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ON THE SPECTRUM AND FINE SPECTRUM OF THE COMPACT RHALY OPERATORS

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Rhaly[8] determined the spectrum of a Rhaly matrix M regarded as an operator on the Hilbert space l_2. It is the purpose of this paper to determine the spectrum of a Rhaly matrix M as an operator on the spaces c_0 and c, under the assumption that diag $\{M\} = o(1/n)$.

1. INTRODUCTION

Let l_∞, c, c_0 be the spaces of bounded sequences, convergent sequences, and null sequences $x = (x_n)$ respectively, normed by $\| x \| = \sup_n | x_n |$ and let $l_1 := \{x : \sum_n |x_n| < \infty\}$. If A is an infinite matrix and $x = (x_n)$ is a sequence, write $(Ax)_n := \sum_k a_{nk} x_k$ if this exists; Ax is the sequence $((Ax)_n)$.

The set of all eigenvalues and the spectrum of a bounded operator T on a Banach space X are denoted by $\pi_0(T, X)$ and $\sigma(T, X)$, respectively.

By Goldberg[3], if $T \in B(X)$, there are three possibilities for $R(T)$, the range of T : (I) $R(T) = X$; (II) $R(T) \neq \overline{R(T)} = X$; (III) $\overline{R(T)} \neq X$; and three possibilities for T^{-1}: (1) T^{-1} exists and is continuous; (2) T^{-1} exists but is discontinuous; (3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by : $I_1, I_2, \ldots, III_3$. If, for example, an operator is in state III_2, then $\overline{R(T)} \neq X$ and T^{-1} exists but is discontinuous.

Applying Goldberg's classification to $T_\lambda := \lambda I - T$, where λ is a complex number, we have three possibilities for T_λ : (I) T_λ is surjective; (II) $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$; (III) $\overline{R(T_\lambda)} \neq X$; and three possibilities for T_λ^{-1} : (1) T_λ is injective and T_λ^{-1} is bounded; (2) T_λ is injective and T_λ^{-1} is unbounded; (3) T_λ is not injective.

If λ is a complex number such that $T_\lambda \in II_1$, then λ belongs to the resolvent set $\rho(T,X)$ otherwise $\lambda \in \sigma(T,X)$. The further classification of $\sigma(T,X)$ gives rise to the fine spectrum of T. If for example, T_λ is in state III_2, then we write $\lambda \in III_2\ \sigma(T,X)$.

Given a sequence $a = (a_n)$ of scalars, the Rhaly matrix $M = R_a$ is the lower triangular matrix with constant row segments

$$M = \begin{bmatrix} a_0 & 0 & 0 & \dots \\ a_1 & a_1 & 0 & \dots \\ a_2 & a_2 & a_2 & \dots \\ \dots & \dots & \dots & \ddots \end{bmatrix}. \qquad \dots (1)$$

For $a = \left(\frac{1}{n+1}\right)$ the spectra of the Cesàro matrix on c_0 and c are studied in Reade[9] and Wenger[11], respectively. Rhaly[7] showed that if $a = \left(\frac{1}{(n+1)^s}\right)$ where $s > 1$, then M is a bounded operator on the Hilbert space l_2 of square summable sequences, and in another paper[8] he showed that M is bounded on l_2 if the a_n's are distinct and $\lim_n (n+1)a_n = 0$, and he also determined its l_2-spectrum and eigenvalues. Leibowitz[5] found the conditions for a Rhaly matrix to be bounded on c_0 or c.

In this study, we shall assume throughout that a is a strictly decreasing sequence of positive real numbers and that $L = \lim_n (n+1)a_n = 0$. Under these hypotheses, we determine the spectrum and fine spectrum of M on c_0 and c, and we give a Mercerian theorem.

We write $S = \{a_n : n = 0, 1, \dots\}$ for the range of a.

2. The Spectrum and Fine Spectrum of M on c_0 for $L = 0$

Theorem 1 — The Rhaly matrix M acts boundedly on c_0 iff $\{(n+1)\, a_n\}$ is bounded.

Proof : See Proposition 3.2 of Leibowitz[5].

Theorem 2 — If $L = 0$ [which is assumed henceforth], then M is a compact operator on c_0.

Proof : The proof is the same as in Proposition 3.1 of Leibowitz[5]. If M^r is the Rhaly matrix with diagonal sequence $(a_0, a_1, \dots, a_r, 0, 0, \dots)$, then the operator M^r is in $B(c_0)$, the dimension of $R(M^r)$ is finite for each r, and $\| M - M^r \| \leq (r+2)\, a_{r+1}$.

Thus

$$\lim_{r \to \infty} \| M - M^r \| = 0.$$

Hence M is a compact operator on c_0 if $L = 0$.

Theorem 3 — $\pi_0(M, c_0) = S$.

