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THE LINEAR DIOPHANTINE EQUATION $ax + by + cz = e$ IN $Q(\sqrt{5})$

HÜSEYIN ALTINDIS AND MUZAFFER ATASOY

Department of Mathematics, Erciyes University, 38039 *Kayseri, Turkey*

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The solution of the linear diophantine equation

$$a_0x + a_1y + a_2z = d$$

in rational integers is very well known, where a_0, a_1, a_2, d are integers. The purpose of this note is to discuss the solutions of this equation when a_0, a_1, a_2, d are integers in $Q(\sqrt{5})$ and the solutions are integers in $Q(\sqrt{5})$.

INTRODUCTION

Definition 1 — A continued fraction expansion of a_0/a_1 is

$$\frac{a_0}{a_1} = r_0 + \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cdots + \cfrac{1}{r_n}}},$$

where $r_0, r_1, r_2, \ldots, r_n$ are rational integers with $r_1, r_2, \ldots, r_n$ positive. The continued fraction is denoted by $[r_0, r_1, r_2, \ldots, r_n]$.

Definition 2 — The continued fraction $[r_0, r_1, r_2, \ldots, r_N]$ is called the Nth convergent of the continued fraction $[r_0, r_1, r_2, \ldots, r_n]$, where N is a nonnegative integer less than or equal to n.

The Nth convergent is denoted by $\frac{p_N}{q_N} = [r_0, r_1, r_2, \ldots, r_N]$.

If $\frac{a_0}{a_1} = \frac{p_n}{q_n}$ then the penultimate convergent is $\frac{p_{n-1}}{q_{n-1}}$ and the following well known relation is satisfied,

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}.$$

Hence a solution of

$$a_0 x + a_1 y + a_2 z = d \qquad \text{... (1)}$$

for $a_2 = 0$ and $(a_0, a_1) = 1$ is

$$x = (-1)^{n+1} dq_{n-1} \text{ and } y = (-1)^n dp_{n-1}. \qquad \text{... (2)}$$

The integer solutions of eqn. (1) exist if and only if $(a_0, a_1) \mid d$. If $(a_0, a_1) = c$, then there exist A_0, A_1 and A_3 integers such that $\frac{a_0}{c} = A_0$, $\frac{a_1}{c} = A_1$, $\frac{d}{c} = A_3$. Equation (1) becomes $A_0 x + A_1 y = A_3$ and $(A_0, A_1) = 1$ so we may assume that $(a_0, a_1) = 1$ (Uspensky and Heaslet[1]).

The integers in $Q(\sqrt{5})$ have the form $r + s\lambda$, where $r, s \in Z$ and $\lambda = \frac{1+\sqrt{5}}{2}$. The element of $Q(\sqrt{5})$ have a unique finite continued fraction representation and every other real number has a unique infinite continued fraction representation. The continued fractions will be referred to as λ_5-fractions (Rosen[2]).

It was shown in Rosen[2], that every finite λ_q-fraction is an element in the algebraic number field $Q(\lambda_q)$ and Leutbecher[4] showed that only in the case $q = 5$, every element in $Q(\sqrt{5}) = Q(\lambda_5)$ has a finite λ_5-fraction, and a real number is an element of $Q(\sqrt{5})$ if and only if it has a finite λ_5-continued fraction representation.

Definition 3 — A λ_5-fraction is an expression of the form

$$r_0\lambda + \cfrac{\varepsilon_1}{r_1\lambda + \cfrac{\varepsilon_2}{r_2\lambda + \cdots}}\,,$$

where $\varepsilon_i = \pm 1$, $r_i \in Z^+$, $i \geq 1$, $r_0 \in Z$ and $\lambda = \frac{1+\sqrt{5}}{2}$. The continued fraction will be written as $\left[r_0\lambda, \frac{\varepsilon_1}{r_1\lambda}, \frac{\varepsilon_2}{r_2\lambda}, \ldots \right]$.

Let ξ be any arbitrary real number and {} the nearest integer to ξ/λ for which $\frac{-1}{2} \leq r_0 - \frac{\xi}{\lambda} < \frac{1}{2}$. It follows that $\left\{ \frac{\xi}{\lambda} \right\} = r_0$ and if $\xi_1 = \frac{\varepsilon_1}{\xi - r_0\lambda}$ then $\xi = r_0\lambda + \frac{\varepsilon_1}{\xi_1}$, where $\varepsilon_1 > 0$ if $r_0\lambda < \xi$ and $\varepsilon_1 < 0$ if $r_0\lambda > \xi$. Similarly $r_1 = \left\{ \frac{\xi_1}{\lambda} \right\}$, and by continuing the above process one obtains $r_{n-1} = \left\{ \frac{\xi_{n-1}}{\lambda} \right\}$. The nearest integer algorithm yields a unique λ_5-fraction (Rosen[2]); $\xi = r_0\lambda, \frac{\varepsilon_1}{r_1\lambda}, \ldots, \frac{\varepsilon_{n-1}}{r_{n-1}\lambda}, \frac{\varepsilon_n}{\xi_n}$.

MAIN RESULT

Theorem 1 — If the elements p, q, r, s which are not units, in $Z(\sqrt{5})$ and $(q, r) = 1$, then the diophantine equation

$$px + qy + rz = s \quad \text{... (3)}$$

has integer solutions in $Q(\sqrt{5})$. If x_0, y_0, z_0 is a particular solution then any other solution has the form

$$x = x_0 + Av$$

$$y = y_0 + Bu + Cv$$

$$z = z_0 + Du + Ev$$

where A, B, C, D, E are known integers in $Q(\sqrt{5})$ determined by the algorithm and u, v are arbitrary integers in $Q(\sqrt{5})$.

