Indian J. pure appl. Math., 27(9) : 853-860, September 1996

ELLIPTIC SURFACES OVER A GENUS 1 CURVE WITH EXACTLY THE PAIR (I_2^*, I_4) OF SINGULAR FIBERS

AHMAD AL-RHAYYEL

Department of Mathematics, Yarmouk University, Irbid-Jordan

(*Received* 7 *December* 1994; *after revision* 26 *December* 1995; *accepted* 9 *April* 1996)

In this paper we give a classification (up to twists) of all minimal elliptic surfaces $(\pi : E \rightarrow C)$ over a genus 1 curve C, with a section and exactly the pair (I_2^*, I_4) of singular fibers.

1. INTRODUCTION

The serious study of elliptic surfaces was started by Kodaira[6]. He listed all possible types of singular fibers, gave their invariants and analyzed an important invariant called the J-map. Beauville[2] studied elliptic surfaces over $\mathbb{P}^1$, in fact he classified the semi-stable cases (i.e., the cases in which all singular fibers are of type I_n). He proved that there are 6 semi-stable cases with the minimal number (= 4) of singular fibers. Schmikler-Hirzebruch[5] wrote Weierstrass equations for all elliptic fibrations with at most three singular fibers. Miranda and Persson[8] listed all extremal rational elliptic surfaces. Stiller[12] classified all elliptic surfaces over a genus 1 curve with exactly one singular fiber necessarily of type I_6^*. Miranda and Persson[9] classified all possible configurations of I_n fibers on elliptic K_3 surfaces. Persson[10] classified all possible configurations of singular fibers on rational elliptic surfaces. Also, Miranda[7] analyzed the same problem by giving a more combinatorial and less geometric analysis.

In this paper we study minimal elliptic surfaces $\pi : E \rightarrow C$ over a genus 1 curve C with a section and exactly the pair (I_2^*, I_4) of singular fibers.

In this paper the notation $|[J^{-1}(x)]| = (n_1, ..., n_t)$ will be used to indicate that $J^{-1}(x)$ consists of t points say $\{x_1, ..., x_t\}$ such that the multiplicity of J at x_i $(m_{x_i}(J))$ is n_i for all $i \in \{1, ..., t\}$.

The plan of the paper goes as follows : First, we review some important ideas used in this paper. Then we give the possible ramification of the J-map and prove its existence, and then construct the J-map and the required surfaces.

Remark 1.1 : To build a minimal elliptic surface (up to a twist) with a section and a given number of singular fibers, it is enough to build the J-map associated to this surface (for more details see section 3 of Miranda[7]).

The following theorem which we call the monodromy theorem is just a restatement of Corollary 3.5 of Miranda[7].

Theorem 1.2 — Let C be a curve and B be a finite subset of $\mathbb{P}^1$, say $|B| = n$, then there is a one-to-one correspondence between

$$\left\{\begin{array}{l} J : C \to \mathbb{P}^1 \text{ such that} \\ \deg(J) = d \text{ and } J \text{ is} \\ \text{branched at most over } B \end{array}\right\} \text{ and } \left\{\begin{array}{l} \text{permutations } \sigma_1, ..., \sigma_n \in S_d \\ \text{such that } \sigma_n \dots \sigma_1 = id, \text{ and} \\ \text{the } \sigma_i\text{'s generate a transitive} \\ \text{subgroup of } S_d \end{array}\right\}$$

where the first set is taken up to isomorphism (fixing $\mathbb{P}^1$) and the second set is taken up to conjugation.

