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UNIFORMLY VALID EXPANSION FOR FINITE AMPLITUDE SURFACE WAVES IN ELECTROHYDRODYNAMICS IN A FLUID OF INFINITE DEPTH

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A weakly nonlinear theory of capillary gravity waves on the surface of a fluid of infinite depth in the presence of an electric field is investigated. We have used slow scales perpendicular to the free surface in addition to the usual slow distance and time scales to obtain uniformly valid perturbation solutions. The equations governing the evolution of the amplitude are obtained for the progressive as well as standing waves. The stability analysis for progressive waves reveals that the wave train of constant amplitude is unstable against modulations. It is interesting to observe that there exist three nonlinear stable as well as unstable regions. We also obtain the nonlinear cut-off wave number which separates the region of stability from that of instability. We find that the electric field has a destabilizing influence on the cut-off wave number.

1. INTRODUCTION

The subject of electrohydrodynamics has drawn considerable interest over the past few decades and has a wide range of interest in various physical situations. Early studies of electrohydrodynamic effects on the motion of dielectric fluids were examined by Lord Rayleigh[1] who considered the effect of surface charges on the vibrations of spherical bubbles. Electrohydrodynamic surface instabilities on the interface of a fluid were successfully studied by Melcher[2-5]. Michael[6] considered the effects of electrostatic forces on the stability of wave motion at the surface of highly conducting fluids using the Poincaré Lighthill-Kuo method. In the first order (i.e. the linear theory) this method reveals that electrostatic forces have a destabilizing effect on the system. In the higher-order approximation (i.e. the non-linear region) this result is also valid. However, as is the case in most expansion techniques, the higher order results do not hold near the cut-off wave number due to the singularities in the

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coefficients of the resulting equations. We always need to modify the analysis for wave numbers near the cut-off wave number; this was examined by Shivamoggi[7]. In the long wavelength approximations, Shivamoggi's analysis reveals that the waves at the free surface grow even at $k = k_c$, the linear cut-off wave number; thus the electrostatic forces continue to have a destabilizing effect on the wave motion at the surface in the nonlinear case.

The method of multiple scales to investigate the nonlinear modulation problems has been successfully used by various authors to examine a variety of problems (Kakutani *et al.*[8]). Kant *et al.*[9] studied the stability of weakly nonlinear dispersive waves on the surface of a fluid layer of finite depth in the presence of an externally applied electric field. They derived the nonlinear Schrödinger equation (NLSE) for the complex amplitude of quasimonochromatic travelling waves and showed that the wavetrain of constant amplitude is unstable against modulation. With certain modifications in the analysis, they showed that near $k = k_c$, the nonlinear cut-off wave number depends sensitively on the initial conditions with respect to the slower scales in time and the nonlinearity changes only slightly the range of stable wave numbers.

It was first pointed out by Lardner[10] that is not necessary to include an aperiodic term in the lowest order velocity potential in order to avoid secular terms in the higher order potentials. In fact for the medium of infinite depth, such a term is only needed in the second-order approximation. It is also observed that in particular the stability criterion can be directly obtained by taking the depth of the fluid layer tending to infinity in the criterion for the finite layer. However one fails to obtain a uniformly valid velocity potential in the body of the fluid by taking such a limit. In order to overcome this difficulty, it is necessary to introduce a slow scale perpendicular to free surface in addition to the usual slow scales in distance and time to obtain uniformly valid perturbation solutions. Thus the analysis given by Kant *et al.*[9] has the limitation that one cannot obtain the case of a layer of infinite depth as its limiting case without introducing non-uniformities in the solutions for the various physical quantities. The purpose of the present paper is to use an extended method of multiple scales to investigate the stability of weakly nonlinear waves on the surface of fluid layer of infinite depth in the presence of an applied electric field and to obtain uniformly valid solutions in the entire region.