PROOF : If $Mx = \lambda x$ then $(a_0 - \lambda) x_0 = 0$ and $(\lambda a_n^{-1} - 1) x_n = a_{n-1}^{-1} \lambda x_{n-1}$ for all $n \geq 1$. So $0 \notin \pi_0(M, c_0)$. If m is the smallest integer for which $x_m \neq 0$, then $\lambda = a_m$ and

$$x_n = \left(\prod_{j=m+1}^{n} \frac{\lambda a_{j-1}^{-1}}{\lambda a_j^{-1} - 1} \right) x_m \qquad \text{... (2)}$$

for $n \geq m + 1$. From (2), we conclude that the eigenvalues of M are simple.

Since $l_2 \subset c_0$, we want to determine whether $\lambda = a_m$ will result in an x for which $\sum |x_n|^2$ converges. As in Theorem 2.4 of Rhaly[8], we turn to Kummer's test with $p_n = 1/na_n^2$ [Olmsted[6], p. 395) and find that for $n \geq m$,

$$\frac{|x_n|^2}{|x_{n+1}|^2} = \frac{(\lambda - a_{n+1})^2}{\lambda^2} \frac{a_n^2}{a_{n+1}^2}. \qquad \text{... (3)}$$

Since $$\lim_{n \to \infty} \left(p_n \frac{|x_n|^2}{|x_{n+1}|^2} - p_{n+1} \right)$$

$$= \lim_{n \to \infty} \frac{\lambda^2 - 2(n+1) a_{n+1} \lambda + (n+1) a_{n+1}^2}{n(n+1) a_{n+1}^2 \lambda^2} = \infty, \qquad \text{... (4)}$$

we have $x \in l_2$. Thus when $\lambda = a_m$, $x \in c_0$. So $\pi_0 (M, c_0) = S$.

Theorem 4 — $\pi_0 (M^*, c_0^* = l_1) = S$.

PROOF : The argument proceeds much as in Theorem 2.3 of Rhaly[8]. It is clear that the matrix of M^* is the transpose of the matrix of M. If $M^*x = \lambda x$, we have $x_n = \lambda a_n^{-1} (x_n - x_{n+1})$.

Thus $0 \notin \pi_0(M^*, c_0)$ ($\lambda = 0$ implies that all $x_n = 0$) and hence

$$M^*x = \lambda x \Leftrightarrow x_{n+1} = \left(1 - \frac{a_n}{\lambda} \right) x_n \text{ for all } n \geq 0.$$

It follows that every $a_m \in S$ is an eigenvalue of M^*, that every eigenvalue λ is simple, and that if $\lambda \in \pi_0 (M^*, c_0)$, then

$$x_n = \prod_{j=0}^{n-1} (1 - a_j/\lambda) x_0. \qquad \text{... (5)}$$

Now we show that if λ is not one of the a_n's then λ is not an eigenvalue. If x satisfies (5), then

$$\frac{|x_n|}{|x_{n+1}|}-1=\frac{1-|1-a_n/\lambda|}{|1-a_n/\lambda|}=\frac{1-|1-a_n/\lambda|^2}{|1-a_n/\lambda|(1+|1-a_n/\lambda|)}$$

$$=\frac{2\,\mathrm{Re}\,(a_n/\lambda)-a_n{}^2/|\lambda|^2}{|1-a_n/\lambda|(1+|1-a_n/\lambda|)}.$$

So the Raabe's test limit is $\lim_{n\to\infty} n\left(\frac{|x_n|}{|x_{n+1}|}-1\right)=0<1.$

Thus $x\notin l_1$ and there are no eigenvalues except the a_n's.

Now we can compute $\sigma(M, c_0)$ by using some properties of compact operators.

Theorem 5 — $\sigma(M, c_0) = S \cup \{0\}$.

PROOF : Since M is a compact operator, $0 \in \sigma(M, c_0)$ (Rudin[10], p. 99), and if $0 \neq \lambda \in \sigma(M, c_0)$ then $\lambda \in \pi_0(M, c_0) \cap \pi_0(M^*, c_0)$. Combining these results with Theorems 3 and 4, we have the proof.

Theorem 6 — $0 \in II_2\, \sigma(M, c_0)$.

PROOF : Since $0 \notin S = \pi_0(M, c_0)$ by Theorem 3, M^{-1} exists. But $0 \in \sigma(M, c_0)$, so M^{-1} is discontinuous. Consequently, $M \in (2)$.

Next, we show that $M \in II$; that is, M has a dense, proper range. Since $0 \notin \pi_0(M^*, c_0)$, the operator M^* is one to one. By Goldberg[3] (p. 59), $\overline{R(M)} = c_0$.

Now let us show that $R(M) \neq c_0$.

The inverse of M has matrix entries

$$b_{nk}=\begin{cases} a_n^{-1} & \text{if} \quad k=n \\ -a_{n-1}^{-1} & \text{if} \quad k=n-1 \\ 0 & \text{otherwise}\end{cases} \qquad \ldots (6)$$

(See [Leibowitz[5], p. 279].) Let $y = (y_n) = \{(-1)^n a_n\}$. If $Mx = y$ then by (6), $x_n = a_n^{-1} y_n - a_{n-1}^{-1} y_{n-1} = 2(-1)^n$ for each $n \geq 1$, and so $x \notin c_0$. Since $y \in c_0$, $R(M) \neq c_0$. Thus $M \in II_2$, and so $0 \in II_2\, \sigma(M, c_0)$.