1.3. The rational elliptic surface X_{321} (see Miranda and Persson[8], p. 546) has exactly three singular fibers : type III* fiber over 1 and the pair (I_2, I_1) of singular fibers over ∞. The J-map $(J_1 : P_1 \to P_1)$ associated to this surface is a degree 3 map given by $J_1(x, y) = (4x^3)/(x-3y)^2(4x-3y)$. Moreover, J_1 is ramified as follows :

$$|[J_1^{-1}(0)]| = (3), \ |[J_1^{-1}(1)]| = (2, 1) \text{ and } |[J_1^{-1}(\infty)]| = (2, 1).$$

1.4. The rational elliptic surface $P : S \to \mathbb{P}^1$ has Weierstrass equation given by

$$Y^2 = X^3 - 3t(t-1)^3 X + 2t(t-1)^5.$$

This surface has J-map given by $J(t) = t$, and has exactly three singular fibers : A fiber of type II over $t = 0$, a fiber of type III* over $t = 1$, and a fiber of type I_1 over $t = \infty$ (see Miranda[7], p. 203).

2. Main Results

There are the following types of singular fibers I_0^*, I_n, I_n^* $(n \geq 1)$, II, III, IV, IV*, III*, II* (see Kodiara[6]). Let $e(F)$ denote the Euler number of the fiber F, then the Euler numbers of the above list are : 6, n, n + 6, 2, 3, 4, 8, 9, 10 respectively.

Lemma 2.1 — Let C be a genus 1 curve, suppose $\pi : E \to C$ is a minimal elliptic surface with a section and exactly two singular fibers, and suppose the degree of the line bundle $\mathbb{L}$ is 1 (i.e., $\mathbb{L}$ is the conormal bundle to the section), then there are 25 possibilities for the pair of singular fibers such that the sum of the Euler numbers is 12.

PROOF : This is immediate from the fact that if (F_1, F_2) is a possible pair of singular fibers, then $e(F_1) + e(F_2) = 12$.

Remark 2.2 : An easy check shows that two possible pairs are (I_2, I_4^*) and (I_2^*, I_4), and clearly these two possibilities are twists (see Miranda[7], p. 203) of each other, so it is enough to consider only one of them say (I_2^*, I_4) (since we are only interested in building these surfaces up to twists).

Lemma 2.3 — Let C be a genus 1 curve, if $\pi : E \to C$ is a minimal elliptic surface with a section and exactly the pair (I_2^*, I_4) of singular fibers, then deg (J) = 6 and the ramification of J must be given by one of the following :

(a) $|[J^{-1}(\infty)]| = (2, 4)$, $|[J^{-1}(1)]| = (2, 4)$ and $|[J^{-1}(0)]| = (3, 3)$.

(b) $|[J^{-1}(\infty)]| = (2, 4)$, $|[J^{-1}(1)]| = (2, 2, 2)$ and $|[J^{-1}(0)]| = (6)$.

(c) $|[J^{-1}(\infty)]| = (2, 4)$, $|[J^{-1}(1)]| = (2, 2, 2)$, $|[J^{-1}(0)]| = (3, 3)$ and $|[J^{-1}(\lambda)]| = (2, 1, 1, 1, 1)$ for some $\lambda \in \mathbb{P}^1$, with $\lambda \notin \{0, 1, \infty\}$.

PROOF : $Deg(J) = \sum_{n \geq 1} n$ (# of I_n-fibers + # of I_n^*-fibers)

= 6 (see Miranda[7], p. 194).

Let R = {ramification points of J} and let $m_x(J)$ denote the multiplicity of J at x. By Hurwitz's formula for the genus of a curve we have

$$0 = -12 + \sum_{x \in R} (m_x(J) - 1)$$

or

$$\sum_{x \in R} (m_x(J) - 1) = 12.$$

Now over ∞ we have the pair (I_2^*, I_4) of singular fibers, hence $\Sigma_{x \in J^{-1}(\infty)} (m_x(J) - 1) = 4$. Over 0 we have smooth fibers and if $x \in J^{-1}(0)$, then 3 divides $m_x(J)$ (see Miranda[7]), hence $|[J^{-1}(0)]| = (3, 3)$ or (6); therefore, $\Sigma_{x \in J^{-1}(0)} (m_x(J) - 1) = 4$ or 5. Over 1 we have smooth fibers and if $x \in J^{-1}(1)$, then 2 divides $m_x(J)$ (see Miranda[7]), hence $|[J^{-1}(1)]| = (2, 2, 2)$ or $(2, 4)$ or (6); thus $\Sigma_{x \in J^{-1}(1)} (m_x(J) - 1) = 3$ or 4 or 5. Consequently an easy check by Hurwitz's formula shows that the ramification of J must be given by one of the above three possibilities.