2. Formulation of the Problem

We consider wave motions at the surface of an incompressible, inviscid and perfectly conducting fluid of infinite depth supported by a conducting electrode maintained at a fixed potential V_0 at a distance $y = b$ above the surface. We suppose the motion is irrotational and confined to the x- and y-directions. If Ω denotes the velocity potential inside the fluid $[-\infty < y < \eta]$ and ϕ denotes the electrostatic potential above the fluid $[\eta < y < b]$, then both Ω and ϕ satisfy Laplace's equation. Here $y = \eta(x, t)$ denotes the elevation of the free surface at time t. The various physical quantities are normalized with respect to the characteristic length $L = (T/\rho g)^{1/2}$ and time $t_c = (L/g)^{1/2}$, where ρ, T are the density and the surface tension of the fluid respectively and g is the acceleration due to gravity. The non-dimensional potential functions are taken to be $g^{-1/2}\ L^{3/2}\Omega$ and $\phi_c\,\phi$, where ϕ_c = $(L^3\, g\, \rho)^{-1/2}\, V_0$. The boundary conditions at the free interface $y = \eta(x, t)$ are :

$$\frac{\partial \eta}{\partial t} + \frac{\partial \Omega}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \Omega}{\partial y}, \qquad \text{... (1)}$$

$$\phi = 0, \qquad \text{... (2)}$$

$$\frac{\phi_c^2}{8\pi}\left(\frac{\partial \phi}{\partial \tilde{n}}\right)^2 - \frac{1}{2}\left[\left(\frac{\partial \Omega}{\partial x}\right)^2 + \left(\frac{\partial \Omega}{\partial y}\right)^2\right] + \frac{\partial^2 \eta}{\partial x^2}\left[1 + \left(\frac{\partial \eta}{\partial x}\right)^2\right]^{-3/2} - \eta - \frac{\partial \Omega}{\partial t} = \text{constant}, \qquad \text{... (3)}$$

where $\tilde{n}$ denotes the unit normal drawn outward to $y = \eta(x, t)$. It may be noted that (see Lamb[12]),

$$\left(\frac{\partial \phi}{\partial \tilde{n}}\right)^2 = \left[\frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial y}\right]^2\left[1 + \left(\frac{\partial \eta}{\partial x}\right)^2\right]^{-1}. \qquad \text{... (4)}$$

Away from the interface, the motions vanish; that is $|\nabla\Omega| \to 0$ as $y \to -\infty$. Also the electrostatic potential $\phi = 1$ at $y = b$.

We wish to investigate motions which are finite perturbations of the steady state. We use the derivative expansion method with multiple scales and assume that all the physical quantities have uniformlly valid asymptotic expansions in powers of a small parameter ε, characterizing the steepness ratio of the wave. We write for any variable

$$f(x, y, t) = \sum_{n=1}^{3} \varepsilon^n f_n + O(\varepsilon^4), \qquad \text{... (5)}$$

where f is any of the variable $\Omega, \phi - \phi_0$ or $\eta(x, t)$, and is a function of slow scales variables in space and time, namely

$$x_n = \varepsilon^n x,\quad y_n = \varepsilon^n y,\quad t^n = \varepsilon^n t,\quad n = 0, 1, 2, \ldots \qquad \text{... (6)}$$

and we allow Ω_n, ϕ_n, η_n to depend in general on all these variables. While writing the expansion for η, it will be noted that it depends only on x and t and not on y. To evaluate boundary conditions (1)-(3), we use Maclaurin's series expansion at $y = 0$ for the quantities involved. This leads to equations of various orders :

Order ε :

$$\nabla_0^2\,\Omega_1 = 0;\quad \nabla_0^2\,\phi_1 = 0, \qquad \text{... (7)}$$

with boundary conditions

$$L_0(\eta_1, \Omega_1) = 0; \quad U_1 = 0; \quad M_0(\phi_1, \Omega_1, \eta_1) = 0 \text{ at } y = 0 \qquad \dots (8)$$

$$\phi_1 = 0, \text{ at } y = b; \; N_0(\Omega_1) = 0, \text{ when } y \to -\infty. \qquad \dots (9)$$