PROOF : Suppose p, q and r are not equal to 0. Let

$$y = \alpha t + \beta u$$

$$z = \gamma t + \delta u \quad \text{... (4)}$$

where $\alpha, \beta, \gamma, \delta$ are integers in $Q(\sqrt{5})$ chosen so that $\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = 1$. Substituting (4) in (3), we get

$$px + (q\alpha + r\gamma)t + (q\beta + r\delta)u = s. \quad \text{... (5)}$$

The solution of this equation in integers x, t, u is entirely equivalent to the solution of eqn. (3) in integers x, y, z in $Q(\sqrt{5})$.

If $(q, r) = 1$, then the solution of the equation $q\beta + r\delta = 0$ is $\beta = -r$, $\delta = q$. Then the equation $q\alpha + r\gamma = 1$ becomes $\delta\alpha - \beta\gamma = 1$ which is solvable in integers α, γ. We use $\alpha, \beta, \gamma, \delta$ in (5) to get

$$px + t = s \quad \text{... (6)}$$

which involves only two unknowns x and t. We find all the solution of (6), which are

$$x = x_0 + Av$$

$$t = t_0 + B_0 v,$$

by using λ_5-fraction and (2). Hence, we obtain all the solution of (3) by replacing this t in (4). This completes the proof.

Example —

$$(2 + \lambda)x + (6\lambda - 1)y + (1 + 3\lambda)z = 3 + \lambda. \quad \text{... (7)}$$

We note $q = 6\lambda - 1$, $r = 1 + 3\lambda$, $(q, r) = 1$ and let $y = \alpha t + \beta u$, $z = \gamma t + \delta u$.

Correspondingly $\beta = -(1+3\lambda)$, $\delta = (6\lambda - 1)$ and α, γ satisfy the equation

$$(6\lambda - 1)\alpha + (1+3\lambda)\gamma = 1.$$

One can verify that

$$\frac{6\lambda - 1}{1+3\lambda} = \left[\lambda, \frac{-1}{5\lambda}, \frac{-1}{\lambda}, \frac{1}{\lambda}, \frac{-1}{3\lambda}, \frac{-1}{\lambda}\right].$$

The right side, when rolled up and using $\lambda^2 = \lambda + 1$, becomes $\frac{113\lambda + 70}{76\lambda + 47}$. The numerator is $(8+13\lambda)(6\lambda - 1)$ and the denominator is $(8+13\lambda)(1+3\lambda)$, $(8 + 13\lambda = \lambda^7)$.

The penultimate convergent is $\left[\lambda, \frac{-1}{5\lambda}, \frac{-1}{\lambda}, \frac{1}{\lambda}, \frac{-1}{3\lambda}, \frac{-1}{\lambda}\right] = \frac{82\lambda + 49}{52 + 38\lambda}$.

Hence $\alpha' = (52\lambda + 38)$ and $\gamma' = -(82\lambda + 49)$ is a solution of

$$(113\lambda + 70)\alpha + (76\lambda + 47)\gamma = 1.$$

It follows that

$$\alpha = (52\lambda + 38)(8 + 13\lambda) = 1586\lambda + 980$$

and $$\gamma = -(82\lambda + 49)(8 + 13\lambda) = -(2359\lambda + 1458)$$

is a solutions of

$$(6\lambda - 1)\alpha + (1 + 3\lambda)\gamma = 1.$$

By taking $\alpha = 1586\lambda + 980$, $\gamma = -(2359\lambda + 1458)$, we obtain

$$y = (1586\lambda + 980)t - (1 + 3\lambda)u$$

$$z = -(2359\lambda + 1458)t + (6\lambda - 1)u.$$

Setting these y and z in (7) we get

$$(2 + \lambda)x + t = 3 + \lambda.$$

It is easy to see that

$$2 + \lambda = 2\lambda + \cfrac{1}{2\lambda - \cfrac{1}{\lambda}}.$$

The right side, when rolled up, becomes $\frac{7\lambda + 4}{2\lambda + 1}$ and $x' = 6\lambda + 4$, $t' = -(22\lambda + 13)$ solves $(2 + \lambda)x' + t' = 1$. It follows that $x = (6\lambda + 4)(3 + \lambda) = 28\lambda + 18$ $t = -(22\lambda + 13)(3 + \lambda) = -(101\lambda + 61)$ solves $(2 + \lambda)x + t = 3 + \lambda$. The general solution of the equation $(2 + \lambda)x + t = 3 + \lambda$ is

$$x = (28\lambda + 18) + v$$

$$t = -(101\lambda + 61) - (2 + \lambda)v, \; v \in Q(\sqrt{5}).$$

Hence all solutions of eqn. (7) is

$$x = (28\lambda + 18) + v$$

$$y = -[(355912\lambda + 219966) + (14564\lambda + 2940)v] - (1 + 3\lambda)u$$

$$z = [(529416\lambda + 327197) + (8535\lambda + 5275)v] + (6\lambda - 1)u$$

with two arbitrary integers u, v.

Remark : In a subsequent paper, we will extend our algorithm to $Q(\lambda_q)$ when solving the Diophantine equation

$$ax + by + cz = k$$

in $Q(\lambda_q)$, where a, b, c and k are integers in $Q(\lambda_q)$.

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