Q.E.D.

Lemma 2.4 — Let C be a genus 1 curve, if $\pi : E \to C$ is a minimal elliptic surface with a section and exactly the pair (I_2^*, I_4) of singular fibers, then the J-map ($J : C \to \mathbb{P}^1$) ramified as in case a of Lemma 2.3 exists. Moreover, there are (up to isomorphism) two curves C and two J-maps ($J : C \to \mathbb{P}^1$) which can be used to build such a surface.

PROOF : It is enough to find (up to conjugation) two sets of permutations σ_0, σ_1 and σ_∞ in S_6 representing the monodromy of J around 0, 1 and ∞ (respectively). Such that : $\sigma_0 \sigma_1 = \sigma_\infty^{-1}$, the triple $(\sigma_0, \sigma_1, \sigma_\infty)$ generates a transitive subgroup of S_6, and such that the cycle structure of σ_0 is (3, 3), that of σ_1 is (2, 4) and that of σ_∞ is (2, 4).

Let σ_0 = (1 2 3) (4 5 6) and let σ_1 = (a b) (c d e f); thus we must find σ_1 so that the cycle structure of $\sigma_0 \sigma_1 = \sigma_\infty^{-1}$ is (2, 4). It is clear that we may assume (a b) = (1 4) and c = 2, hence d = 3 or 5 or 6. If d = 3, then clearly we must have (c d e f) = (2 3 5 6), and we get

$$\sigma_1 = (1\ 4)\ (2\ 3\ 5\ 6) \text{ and } \sigma_0 \sigma_1 = \sigma_\infty^{-1} = (1\ 5\ 4\ 2)\ (3\ 6). \qquad \ldots (2.4.1)$$

If d = 5, then we get

$$\sigma_1 = (1\ 4)\ (2\ 5\ 3\ 6) \text{ and } \sigma_0 \sigma_1 = (1\ 5)\ (2\ 6\ 3\ 4). \qquad \ldots (2.4.2)$$

If d = 6, then we get

$$\sigma_1 = (1\ 4)\ (2\ 6\ 3\ 5) \text{ and } \sigma_0 \sigma_1 = (1\ 5\ 3\ 6)\ (2\ 4). \qquad \ldots (2.4.3)$$

Let α = (1 4) (2 5) (3 6), and notice that α(2.4.2) α^{-1} = (2.4.3) i.e., the permutations in (2.4.2) are conjugate to the permutations in (2.4.3). It is clear that the permutations in (2.4.1) are not conjugate to those in (2.4.2); thus we have two conjugacy classes of these permutations in S_6, and hence we have (up to isomorphism of curves) two genus 1 curves which can be used as a base curve for our surfaces, and hence we have two isomorphism classes of such surfaces.

Lemma 2.5 — Let C be a genus 1 curve, if $\pi : E \rightarrow C$ is a minimal elliptic surface with a section and exactly the pair (I_2^*, I_4) of singular fibers, then the J-map (J : $C \rightarrow \mathbb{P}^1$) ramified as in case b of Lemma 2.3 exists and is unique, and hence the curve C is unique.

PROOF : It is enough to find three permutations σ_0, σ_1 and σ_∞ in S_6 representing the monodromy of J around 0, 1 and ∞ respectively, such that : $\sigma_0 \sigma_1 = \sigma_\infty^{-1}$, the triple ($\sigma_0, \sigma_1, \sigma_\infty$) is unique up to conjugation and generates a transitive subgroup of S_6, and such that the cycle structure of σ_0 is (6), that of σ_1 is (2, 2, 2) and that of σ_∞ is (2, 4).