Order ε^2 :

$$\nabla_0^2 \Omega_2 = -R_0 \Omega_1; \quad \nabla_0^2 \phi_2 = -R_0 \phi_1, \qquad \dots (10)$$

with boundary conditions

$$L_0(\eta_2, \Omega_2) = \left[\Delta_{11} + \eta_1 \Delta_{02} - (D_{01}\eta_1) D_{01} \right] \Omega_1 - \Delta_1 \eta_1,$$

$$U_2 = -S_0(\phi_1),$$

$$M_0(\phi_2, \Omega_2, \eta_2) = [\Delta_1 + \eta_1 \Delta_0 \Delta_{01}] \Omega_1 - 2D_{01} D_{11} \eta_1 - \alpha\, b\, \Delta_{11} \phi_1$$

$$+ \frac{1}{2}\left[\left(D_{01}\Omega_1\right)^2 + \left(D_{01}\Omega_1\right)^2 \right] - \frac{\alpha}{2}\left[\left(b\Delta_{01}\phi_1\right)^2 \right.$$

$$\left. + 2b\eta_1 \Delta_{02}\phi_1 - \left(D_{01}\eta_1\right)\left(2bD_{01}\phi_1 + D_{01}\eta_1\right) \right] \text{ at } y = 0 \qquad \dots (11)$$

$$\phi_2 = 0, \text{ at } y = b; \; N_0(\Omega_2) = 0, \text{ when } y \to -\infty. \qquad \dots (12)$$

Order ε^3 :

$$\nabla_0^2 \Omega_3 = -R_0 \Omega_2; \quad \nabla_0^2 \phi_3 = -R_0 \phi_2, \qquad \dots (13)$$

with boundary conditions

$$L_0(\eta_3, \Omega_3) = -\nabla_1 \eta_2 - \nabla_2 \eta_1 + \left[\eta_1 \Delta_{02} + \Delta_{11} \right] \Omega_2$$

$$+ \left[\eta_2 \Delta_{02} + \frac{\eta_1^2}{2} \Delta_{03} + \Delta_{21} + 2\eta_1 \Delta_{01} \Delta_{11} \right] \Omega_1$$

$$- \left(D_{01}\Omega_1\right)\left[D_{11}\eta_1 + D_{01}\eta_2 \right]$$

$$- \left(D_{01}\eta_1\right) \left[D_0 \Omega_2 + \left(\eta_1 D_{01} \Delta_{01} + D_{11} \right) \Omega_1 \right],$$

$$U_3 = -S_0(\phi_2) - \eta_2 \Delta_{01} \phi_1,$$

$$M_0(\phi_3, \Omega_3, \eta_3) = [\Delta_1 + \eta_1 \Delta_0 \Delta_{01}] \Omega_2 + [\Delta_2 + \eta_1 \Delta_1 \Delta_{01} + \eta_1 \Delta_0 \Delta_{11}$$

$$+ \eta_2 \Delta_0 \Delta_{01} + \frac{\eta_1^2}{2} \Delta_0 \Delta_{02} \Big] \Omega_1 + (D_{01}\Omega_1)\,[D_{01}\Omega_2 + D_{11}\Omega_1$$

$$+ \eta_1 D_{01} \Delta_{01} \Omega_1] + (\Delta_{01}\Omega_1)\,[\Delta_{01}\Omega_2 + \Delta_{11}\Omega_1 + \eta_1 \Delta_{02}\Omega_1]$$

$$- \left[D_{11} + 2D_{01} D_{21} - \frac{3}{2}\left(D_{01}\eta_1\right)^2 D_{02} \right] \eta_1 - 2 D_{01} D_{11} \eta_2 -$$

