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SEMI-SYMMETRIC AND SEMI-PARALLEL GEODESIC SPHERES AND TUBES

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We prove that a connected Riemannian manifold of dimension n is a real space form if and only if all sufficiently small geodesic spheres or all tubes with sufficiently small radius about all geodesics are semi-parallel (for $n \geq 3$) or Ricci-semi-symmetric (for $n \geq 4$).

1. INTRODUCTION

As is well-known, geodesic spheres with sufficiently small radius in a Riemannian manifold (M^n, g) of dimension $n > 2$ are locally symmetric if and only if (M^n, g) is of constant curvature[4]. A similar property may be derived for geodesic tubes, that is, tubes about geodesics (see also below). Locally symmetric spaces are semi-symmetric, that is, $R_{XY} \cdot R = 0$ where R is the Riemannian curvature tensor and R_{XY} acts as a derivation for all tangent vector fields X, Y. This last condition is weaker as follows, for example, from Szabó[11]. (See also (Boeckx[2]) for a survey, recent results and further references.) The results in this note stem from the following question : which are the Riemannian manifolds all of whose small geodesic spheres or geodesic tubes are semi-symmetric ? One of our results is that this condition again implies that (M^n, g) has constant curvature for $n \geq 4$, the condition holding trivially for $n = 3$. In fact, we will prove a stronger result : real space *forms of dimension* $n \geq 4$ are characterized by the Ricci-semi-symmetry condition $\tilde{R}_{XY} \cdot \tilde{\rho} = 0$ for the geodesic spheres or geodesic tubes $\tilde{M}, \tilde{\rho}$ being the Ricci tensor of type (0, 2) on $\tilde{M}$ and X, Y tangent vector fields on $\tilde{M}$.

The results in Chen and Vanhecke[4] and Gheysens and Vanhecke[7] indicate a strong similarity between the intrinsic geometry determined by $\tilde{\rho}$ and the extrinsic properties related to the second fundamental form σ of the geodesic spheres and tubes. Therefore, we also investigate the condition $\tilde{R}_{XY} \cdot \sigma = 0$ which means that the

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$\tilde{M}$ are semi-parallel hypersurfaces (Deprez[5]). We will show that, for $n \geq 3$, this condition also characterizes spaces of constant curvature.

Complex and quaternionic space forms will be characterized in a similar way in a forthcoming paper.

2. Preliminaries

Let (M, g) be an n-dimensional, connected, smooth Riemannian manifold with $n > 2$. Denote by ∇ the Levi-Civita connection and by R and ρ the corresponding Riemannian curvature tensor and Ricci tensor, respectively. We use the following sign convention :

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for tangent vector fields X, Y on M.

Next, let $m \in M$, and let γ be a geodesic parametrized by arc length s such that $\gamma(0) = m, \gamma'(0) = u$. Moreover, let $\{e_1, ..., e_n\}$ be an orthonormal basis of the tangent space $T_m M$ at m with $e_1 = u$ and extend this basis to a parallel basis $\{E_1, ..., E_n\}$ along γ. Further, let $G_m(r)$ be a geodesic sphere centered at m and with radius r. We always suppose that $r < i(m)$, the injectivity radius at m. Then $G_m(r)$ is a hypersurface and $p = \exp_m(ru)$ is a point of $G_m(r)$. According to Chen and Vanhecke[4] we have the following expansions for the curvature tensor $\tilde{R}$, the Ricci tensor $\tilde{\rho}$ and the second fundamental form σ of $G_m(r)$ with respect to the parallel basis $\{E_1, ..., E_n\}$:

$$\tilde{R}_{abcd}(p) = \frac{1}{r^2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + O(1), \qquad \text{... (1)}$$

$$\tilde{\rho}_{ab}(p) = \frac{n-2}{r^2}\delta_{ab} + \left(\rho_{ab} - \frac{1}{3}\rho_{uu}\delta_{ab} - \frac{n}{3}R_{aubu}\right)(m) + O(r), \qquad \text{... (2)}$$

$$\sigma_{ab}(p) = \frac{1}{r}\delta_{ab} - \frac{r}{3}R_{aubu}(m) + O(r^2) \qquad \text{... (3)}$$

for $a, b, c, d = 2, ..., n$, where $R_{abcd} = g(R_{E_aE_b}E_c, E_d)$ and similarly for the other tensors. We refer to Chen and Vanhecke[4], Gray[8], Gray and Vanhecke[9] and Vanhecke[12] for more details.