Let σ_0 = (1 2 3 4 5 6) and let σ_1 = (a b) (c d) (e f), and notice that no two consecutive numbers in σ_0 can form a transposition, hence (a b) = (1 3) or (1 4) or (1 5). If (a b) = (1 3), then we may assume c = 2, hence d = 5 or 6, but clearly $d \neq 6$, hence d = 5 and (e f) = (4 6) and we get

$$\sigma_1 = (1\ 3)\ (2\ 5)\ (4\ 6) \text{ and } \sigma_0 \sigma_1 = (1\ 4)\ (2\ 6\ 5\ 3). \qquad \ldots (2.5.1)$$

If (a b) = (1 4), then it is easy to see that we must have (c d) = (2 6) and (e f) = (3 5), and hence we get

$$\sigma_1 = (1\ 4)\ (2\ 6)\ (3\ 5) \text{ and } \sigma_0 \sigma_1 = (1\ 5\ 4\ 2)\ (3\ 6). \qquad \ldots (2.5.2)$$

If (a b) = (1 5), then it is easy to see that we must have (c d) = (2 4) and (e f) = (3 6), and hence we get

$$\sigma_1 = (1\ 5)\ (2\ 4)\ (3\ 6) \text{ and } \sigma_0 \sigma_1 = (1\ 6\ 4\ 3)\ (2\ 5). \qquad \ldots (2.5.3)$$

Let α = (1 5 3) (2 6 4) and β = (1 3 5) (2 4 6) = α^{-1}, then α (2.5.1) α^{-1} = (2.5.2) and β (2.5.1) β^{-1} = (2.5.3) (i.e., the triple $(\sigma_0, \sigma_1, \sigma_\infty)$ is unique up to conjugation), and clearly $\sigma_0 \sigma_1 \sigma_\infty = id$, and the triple $(\sigma_0, \sigma_1, \sigma_\infty)$ of permutations generates a transitive subgroup of S_6, and each permutation in this triple has the right cycle structure, hence the curve C is unique (up to isomorphism over C) and the J-map ramified as given exists and is unique. Q.E.D.

Lemma 2.6 — Let C be a genus 1 curve, if $\pi : E \to C$ is a minimal elliptic surface with a section and exactly the pair (I_2^*, I_4) of singular fibers, then the J-map $(J : C \to \mathbb{P}^1)$ ramified as in case C of Lemma 2.3 exists. Moreover, there are (up to isomorphism) five pairs (J, C) which can be used to build such a surface.

PROOF : It is enough to find (up to conjugation) five sets of permutations $\sigma_0, \sigma_1, \sigma_\lambda$ and σ_∞ in S_6 representing the monodromy of J around 0, 1, λ and ∞ respectively such that : $\sigma_0 \sigma_1 \sigma_\lambda = \sigma_\infty^{-1}$, the permutations $\sigma_0, \sigma_1, \sigma_\lambda$ and σ_∞ generate a transitive subgroup of S_6 and such that the cycle structure of σ_0 is (3, 3) that of σ_1 is (2, 2, 2), that of σ_λ = (2) and that of σ_∞ is (2, 4).

To this end let σ_0 = (1 2 3) (4 5 6) and $\sigma_1 = (a\ b)\ (c\ d)\ (e\ f)$ and $\sigma_\lambda = (g\ h)$. Notice that one of the 2-cycles of σ_1 must come from the two disjoint cycles of σ_0, so assume $(a\ b)$ = (1 4), and since 2 has to appear in σ_1, we may assume $c = 2$; thus σ_1 = (1 4) (2 3) (5 6) or (1 4) (2 5) (3 6) or (1 4) (2 6) (3 5), and hence we have three cases to consider :

Case 1 : σ_1 = (1 4) (2 3) (5 6), hence $\sigma_0 \sigma_1$ = (1 5 4 2), and clearly to get the right cycle structure of σ_∞ we must have σ_λ = (3 6).