(Equation continued on page 811)

$$M_0\left(\phi_3, \Omega_3, \eta_3\right) = \frac{\partial c_\Omega}{\partial t_1} + \frac{\partial D_1}{\partial t_2} - \alpha\left[-ib\frac{\partial c_\phi}{\partial x_1} + \frac{ik}{S_1^2}(1+kb)\left\{ H\frac{\partial \overline{H}}{\partial x_1} - \overline{H}\frac{\partial H}{\partial x_1}\right\}\right]$$

$$+ \left[\frac{1}{k}\frac{\partial^2 H}{\partial t_1^2} - \frac{i\omega}{k}\frac{\partial H}{\partial t_2} - 2ik\frac{\partial H}{\partial x_2} - \left(1 + 2kb(1+\sigma_1)\right)\frac{\partial^2 H}{\partial x_1^2}\right.$$

$$+ \frac{\omega}{k^2}\left(1 - kb(1+\sigma_1)\right)\frac{\partial^2 H}{\partial x_1 \partial t_1} - i\omega k p_2 H^2 \overline{H} - \omega^2 c_\eta H$$

$$+ \frac{3}{2}(\omega^2 k - k^4) H^2 \overline{H} - \alpha\{ kb\sigma_1 c_\phi H + b(1+\sigma_1)\frac{\partial^2 H}{\partial x_1^2}$$

$$- \frac{ik^3}{\omega}(3 - 2\sigma_1\sigma_2) p_2 H^2 \overline{H} - k^3 (3 - 2\sigma_1\sigma_2 + \frac{9}{2}\sigma_1$$

$$\left.- 2\sigma_1^2\sigma_2) H^2\overline{H} \;\}\right] e^{i\theta_0} + c.c. + NST, \text{ at } y = 0 \quad \ldots (40)$$

$$\phi_3 = 0 \text{ at } y = b, \quad \ldots (41)$$

$$\frac{\partial \Omega_3}{\partial y_0} = -\frac{\partial D_1}{\partial y_2} - \frac{\partial c_\Omega}{\partial y_1}, \text{ when } y \to -\infty \quad \ldots (42)$$

where *NST* stands for non-secular terms.

Since the homogeneous part of the third-order problem has a nontrivial solution, the inhomogeneous part has a solution if and only if, a solvability condition is satisfied. This condition yields

$$\frac{2i\omega}{k}\frac{\partial H}{\partial t_2} + i\left[\alpha(1+\sigma_1)\left\{kb(\sigma_1 - 1) - 1\right\} + \alpha + 2k + \frac{\omega^2}{k^2}\right]\frac{\partial H}{\partial x_2}$$

$$- \frac{1}{k}\frac{\partial^2 H}{\partial t_1^2} - \frac{2\omega}{k^2}\left(1 - kb(1+\sigma_1)\right)\frac{\partial^2 H}{\partial x_1 \partial t_1} - \left[\alpha b(1+\sigma_1)\left(1 + kb(1-\sigma_1)\right)\right.$$

$$\left. + \frac{\omega^2}{k^3} - 1 - kb(1+\sigma_1)\left\{2 + \frac{\omega^3}{k^3}\right\}\right]\frac{\partial^2 H}{\partial x_1^2}$$

$$= \alpha k^2(1-\sigma_1^2) c_\eta H + (1 + k^2 - 2\alpha k\sigma_1) b\, c_\phi H + \omega H \frac{\partial D_1}{\partial x_1}$$

$$+ k\left[4k + \frac{5}{2}k^3 + \alpha k^2\left\{3 + 2\sigma_1(1 - \sigma_2 - \sigma_1\sigma_2)\right\} - \right.$$

(Equation continued on page 816)

$$-\alpha k^2 \sigma_1 \left\{ \sigma_1 + 2\sigma_2(1+\sigma_1) \right\} + \frac{ikp_2}{\omega} \left\{ 2(1+k^2) + k\left(3 - 2\sigma_1(\alpha+\sigma_2)\right) \right.$$

$$\left. \left. - \alpha\,\sigma_1(\sigma_1 + 2\sigma_2) \right\} \right] H^2 \overline{H}. \quad \text{... (43)}$$

The function D_1 satisfies the boundary conditions (31) and (42). It is consistent with these conditions to take $D_1 = 0$, which is in contrast with the corresponding situation for bounded media. From the third order boundary conditions, we can obtain c_Ω but is not needed in eqn. (43). Equations (27) and (43) describe the evolution of the two dimensional wave packets near and away from the cut-off wave number separating stable from unstable motions.