When (M^n, g) is a space of constant sectional curvature c, we can write down complete formulas for $\tilde{R}$, $\tilde{\rho}$ and σ. We have

$$\sigma = \sqrt{c}\cot(\sqrt{c}r)\, g, \qquad \text{... (4)}$$

$$\tilde{R}_{XYZW} = c(1 + \cot^2(\sqrt{c}r))\ \{g(X, Z)\, g(Y, W) - g(X, W)\, g(Y, Z)\}, \qquad \text{... (5)}$$

$$\tilde{\rho} = (n-2)\, c(1 + \cot^2(\sqrt{c}r))\, g \qquad \text{... (6)}$$

for $c > 0$ and where g denotes the induced metric. For $c < 0$ we replace the trigonometric functions by hyperbolic functions and the formulas for $c = 0$ are obtained by taking the limit for $c \to 0$.

Now, we consider tubes about curves and geodesics. We refer to papers referred to under references [7], [8], [10], [12], [13] for more details. Let $\sigma : [a, b] \to M$ be a smooth embedded curve and let P_r denote the tube of radius r about σ, that is,

$$P_r = \{p \in M \mid d(p, \sigma) = r\}$$

where we take r smaller than the distance from σ to its nearest focal point. Let σ be parametrized by the arc length t and denote by $\{e_1, e_2, ..., e_n\}$ an orthonormal basis of $T_{\sigma(a)}M$ such that $e_1 = \dot{\sigma}(a)$. Further, let $E_1 = \dot{\sigma}$ and denote by $E_2, ..., E_n$ the normal vector fields along σ which are parallel with respect to the normal connection of the normal bundle $\sigma^{\perp}$ and such that $E_i(a) = e_i$, $i = 2, ..., n$. Next, let $p \in P_r$ and denote by γ the geodesic through p which cuts σ orthogonally at $m = \sigma(t)$. We parametrize γ by arc length such that $\gamma(0) = m$ and take $(E_2, ..., E_n)$ such that $E_2(t) = \gamma'(0) = u$. Finally, let $\{F_1, ..., F_n\}$ be the orthonormal frame field along γ obtained by parallel translation of $\{E_1(t), ..., E_n(t)\}$ along γ.

For the hypersurface P_r we then have the following expansions with respect to this parallel frame field (Gheysens and Vanhecke[7] and Vanhecke and Willmore[13]) :

$$\widetilde{R}_{1a1b}(p) = -\frac{1}{r}\kappa_u(m)\,\delta_{ab} + (R_{1a1b} - R_{1u1u}\,\delta_{ab} - \kappa_u^2\,\delta_{ab})(m) + O(r), \quad \text{... (7)}$$

$$\widetilde{R}_{1abc}(p) = O(1), \quad \text{... (8)}$$

$$\widetilde{R}_{abcd}(p) = \frac{1}{r^2}(\delta_{ac}\,\delta_{bd} - \delta_{ad}\,\delta_{bc}) + O(1), \quad \text{... (9)}$$

$$\widetilde{\rho}_{11}(p) = O(r^{-1}), \quad \text{... (10)}$$

$$\widetilde{\rho}_{1a}(p) = O(1), \quad \text{... (11)}$$

$$\widetilde{\rho}_{ab}(p) = \frac{n-3}{r^2}\delta_{ab} - \frac{1}{r}\kappa_u(m)\,\delta_{ab} + \left(\rho_{ab} - \frac{n-1}{3}R_{aubu} - \frac{1}{3}\rho_{uu}\,\delta_{ab} - \frac{2}{3}R_{1u1u}\,\delta_{ab} - \kappa_u^2\,\delta_{ab}\right)(m) + O(r), \quad \text{... (12)}$$

$$\sigma_{11}(p) = -\kappa_u(m) - r(R_{1u1u} + \kappa_u^2)(m) + O(r^2), \quad \text{... (13)}$$

$$\sigma_{1a}(p) = O(r), \quad \text{... (14)}$$

$$\sigma_{ab}(p) = \frac{1}{r}\delta_{ab} - \frac{r}{3}R_{aubu}(m) + O(r^2) \quad \text{... (15)}$$

for $a, b, c, d \in \{3, 4, ..., n\}$ and where $\kappa_u = g(\dot{\sigma}', u)$.

Note that σ is a geodesic if and only if $\kappa_u = 0$ for all normal u. In that case P_r is called a geodesic tube.

Next, let (M^n, g) be a space of constant sectional curvature c. Then we have for a geodesic tube P_r (Gheysens and Vanhecke[7]) :

$$\widetilde{R}_{1a1b}(p) = 0, \qquad \text{... (16)}$$

$$\widetilde{R}_{1abc}(p) = 0, \qquad \text{... (17)}$$

$$\widetilde{R}_{abcd}(p) = \frac{c}{\sin^2(\sqrt{c}\,r)}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}), \qquad \text{... (18)}$$

$$\widetilde{\rho} = (n-3)\frac{c}{\sin^2(\sqrt{c}\,r)}(g - \eta \otimes \eta), \qquad \text{... (19)}$$

$$\sigma = \lambda g + \mu \eta \otimes \eta \qquad \text{... (20)}$$

where $\lambda = \sqrt{c}\cot(\sqrt{c}\,r)$, $\lambda + \mu = -\sqrt{c}\tan(\sqrt{c}\,r)$ and $\eta(X)(p) = g(X, F_1)(p)$ for any tangent X to P_r at p and $c > 0$. We have similar formulas for $c < 0$ and those for $c = 0$ can be obtained by taking the limit for $c \to 0$.