Case 2 : σ_1 = (1 4) (2 5) (3 6), hence $\sigma_0 \sigma_1$ = (1 5 3 4 2 6), and clearly to get the right cycle structure of σ_∞ we must have σ_λ = (1 2).

Case 3 : σ_1 = (1 4) (2 6) (3 5), hence $\sigma_0 \sigma_1$ = (1 5) (2 4) (3 6) and clearly to get the right cycle structure of $\sigma_\infty, \sigma_\lambda$ must join two of the transpositions of $\sigma_0 \sigma_1$ and leave the other unchanged. Now it is easy to check that there are (up to conjugation) three possibilities for these permutations according to the following three values of σ_λ : (1 2) or (1 4) or (2 5).

Thus we conclude that there are (up to conjugation) five sets of permutations $\sigma_0, \sigma_1, \sigma_\lambda$ and σ_∞ in S_6 satisfying all conditions stated at the begining of the proof, and hence there are (up to isomorphism) five pairs (J, C) which can be used to build the required surfaces, where C is a genus 1 curve and $J : C \to \mathbb{P}^1$ is a degree 6 map ramified as given in Lemma 2.3(c) above. Q.E.D.

3. J-MAPS

In this section we construct the J-maps associated to the desired surfaces.

Remark 3.1 : Let C be a genus 1 curve, then clearly a degree 2 map $f : C \to \mathbb{P}^1$ exists, and by Harwitz's formula for the genus of a curve f must be branched over exactly 4 points of $\mathbb{P}^1$. Let $J_1 : \mathbb{P}^1 \to \mathbb{P}^1$ be the degree 3-map given in (1.3), then clearly $J = J_1 \circ f : Q \to \mathbb{P}^1$ is a degree 6-map. Now choose C so that

(a) $J : C \to \mathbb{P}^1$, is ramified as in Lemma 2.3(a).

(b) $J : C \to \mathbb{P}^1$, is ramified as in Lemma 2.3(b).

(c) $J : C \to \mathbb{P}^1$, is ramified as in Lemma 2.3(c).

Thus we built the J-map for three of our cases.

Lemma 3.2 — There are up to isomorphism over $\mathbb{P}^1$ two curves C of genus 1 for which there is a degree 3-map $J : C \to \mathbb{P}^1$, ramified over 4 points of $\mathbb{P}^1$ say $\{s_0, s_1, s_2, s_\infty\}$, with $|[g^{-1}(s_0)]| = (3) = |[g^{-1}(s_1)]|$ and $|[g^{-1}(s_2)]| = |[g^{-1}(s_\infty)]| = (2, 1)$ and hence g has no other ramification.

PROOF : An easy monodromy argument shows that the result is true. Moreover, it shows that we get (up to conjugation) two sets of permutations, namely;

$$\sigma_0 = (1\ 2\ 3) = \sigma_1,\ \sigma_2 = (1\ 2) \text{ and } \sigma_\infty = (2\ 3) \qquad \text{... (3.2.1)}$$

and

$$\sigma_0 = (1\ 2\ 3),\ \sigma_1 = (1\ 3\ 2) \text{ and } \sigma_2 = (1\ 2) = \sigma_\infty. \qquad \text{... (3.2.2)}$$

Hence we conclude that there are two curves C which can be used to build such a map g.