3.1. *Modulational Instability of Travelling Waves*

We now discuss the case when the frequency is different from zero, that is, $\omega \neq 0$. Introducing the transformations $\xi = \varepsilon^{-1}(x_2 - V_g t_2) = x_1 - V_g t_1 = \varepsilon(x - V_g t)$ and $\tau = t_2 = \varepsilon t_1 = \varepsilon^2 t$, we obtain the well known non-linear Schrödinger equation from eqns. (27) and (43) :

$$i\frac{\partial H}{\partial \tau} + P\frac{\partial^2 H}{\partial \xi^2} = QH^2\overline{H}, \quad \text{... (44)}$$

where

$$P = \frac{1}{2\omega}\left[-V_g^2 + 3k - \alpha\sigma_1 + \alpha kb(\sigma_1^2 - 1)(2 - kb\,\sigma_1) \right], \quad \text{... (45)}$$

$$Q = \frac{k^3}{4\omega}\left[8 + 5k^2 + 2\alpha k\left(3 + \sigma_1(2 - \sigma_1 - \sigma_2 - 3\sigma_1\sigma_2)\right) \right.$$

$$+ \frac{2k^2}{D(2\omega, 2k)}\left\{ 6k + 2\alpha\sigma_1 + 9\sigma_1 - 4\alpha\sigma_2(1+\sigma_1) - \alpha(\sigma_1^2 - 3) \right\}$$

$$\left\{ 2(1+k^2) + \alpha k(3 - \sigma_1(2 + \sigma_1 + 9\sigma_2)) \right\}$$

$$\left. + \frac{\alpha}{2\omega b(\alpha - b)}\left\{ \alpha(2+kb)\left(2\sigma_1 + kb(\sigma_1^2 - 1)\right) - 4\sigma_1(\alpha - b) \right\} \right]. \quad \text{... (46)}$$

It may appear that eqn. (44) can be obtained from the results of finite depth by taking the limit of infinite depth. However this limit cannot be taken readily without introducing non-uniformities into the perturbation series. It is, therefore, to be emphasized that in an infinite medium, the nonlinearity does not impose a slow steady flow in the first order ($D_1 = 0$); it does so in the second order (C_Ω # 0) solutions. It is also known that the solutions of eqn. (44) are unstable against modulation if $PQ < 0$. From numerical calculations, Fig. 1 indicates the plot of wave number k versus the depth b (for $\alpha = 3$) at which the modulational instability sets in. It is interesting to note that when $\alpha < b$, the wave number k_m decreases with the increase in b, while it has an opposite effect in the region for $\alpha > b$. This is due to the fact that Q changes its sign in eqn. (46). The coefficient Q in eqn. (46) becomes

