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CORE THEOREMS FOR REAL BOUNDED SEQUENCES

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Choudhary[2] generalized the famous Knopp's core theorem. In this paper, using his idea, we also generalise the results obtained by Orhan[10].

1. INTRODUCTION

Let m and c respectively denote the linear spaces of real bounded and convergent sequences normed by $\| x \| = \sup_k |x_k|$. We write

$$m_0 = \left\{ x \in m : \sup_n \left| \sum_{k=1}^{n} x_k \right| < \infty \right\}.$$

Let $A = (a_{nk})$ be a real infinite matrix and $x = (x_k)$ be a real sequence. If

$$(Ax)_n := \sum_k a_{nk} x_k$$

exists (i.e., the series on the right-hand side converges for each n) we then write $Ax := \{(Ax)_n\}$. If $\lim Ax = \lim x$ for each $x \in c$, we say that A is regular[1, 8] and write $A \in (c, c; p)$.

We now turn our attention to some sublinear functionals on m. It is well known that the functional

$$q(x) = \inf_{n_1, n_2, \ldots, n_r} \limsup_k \frac{1}{r} \sum_{i=1}^{r} x_{k+n_i}$$

is sublinear on m (see Lorentz[7], Peterson[11]). We also consider the following functionals on m :

$$l(x) = \liminf x_n; \; L(x) = \limsup x_n$$

$$w(x) = \inf_{z \in m_0} L(x+z)$$

$$l^*(x) = \liminf_n \sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} x_r$$

$$L^*(x) = \limsup_n \sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} x_r$$

$$w^*(x) = \inf_{z \in m_0} L^*(x+z).$$

If $q(x) = -q(-x) = s$, then x is called almost convergent to s (Lorentz[7]) and in this case we write F-lim $x = s$. By F we denote the set of all almost convergent sequences. It is also well known that $q(x) = L^*(x)$ (Das and Mishra[4]).

If f, g are any two of the above functionals, we shall write $fA \leq gB$ to denote that, for every bounded x, Ax and Bx are defined and bounded and also $f(Ax) \leq g(Bx)$.

The famous Knopp's core theorem determines a class of regular matrices for which

$$L(Ax) \leq L(x)$$

for all $x \in m$; that is $\mathcal{K}$-core $Ax \subseteq \mathcal{K}$-core x, where Knopp's core (or $\mathcal{K}$-core) of $x \in m$ is defined to be the interval $[l(x), L(x)]$, Cooke[1], Maddox[9], Simons[12]. Devi[5] has established the inequality

$$L(Ax) \leq w(x), \text{ (for all } x \in m)$$

which is sharper than that of Knopp's Core theorem. These results have been generalized by Choudhary[2].

Das[3] defined Banach core (or $\mathcal{B}$-core) of $x \in m$ to be the closed interval $[-q(-x), q(x)]$. Since $q(x) \leq L(x)$, it follows that $\mathcal{B}$-core $x \subseteq \mathcal{K}$-core x.

Orhan[10] has proved some inequalities such as

$$LA \leq L^*, \; L^*A \leq L^*, \; L^*A \leq L, \; L^*A \leq w^*. \qquad \text{... (1)}$$

Since we need Orhan's results we just state them here. Before giving these results, we give the following definitions :

A matrix $A = (a_{nk})$ is called F-regular if F-lim $Ax = F$-lim x for each $x \in F$. The matrix A is called almost regular if maps c into F and F-lim $Ax = \lim x$, for all $x \in c$. The matrix A is called strongly regular if it maps F into c and lim $Ax = F$-lim x, for all $x \in F$. It is known that A is strongly regular[7] if and only if it is regular and that

$$\sum_k |a_{nk} - a_{n,k+1}| \to 0, \quad (n \to \infty).$$

Theorem A (Orhan[10], Theorem 1) — Let $\| \mathcal{A} \| := \sup_{n,i} \sum_k | a_{nk}(i) | < \infty$. Then for any $x \in m$

$$\lim_n \sup \sup_i \sum_k a_{nk}(i) x_k \leq L(x)$$

if and only if $(\mathcal{A})$ is regular and

$$\sum_k | a_{nk}(i) | \to 1, \quad (n \to \infty, \text{ uniformly in } i)$$

where $(a_{nk}(i))$ is a sequence of real infinite matrices.

Theorem B (Orhan[10], Theorem 3) — $LA \leq L^*$ if and only if A is strongly regular and

$$\sum_k | a_{nk} | \to 1, \quad (n \to \infty).$$

Theorem C (Orhan[10], Theorem 4) — $L^* A \leq L^*$ if and only if A is F-regular and

$$\lim_n \sup_i \sum_k \left| \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk} \right| = 1. \quad \ldots (2)$$

Theorem D (Orhan[10], Theorem 6) — $L^* A \leq w^*$ if and only if A is F-regular and (1) holds.