Remark 3.3 : If $g : C \to \mathbb{P}^1$ is a degree 3-map as in Lemma 3.2 above, then the set $D = \{g^{-1}(p) : p \in \mathbb{P}^1\}$ is a divisor of deg 3, and hence $\phi_{|D|}$ embeds C in $\mathbb{P}^2$; thus g is obtained by projecting C from a point q off C to aline. Since $g^{-1}(s_0)$ is one point of multiplicity 3 and so is s_1, then the projection point q must be a point of intersection of two flex lines, now by changing coordinates (if necessary) we may assume the flex lines to be line $z = 0$ and $y = 0$, hence (change coordinates if necessary) we may assume that the curve C is given by

$$Y^2Z + aYZ + XYZ - X^3 = 0,$$

hence g is given by $g([x : y : z]) = [y : z]$.

Now complete the square in Y, replace X by $(x - 3)/36$, set $y' = y + (a + x)/2$, set $a = u/27$ and $y = 216y'$ to get C is given by

$$Y^2 = X^3 + (4u - 27)X + (16u^2 - 72u + 54)$$

and of course this is a one parameter family of elliptic curves. Moreover, under the above changes g is given by

$$g([x : y : z]) = [y - 3x : z].$$

Remark 3.4 : The map $h : \mathbb{P}^1 \to \mathbb{P}^1$, given by $h(z) = (z - \alpha)^2$, where α is a constant, is a degree 2 map branched over two points of $\mathbb{P}^1$, and clearly ∞ is one of them.

Remark 3.5 : Let C be a genus 1 curve. If $J_2 : C \to \mathbb{P}^1$ is defined by $J_2 = h \circ g$, where g and h are as defined above in Remark 3.3 and 3.4 respectively, then J_2 is a degree 6 map, and by a suitable choice of C we may assume that the ramification of J_2 is given by one of the following two cases :

(a) $|[J_2^{-1}(\infty)]| = (2, 4) = |[J_2^{-1}(1)]|$, and $|[J_2^{-1}(0)]| = (3, 3)$.

(b) $|[J_2^{-1}(\infty)]| = (2, 4)$, $|[J_2^{-1}(1)]| = (2, 2, 2)$, $|[J_2^{-1}(0)]| = (3, 3)$ and $|[J_2^{-1}(\lambda)]| = (2, 1, 1, 1, 1)$, where $\lambda \in \mathbb{P}^1$ with $\lambda \notin \{0, 1, \infty\}$.

i.e., J_2 is ramified as in case (a) and case (c) of Lemma 2.3 respectively.

4. The Surfaces

In this section we construct some of our surfaces (up to twists), in fact we will construct all surfaces in which the associated J-map has ramification as given in cases (a) and (b) of Lemma 2.3, and we will give two isomorphism classes of surfaces whose J-map is ramified as given in case C of Lemma 2.3.

Theorem 4.1 — If C is a genus 1 curve as given in Lemma 2.4 (there are up to isomorphism two such curves), then we can build two isomorphism classes of minimal elliptic surfaces $(\pi : E \to C)$ each with a section and exactly the pair (I_2^*, I_4) of singular fibers.

Proof : We know that to build these two surfaces it is enough to build the two J-maps associated to these surfaces, and we already built these two J-maps in Remark 3.1(a) and in case of Remark 3.5(a). Now we consider two cases according to the two J-maps.

Case 1 : Let (J, C), $(J = J_1 \circ f)$, be as given in Remark 3.1(a) above, let $\pi : E \to C$ be the pull-back of the surface X_{321} via f, then $\pi : E \to C$ is a minimal elliptic surface with a section and the following singular fibers : An I_0^*-fiber over 1 and the pair (I_2, I_4) of singular fibers over ∞ (see Miranda and Persson[8], Table 7.1, p. 555), and by using the process of a transfer of a *, we get a minimal elliptic surface with a section and exactly the pair (I_2^*, I_4) of singular fibers.