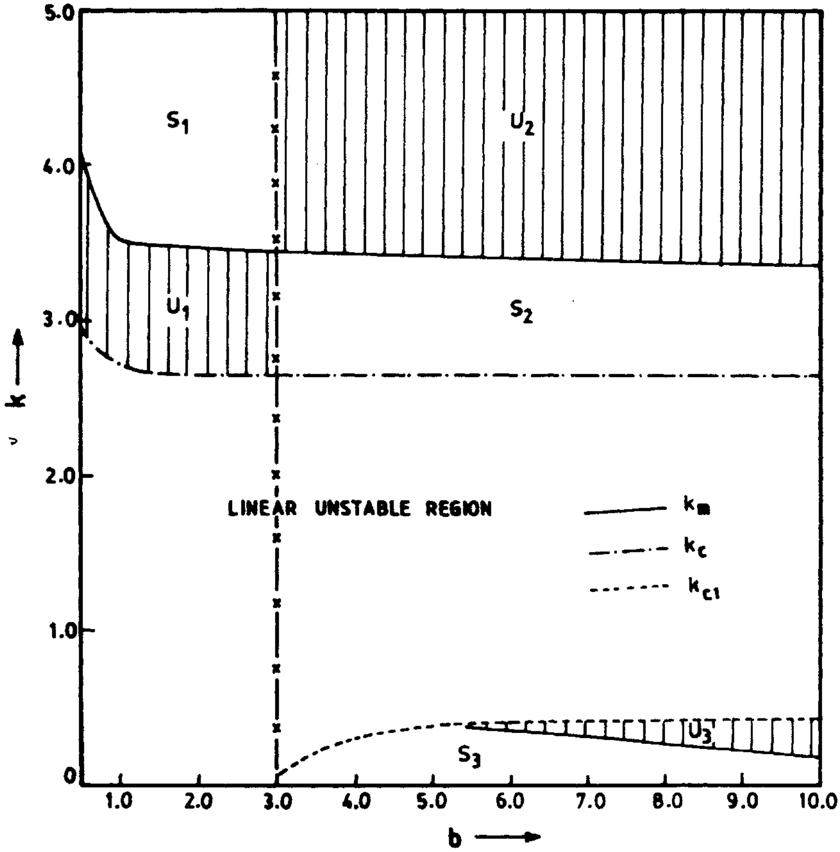

FIG. 1. The critical wave number k_m is plotted against b for $\alpha = 3$. The regions S_i's and U_i's represent the various regions of modulational stability and instability respectively. The analysis is not valid along the curve $-\times-\times-$ in the nonlinear theory.

unbounded as α approaches b, therefore, the analysis is not valid near $\alpha = b$. Similar behaviour of $(k - b)$ plot is found for other value of α.

3.2. *Nonlinear Cut-off Wave Number*

We now discuss amplitude modulation of waves near the linear cut-off wave number k_c. In this case $\omega \sim 0$. As $\omega \to 0$, $V_g \to \infty$. Thus eqn. (27) yields

$$\frac{\partial H}{\partial x_1} = 0. \qquad \text{... (47)}$$

From eqns. (44) and (47), we obtain

$$i\frac{\partial H}{\partial X} + \hat{P}\frac{\partial^2 H}{\partial t_1^2} = \hat{Q}\, H^2 \bar{H}, \qquad \text{... (48)}$$

where

$$X = t_1 - \left(\frac{dk}{d\omega} \right) x_1, \qquad \text{... (49)}$$

$$\hat{P} = -\left[k_c^2 - 1 + \alpha b k_c^2 (\sigma_{1c}^2 - 1) \right]^{-1}, \qquad \text{... (50)}$$

$$\hat{Q} = -k_c^3 \left[k_c^2 - 1 + \alpha b k_c^2 (\sigma_{1c}^2 - 1) \right]^{-1}$$
$$\times \left[\frac{\alpha}{b(\alpha - b)} \left\{ \alpha \left(2\sigma_{1c} + k_c b(\sigma_{1c}^2 - 1) \right)^2 - 4(\alpha - b) \sigma_{1c}^2 \right\} + \frac{3}{2} + \alpha k_c \left(\frac{9}{2} \sigma_{1c} - 4\sigma_{1c}^2 \sigma_{2c} \right) - \frac{\alpha^2 k_c^2}{2} \left\{ \sigma_{1c}^2 - 3 + 4\sigma_{1c} \sigma_{2c} \right\}^2 \left\{ 1 + 4k_c^2 - 2\alpha k_c \sigma_{2c} \right\}^{-1} \right] \qquad \text{... (51)}$$

$$\sigma_{nc} = \coth\,(nk_c b). \qquad \text{... (52)}$$

Equation (48) is a NSLE in which the roles of space and time variables are interchanged. In order to discuss the wave train solutions of constant amplitude, we write,

$$H = H_0 \exp\,[i(KX - \Lambda t_1)], \qquad \text{... (53)}$$

where H_0 is a constant. Equations (48) and (53) yields

$$\Lambda^2 = -\frac{K + \hat{Q} \,|\, H_0 \,|^2}{\hat{P}}. \qquad \text{... (54)}$$