3. Semi-symmetric and Semi-parallel Geodesic Spheres

It is well-known that small geodesic spheres in real space forms are symmetric spaces. This can be checked easily by using (5). For an elegant proof using reflections, we refer to Berndt and Vanhecke[1]. Hence, $G_m(r)$ satisfies $\widetilde{R}_{XY} \cdot \widetilde{R} = 0$ for all X, Y tangent to $G_m(r)$. So, all $G_m(r)$ are semi-symmetric and via contraction we also get $\widetilde{R}_{XY} \cdot \widetilde{\rho} = 0$, that is, the $G_m(r)$ are Ricci-semi-symmetric. Moreover, (4) and (5) yield at once $\widetilde{R}_{XY} \cdot \sigma = 0$ which, according to Deprez[5], means that the geodesic spheres are semi-parallel.

We note that a two-dimensional (M, g) is always semi-symmetric. Hence, the geodesic spheres of a three-dimensional Riemannian manifold are always semi-symmetric.

Now we shall prove that these conditions characterize spaces of constant curvature of dimension $n > 3$ or $n \geq 3$, respectively.

Theorem 3.1 — Let (M^n, g), $n \geq 4$, be a Riemannian manifold such that all its small geodesic spheres are Ricci-semi-symmetric. Then (M, g) is of constant curvature.

Proof : First we have

$$0 = (\widetilde{R}_{ab} \cdot \widetilde{\rho})_{cc} = -2 \sum_{i=2}^{n} \widetilde{R}_{abci}\widetilde{\rho}_{ci}$$

for a, b, c = 2, ..., n. Using the power series expansions (1) and (2) and considering the coefficient of r^{2} yields the condition

$$\delta_{ac}\left(\rho_{bc}-\frac{n}{3}R_{bucu}\right)-\delta_{bc}\left(\rho_{ac}-\frac{n}{3}R_{aucu}\right) = 0. \qquad \text{... (21)}$$

Next, take $b = c \neq a$ in (21). Then we get

$$3\rho_{ab} = n\,R_{aubu}. \qquad \text{... (22)}$$

Since this must hold for any u orthogonal to (e_a, e_b) we get by contraction $(n-3)\,\rho_{ab} = 0$ and hence, because $n > 3$, $\rho_{ab} = 0$. Then from (22) we obtain $R_{aubu} = 0$ which, by means of Cartan's characterization of real space forms (see, for example, Cartan[3] and also Fialkov[6]), implies that (M, g) has constant curvature.

Theorem 3.2 — Let (M^n, g), $n \geq 3$, be a Riemannian manifold such that all its small geodesic spheres are semi-parallel. Then the manifold (M, g) is of constant curvature.

PROOF : Proceeding in the same way as in Theorem 3.1, but now using expansions (1) and (3), the coefficient or r^{-1} in $(\widetilde{R}_{ab}\cdot\sigma)_{cc} = 0$ leads to the condition

$$\delta_{ac}\,R_{cubu}-\delta_{bc}\,R_{cuau} = 0.$$

By taking $a \neq c = b$ we then get $R_{aubu} = 0$, which again yields the required result.

Hence we have :

Corollary 3.3 — Let (M^n, g), $n \geq 4$, be a Riemannian manifold. Then the following statements are equivalent for sufficiently small radii :

(i) (M, g) is a real space form;

(ii) all geodesic spheres are symmetric;

(iii) all geodesic spheres are semi-symmetric;

(iv) all geodesic spheres are Ricci-semi-symmetric;

(v) all geodesic spheres are semi-parallel.

4. Semi-symmetric and Semi-parallel Geodesic Tubes

Now we prove similar characterizations by considering tubes of sufficiently small radius about curves. First we derive the following result.

Theorem 4.1 — Let (M^n, g), $n \geq 4$, be a Riemannian manifold such that all small tubes about a curve σ are Ricci-semi-symmetric. Then σ is a geodesic.

PROOF : Consider the power series expansion for $(\widetilde{R}_{1b}\cdot\widetilde{\rho})_{1d} = 0$ and use (7), (8), (10), (11) and (12) to compute the coefficient of r^{-3}. This gives the condition

$$(n-3)\,\kappa_u\,\delta_{bd.} = 0$$

which yields the result. ■

Theorem 4.2 — Let (M^n, g), $n \geq 3$, be a Riemannian manifold such that all small tubes about a curve σ are semi-parallel. Then σ is a geodesic.

