.

Theorem 2 of Murthy and Pathak[25] is a special case of their Theorem 1, by setting $r = s = 1$, $T_1 = E^{p(x)}$, $T_2 = F^{q(y)}$.

Corollary 30 (Jaggi and Dass[20]) — Let f be a selfmap defined on a metric space (X, d) satisfying the following :

(i) For some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$,

$$d(fx, fy) \leq \frac{\alpha d(x, fx)\, d(y, fy)}{d(x, fy) + d(y, fx) + d(x, y)} + \beta d(x, y)$$

for all x, y in X, $x \neq y$.

(ii) There exists a point x_0 in X such that $\{f^n x_0\}$ has a convergent subsequence with limit z in X.

Then z is the unique fixed point of f.

PROOF : Set $T_1^r = T_2^s = f$ in Corollary 29.

We shall now reprove a result using another general principle from Park[32].

Theorem P2 (Theorem 2.1 of Park[32]) — Let S and T be selfmaps of a metric space X. If

(i) There exists a sequence $\{x_n\}$ in X such that $x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}$, and $\{x_n\}$ has a cluster point z in X,

(ii) S, T, ST, and TS are continuous at z, and

(iii) S and T satisfy $d(Sx, Ty) < d(x, y)$ for each distinct $x, y \in \{\bar{x}_n\}$ satisfying either $x = Ty$ or $y = Sx$, then either

(1) S or T has a fixed point in $\{x_n\}$, or

(2) z is a common fixed point of S and T and $\lim x_n = z$.

Corollary 31 (Paliwal[31], Theorem 2) — Let T_1 and T_2 be two continuous selfmaps of a metric space (X, d) such that

$$d(T_1^r x, T_2^s y) \leq \frac{\alpha d(x, T_1^r x)\, d(y, T_2^s y)}{d(x, T_2^s y) + d(y, T_1^r x) + d(x, y)} + \beta d(x, y) \qquad \text{... (25)}$$

for each $x, y \in X$, $x \neq y$, where r and s are positive integers, and $\alpha > 0$, $\beta \geq 0$, $\alpha + \beta = 1$. If, for some $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_{2n+1} = T_1^r x_{2n}$, $x_{2n+2} = T_2^s x_{2n+1}$, has a subsequence converging to a point z in X, then T_1 and T_2 have a unique common fixed point.

PROOF : For notational simplicity set $S = T_1^r$, $T = T_2^s$. From Theorem P2, either (1) or (2) holds.

Consequently, the conclusion of Corollary 31 must be changed, since a fixed point of one map in the orbit of x_0 does not imply that it is a fixed point of the other. If condition (2) holds, then, as in Paliwal[31], the common fixed point is unique.

Murthy and Pathak[25] claim that the above result of Paliwal[31] is false. However, the example they construct is one which does not satisfy the contractive definition of Paliwal[31].

Bhagwat and Singh[3], Chatterji[6], and Rathore, Deshmukh, and Rajput[35] each attempt to prove a fixed point theorem for a pair of maps satisfying a rational type contractive condition. However, each of them makes the same error. Bhagwat and Singh[3], p. 996, line 4 state that $d(\xi, T_1\xi) \leq d(\xi, T_1\xi)$ is a contradiction, which is false. In Chatterji[6] p.450, line 4, the author claims strict inequality, but the correct correct conclusion is $\leq$. In Rathore *et al.*[35], p. 155, line 1, it is claimed that $d(Pz, z) \leq d(Pz, z)$ implies that $z = Pz$, which is false.

In the case of Chatterji[6], if one chooses $X = \{0, 1\}$, $F0 = 1$, $F1 = 0$, then F satisfies all of the conditions of the Theorem, but does not have a fixed point.

Finally we conclude by noting that theorems involving a sequence of maps satisfying a rational inequality are also special cases of the general principles of this paper.

Corollary 32 (Pachpatte[28], Theorem 3) — Let T_0, $\{T_n\}$, $n > 0$, be selfmaps of a complete metric space (X, d) satisfying

$$d(T_0x, T_ny) \leq \frac{q \max\left\{ [d(x,y)]^2, [d(x, T_0x)]^2, [d(y, T_ny)]^2, \frac{1}{2}[d(x, T_ny)]^2, [d(y, T_0x)]^2 \right\}}{d(x, T_0x) + d(y, T_ny)} \qquad \text{... (26)}$$

for each $x, y \in X$ for which $d(x, T_0x) + d(y, T_ny) \neq 0$, and each $n = 1, 2, \ldots$, $q \in (0, 1)$. Then there exists a common fixed point z, and for an arbitrary $x_0 \in X$, the sequence $x_0, x_1 = T_0x_0, x_{2n} = T_n x_{2n-2}, x_{2n+1} = T_0x_{2n}$, converges to z. Further if $d(x, T_0x) + d(y, T_ny) = 0$ implies that $d(T_0x, T_ny) = 0$, then z is the unique fixed point of T_n for $n = 0, 1, 2, \ldots$.

PROOF : An alternative proof is as follows. Fix n. Then (26) is a special case of Corollary 28. Therefore z is a common fixed point of T_0 and T_n. For each n we have the same conclusion. Therefore z is a common fixed point of the T_n.

Employing this idea, the following theorems are special cases of the work of this paper : Theorem 1 of Pachpatte[30], Theorem 2 of Pachpatte[26], and Theorem 5 of Khan[22].

Some other examples of rational type fixed point conditions, for which the general techniques of Hicks and Rhoades[17] and Park[32] apply, have appeared in Rhoades[39].

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