If $\hat{P}$ and $\hat{Q}$ are both negative, then for Λ to be imaginary, we require $K < |\hat{Q}|\,|H_0|^2$. The nonlinear cut-off wave number is therefore given by

$$K_n = k_c + \frac{\varepsilon^2 k_c^2 \,|\, H_0 \,|^2}{k_c^2 - 1 + \alpha b k_c^2 (\sigma_{1c}^2 - 1)}$$
$$\times \left[\frac{\alpha}{b(\alpha - b)} \left\{ \alpha \left(2\sigma_{1c} + k_c b(\sigma_{1c}^2 - 1) \right)^2 - 4\,(\alpha - b)\, \sigma_{1c}^2 \right\} + \frac{3}{2} + \alpha k_c \left(\frac{9}{2} \sigma_{1c} - 4\sigma_{1c}^2 \sigma_{2c} \right) - \frac{\alpha^2 k_c^2 \,[\sigma_{1c}^2 - 3 + 4\sigma_{1c} \sigma_{2c}]^2}{2(1 + 4k_c^2 - 2\alpha k_c \sigma_{2c})} \right]. \qquad \text{... (55)}$$

The nonlinear correction to the wave number given on the right hand side of this equation can be positive or negative depending upon the signs of $\hat{P}$ and $\hat{Q}$, thus resulting in stability or instability. For $\alpha = 3$, and $k = k_c$ it is found that $\hat{P}$ is always negative while $\hat{Q} > 0$ for $b < 3$ and $\hat{Q} < 0$ for $b > 3$; thus implying that for all values of K the wave train of constant amplitude are always unstable near the linear cut-off wave number for $b < 3$. On the other hand for $b > 3$ the nonlinear cut-off wave number is given by $k_n = k_c - \varepsilon^2 \hat{Q} |H_0|^2$. While near the cut-off wave number k_{c1} for $\alpha = 3$, $\hat{P}$ is always positive and $\hat{Q}$ changes its sign from negative to positive for $b > 5.4$. This, implies that the wave trains of constant amplitudes are always unstable near the linear cut-off wave number k_{c1} for $b > 5.4$ and $\alpha = 3$. On the other hand $\Omega^2 < 0$ only for $K > \hat{Q} |H_0|^2$, that is wave trains in this case are stable near the cut-off wave number k_{c1} for all values of $K < |\hat{Q}| |H_0|^2$ and unstable otherwise. The nonlinearity slightly changes the range of unstable waves, and the band width of the spectrum is of $O(\varepsilon^2)$ in the wave number space for standing waves.

4. Discussion

We have used the method of multiple scales to derive the nonlinear Schrödinger equations describing the evolution of a two dimensional wave packet on the surface of a fluid layer of infinite depth in the presence of an applied electric field, taking into account the acceleration due to gravity and the surface tension of the fluid. The first NLSE for the complex amplitude of quasimonochromatic travelling wave contains only first derivatives in time and is examined to investigate the stability of finite amplitude uniform wave trains. The stability criterion is a function of two dimensionless parameters $k = K(T/\rho g)^{1/2}$ and $\alpha = \phi_c^2/(4\pi b^2)$. For an electric field parameter $\alpha = 3$, ther $k - b$ plane is divided by four transition curves into three regions of stability and three regions of instability. The analysis is not valid for $\alpha = b$, because $Q \to \infty$. The result also indicates that although the NLSE so obtained is valid for wide range of wave numbers, it does not hold near the cut-off wave numbers separating stable from unstable motions, as is common in all such analyses (see Kawahara).

The carrier wave is not a progressive wave but instead a standing wave near $k = k_c$ and $\omega = 0$. For this case the group velocity, the group velocity rate and the interaction parameter become unbounded. A slightly modified analysis reveals that the complex amplitude of the standing wave near the cut-off wave number is governed by a nonlinear Scrödinger equation in which the role of time and space variables are interchanged. This nonlinear Scrödinger eqation is used to determine the dependence of the cut-off wave number on the disturbance amplitude. The nonlinear cut-off wave number depends sensitively on the initial conditions with respect to the slower scale in time t_1 and the nonlinearity changes slightly the range of stable wave numbers.

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