Case 2 : Let (J_2, C), $(J_2 = h \circ g)$, be as given in Case (a) in Remark 3.5 above. Let $P : S \to \mathbb{P}^1$ be the rational elliptic surface defined in (1.4) and let $\pi : E \to C$ be the pull-back of $(P : S \to \mathbb{P}^1)$ via J_2, then this pull-back is a minimal elliptic surface with a section and the following singular fibers (see Miranda and Persson[8], Table 7.1, p. 555) : a total of 3 I_0^*-fibers, and the pair (I_2, I_4) of singular fibers, now using the process of deflating two *'s and the process of a transfer of a *, we get the desired surface. Q.E.D.

Theorem 4.2 — If C is the unique genus 1 curve given in Lemma 2.5, then we can build (up to isomorphism) a unique minimal elliptic surface $(\pi : E \to C)$ with a section and exactly the pair (I_2^*, I_4) of singular fibers.

Proof : Let $J : C \to \mathbb{P}^1$ $(J = J_1 \circ f)$ be the unique J-map given in Remark 3.1(b) above. Let $\pi : E \to C$ be the pull-back of the surface X_{321} via f, then $\pi : E \to C$ is a minimal elliptic surface with a section and the following singular fibers : An I_0^*-fiber, and pair (I_2, I_4) of singular fibers (see Miranda and Persson[8], Table 7, p. 555); thus by using the process of a transfer of a *, we get the desired surface. Q.E.D.

Next we construct two isomorphism classes of minimal elliptic surfaces each with a section and exactly the pair (I_2^*, I_4) of singular fibers, such that the J-map associated to such a surface is ramified as in case (c) of Lemma 2.3.

Remark 4.3 : Let (J, C) be as given in Remark 3.1(c) $(J = J_1 \circ f)$. Let $\pi : E \to C$ be the pull-back of the surface X_{321} via f, then $(\pi : E \to C)$ is a minimal elliptic surface with a section and the pair (I_2, I_4) of singular fibers (see Miranda and Persson[8], Table 7.1, p. 555); therefore, by using the process of a transfer of a *, we get one isomorphism class of the desired surface.

Remark 4.4 : Let (J_2, C) be as given in case (b) of Remark 3.5, let $P : S \to \mathbb{P}^1$ be the rational elliptic surface given in (1.4), and let $\pi : E \to \mathbb{P}^1$ be the pull-back of $P : S \to \mathbb{P}^1$ via J_2, then $\pi : E \to C$ is a minimal elliptic surface with a section and the following singular fibers (see Miranda and Persson[8], Table 7.1, p. 555). A total of 3 I_0^* and the pair (I_2, I_4). Thus by using the process of deflating two *'s and the process of a transfer of a *, we get the desired surface.

REFERENCES

1. W. Barth, P. Peters and A. Van de Ven, Compact complex surfaces, *Ergebnisse der mathematik und inher Grenzgebiete*, 3. Band 4, Springer-Verlag, 1984.
2. A. Beauville, *C. R. Acad. Sci. Paris*, t. 294 (1982), 657-60.
3. P. Griffith and J. Hariss, *Principles of Algebraic Geometry*. John Wiley and Sons, New York, 1987.
4. R. Hartshorne, *Algebraic Geometry*. Grad. test in Math., Vol. 52, Springer-Verlag, 1977.
5. U. Schmickler-Hirzebruch, Elliptische Flachen uber $\mathbb{P}$, $\mathbb{C}$ mit drei Ausnahmefasen und die hypergeometrische Differentialgleichung. Schriftreine des Mathematischen Institus der Universitat Munster, February, 1985.
6. K. Kodiara, *Annals Math.* **77** (1963), No. 3.
7. R. Miranda, *Math. Z.* **205** (1990), 191-211.
8. R. Miranda and U. Persson, *Math. Z.* **193** (1986), 537-58.
9. R. Miranda and U. Persson, *Math. Z.* **201** (1989), 339-61.
10. U. Persson, *Math. Z.* **205** (1990), 1-47.
11. A. Griffths Phillip, *Introduction to Algebraic Curves*, American Mathematical Society, 1989.
12. P. Stiller, *Manuscripta Mathematica* **60** (1988), Fasc 3.