Theorem 7 — If $\lambda = a_m$ $(m = 0, 1, \ldots)$, then $\lambda \in III_3\, \sigma(M, c_0)$.

PROOF : Let $T_\lambda = \lambda I - M$. By Theorem 3, T_λ is not one to one, so $T_\lambda \in (3)$.

By Theorem 4, $T_\lambda{}^*$ is not one to one, so T_λ does not have a dense range; i.e., $T_\lambda \in III$ (Goldberg[3], p. 59). Thus $T_\lambda \in III_3$ and so $\lambda \in III_3\, \sigma(M, c_0)$.

3. The Spectrum and the Fine Spectrum of M on c for $L = 0$.

Theorem 8 — The Rhaly matrix M acts boundedly on c iff $\{(n + 1)a_n\}$ converges.

PROOF : See Proposition 3.3 of Leibowitz[5].

Theorem 9 — If $L = 0$ (which is assumed henceforth), then M is a compact operator on c.

PROOF : The proof follows the same reasoning as in Theorem 2.

Theorem 10 — $\pi_0(M, c) = S$.

PROOF : The proof follows the same reasoning as in Theorem 3.

Theorem 11 — $\pi_0(M^*, c^* \cong l_1) = S \cup \{0\}$.

PROOF : For the proof, we need a lemma. In the Lemma, we do not assume that $L = 0$.

The proof uses a representation from (Wilansky[12], p. 267):

If $T: c \to c$ is a bounded matrix operator with matrix A, then $T^*: c^* \to c^*$ acting on $\mathbb{C} \oplus_1 l_1$ has matrix of the form $\begin{bmatrix} \chi & 0 \\ b & A^t \end{bmatrix}$ where χ is the limit of the sequence of row sums of A minus the sum of the limits of the columns of A, and b is the column vector whose kth entry is the limit of the kth column of A for each k.

Lemma — Let $M: c \to c$ be a Rhaly matrix. Then the matrix of $M^* \in B(l_1)$ is

$$M^* = \begin{bmatrix} L & 0 \\ 0 & M^t \end{bmatrix}. \qquad \text{... (7)}$$

PROOF : For each k, $b_k = 0$ since $a \in c_0$ and $\chi = L$ for the same reason.

Thus

$$M^* = \begin{bmatrix} L & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & 0 & a_1 & a_2 & a_3 & \cdots \\ 0 & 0 & 0 & a_2 & a_3 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \ddots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \end{bmatrix}$$

We can now prove Theorem 11. Since $\lim (n+1)a_n = 0$ by our assumption, $L = 0$ in (7).

If we have $M^*x = \lambda x$ then $\lambda x_0 = 0$ and $x_n = a_{n-1}^{-1} (\lambda x_n - \lambda x_{n+1})$ for $n \geq 1$. So $\lambda = 0$ is an eigenvalue, with $x = x_0 e_0$ the corresponding eigenvector (for $x_0 \neq 0$). If $\lambda \neq 0$, then

$$x_n = \prod_{j=0}^{n-2} \left(1 - a_j/\lambda\right) x_1 \qquad \text{... (8)}$$

for $n > 1$. The rest of the argument is now a minor modification of the proof of Theorem 4.

Theorem 12 — $\sigma(M, c) = S \cup \{0\}$.

PROOF : The proof follows the same reasoning as in Theorem 5.

Theorem 13 — $\sigma(M, l_\infty) = S \cup \{0\}$.

PROOF : By Cartlidge[2], if a matrix operator A is bounded on c then $\sigma(A, c) = \sigma(A, l_\infty)$. So this is a corollary of Theorem 12.

Theorem 14 — $0 \in III_2\, \sigma(M, c)$.

PROOF : By Theorem 10, the inverse operator M^{-1} exists. Formula (7) shows that every $y \in R(M^*)$ has first component zero, so M^* is not onto. Hence by Goldberg[3] (p. 59), M^{-1} is not bounded.

On the other hand, 0 is an eigenvalue of M^* and so M^* is not one to one. So M does not have a dense range. Thus we have proved that $M \in III_2$.

Theorem 15 — If $\lambda = a_m$ (m = 0, 1, ...), then $\lambda \in III_3\, \sigma(M, c_0)$.

PROOF : The proof follows the same reasoning as in Theorem 7.

Theorem 16 — Suppose that $\lambda \neq 0$ and that $\lambda \neq a_m/(a_m - 1)$ for any m. Then the convergence field of $A = \lambda I + (1 - \lambda) M$ is c.

PROOF : If $\lambda = 1$, there is nothing to prove, so suppose that $\lambda \neq 1$. By Theorem 11 and the choice of λ, $\frac{\lambda}{\lambda - 1} I - M$ has a bounded inverse in $B(c)$.

So $A^{-1} = (\lambda - 1)^{-1} \left[\frac{\lambda}{\lambda - 1} I - M\right]^{-1} \in B(c)$. Since A is a triangle and $A \in B(c)$, the fact that A^{-1} is also conservative implies that $c_A = c$ (Wilansky[12], p. 14).

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