PROOF : The coefficient of r^{-2} in the power series expansion of $(\tilde{R}_{1b} \cdot \sigma)_{1d} = 0$ follows at once from (7), (8), (13), (14), (15) and yields the condition $\kappa_u \delta_{bd} = 0$. Hence $\kappa_u = 0$ and so σ must be a geodesic. ■

Because of these two theorems we restrict ourselves to geodesic tubes. First, let (M^n, g) be a real space form. Then it is well-known that all small geodesic tubes P_r are locally symmetric. This also follows from (16), (17), (18). See also Berndt and Vanhecke[1] for a short proof using reflection theory. Then it also follows easily from (20) that all geodesic P_r are semi-parallel.

Now, we prove the converse theorems. We will do this under weaker hypotheses. Indeed, it is sufficient to consider the conditions $\tilde{R}_{XY} \cdot \tilde{\rho} = \tilde{R}_{XY} \cdot \sigma = 0$ for X, Y tangent to P_r and orthogonal to F_1, the parallel translate along γ of σ. Such vectors will be called horizontal.

Theorem 4.3 — Let (M^n, g), $n \geq 4$, be a Riemannian manifold such that for all small geodesic tubes P_r we have $\tilde{R}_{XY} \cdot \tilde{\rho} = 0$ for all horizontal vectors X, Y. Then (M, g) is of constant curvature.

PROOF : We use the notations of Section 2 and consider the power series expansion of $(\tilde{R}_{ab} \cdot \tilde{\rho})_{cc} = 0$ for a, b, $c \geq 3$. Then the coefficient of r^{-2} yields the condition

$$\delta_{ac}\left(\rho_{bc} - \frac{n-1}{3} R_{bucu}\right) - \delta_{bc}\left(\rho_{ac} - \frac{n-1}{3} R_{aucu}\right) = 0. \qquad \text{... (23)}$$

Now, since $n \geq 4$, we may take $a = c \neq b$ in (23). Then we get

$$3\rho_{bc} = (n-1) R_{bucu}. \qquad \text{... (24)}$$

Since this must hold for all P_r, we get by a summation procedure

$$(n-1)\,\rho_{bc} = 3(n-2)\,\rho_{bc}$$

and hence $\rho_{bc} = 0$. Then (24) yields $R_{bucu} = 0$ and so Cartan's theorem implies again that (M, g) is a real space form. ■

Theorem 4.4 — Let (M^n, g), $n \geq 4$, be a Riemannian manifold such that for all small geodesic tubes P_r we have $\tilde{R}_{XY} \cdot \sigma = 0$ for all horizontal vectors X, Y. Then (M, g) is of constant curvature.

PROOF : We proceed as in Theorem 4.3 and write down the power series expansion for $(\tilde{R}_{ab} \cdot \rho)_{cc} = 0$, a, b, $c \geq 3$. The coefficient of r^{-1} yields the condition

$$\delta_{ac} R_{cubu} - \delta_{bc} R_{cuau} = 0.$$

Since $n \geq 4$ we can take $a = c \neq b$ and this yields in a similar way as before the required result. ■

When $n = 3$, P_r is two-dimensional and hence there is only one horizontal direction. Hence $\tilde{R}_{XY} \cdot \sigma = 0$ always holds for horizontal X, Y. Therefore, we prove.

Theorem 4.5 — Let (M^n, g), $n \geq 3$, be a Riemannian manifold such that all small geodesic P_r are semi-parallel. Then (M, g) is of constant curvature.

PROOF : It follows from Theorem 4.4 that we may restrict to the case $n = 3$. Then we calculate the expansion for $(\tilde{R}_{13} \cdot \sigma)_{13} = 0$. Considering the coefficient of r^{-1} yields $R_{1313} - R_{1u1u} = 0$. This means that for any orthonormal triple (X, Y, Z) of tangent vectors of M we must have

$$R_{XYXY} - R_{XZXZ} = 0. \qquad \ldots (25)$$

Replacing Y by $\frac{1}{\sqrt{2}}(Y+Z)$ and Z by $\frac{1}{\sqrt{2}}(Y-Z)$ in (25) then yields

$$R_{XYXZ} = 0$$

which again implies the required result. ■

So, the above results yield :

Corollary 4.6 — Let (M^n, g), $n \geq 4$, be a Riemannian manifold. Then the following statements are equivalent for sufficiently small radii :

(i) (M, g) is a real space form;

(ii) all geodesic tubes are locally symmetric;

(iii) all geodesic tubes are semi-symmetric;

(iv) all geodesic tubes are Ricci-semi-symmetric;

(v) all geodesic tubes are semi-parallel.

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