In this paper following Choudhary[2], we generalize inequalities (1).

2. The Main Results

Before giving the main results we first state a result due to Choudhary[2].

Lemma 1 — Let n be fixed. In order that, whenever Bx bounded, $(Ax)_n$ should be defined, it is necessary and sufficient that

(i) $c_{nk} = \sum_{\nu=k}^{\infty} a_{n\nu} b_{\nu k}^{-1}$ exists for all k;

(ii) $\sum_{k=0}^{\infty} | c_{nk} | < \infty$ (for all n);

and that

(iii) $\sum_{k=0}^{j} \left| \sum_{\nu=j+1}^{\infty} a_{n\nu} b_{\nu k}^{-1} \right| \to 0 \ (j \to \infty)$

should hold for the n considered. If these conditions are satisfied then, for bounded Bx,

$$(Ax)_n = (Cy)_n \qquad \text{... (3)}$$

where $y := Bx$.

We recall that the matrix B is called normal if it is lower semi triangular matrix with non-zero diagonal entries. In this case B has reciprocal. Denote its reciprocal by $B^{-1} = (b_{nk}^{-1})$. Note that if B is a normal matrix, then the space $m_B := \{x : Bx \in m\}$ is isometrically isomorphic to m. Hence given a sequence $y \in m$, then there exists a unique $x \in m_B$ so that $y = Bx$.

Now we state our first result.

Theorem 2 — Let $B = (b_{nk})$ be a normal matrix and A be any matrix. In order that, whenever Bx is bounded, Ax should exist and bounded and that

$$L^*(Ax) \leq L^*(Bx) \qquad \text{... (4)}$$

it is necessary and sufficient that

(a) $C = AB^{-1}$ exists;

(b) C is F-regular;

(c) for any fixed n

$$\sum_{k=0}^{j} \left| \sum_{v=j+1}^{\infty} a_{nv} b_{vk}^{-1} \right| \to 0 \ (j \to \infty);$$

(d) $\lim\limits_{n} \sup\limits_{i} \sum\limits_{k} \left| \frac{1}{n+1} \sum\limits_{r=i}^{i+n} c_{rk} \right| = 1.$

PROOF : *Necessity* — Assume now that (4) holds. Write $y := Bx$. Let $(Ax)_n$ be exist for each n whenever y is bounded. Then, by Lemma 1, (a) and (c) of the Theorem hold. Moreover, for every bounded y we have (3). Hence, by (4), we get

$$L^*(Cy) \leq L^*(y) \qquad \text{... (5)}$$

for every bounded y.

Now, it follows from Theorem C that (b) and (d) hold.

Sufficiency — Observe that conditions (a)-(d) imply that the conditions of Lemma 1 are satisfied. So (3) holds and that Cy is bounded whenever $y \in m$. Now (b) and (d) imply that

$$L^*(Cy) \leq L^*(y)$$

provided y is bounded[10]. Writting $y = Bx$ we immediately get (4) whence the result.

Recall that the matrix A is called row-finite if every row contains only a finite number of non-zero elements. In this case condition (c) of Theorem 2 is zero for sufficiently large j; hence (c) is evidently satisfied. So, Theorem 2 reduces to the following.

Theorem 3 — Let B be a normal matrix. Then for a row-finite matrix A,

$$L^*(Ax) \le L^*(Bx), \quad (\text{for all } x \in m_B),$$

if and only if (b) and (d) hold.

Theorem 4 — Let B be a normal matrix. In order that, whenever Bx is bounded, Ax should exist and satisfy

$$L^*(Ax) \le w^*(Bx) \qquad \dots (6)$$

it is necessary and sufficient that conditions (a)-(d) of Theorem 2 are satisfied.

PROOF : *Necessity* — Let Ax be defined whenever y is bounded. Hence Lemma 1 holds, so we have (3). Now one can get that

$$w^*(Bx) = \inf_{z \in m_0} L^*(Bx + z)$$

$$\le L^*(Bx + z).$$

Taking $z = \theta$, we see that $w^*(Bx) \le L^*(Bx)$. Combining this with (6), we have

$$L^*(Ax) \le L^*(Bx).$$

This inequality and (6) imply that

$$-L^*(-Bx) \le -L^*(-Ax) \le L^*(Ax) \le w^*(Bx) \le L^*(Bx).$$

Combining this with (3) and using the fact that $l \le l^* \le L^* \le L$ (Orhan[10]), we get

$$l(y) \le l^*(y) \le l^*(Cy) \le L^*(Cy) \le w^*(y) \le L^*(y) \le L(y)$$

from which we conclude that $Cy = Ax$ is bounded. Now Theorem 2 (or equally Theorem D implies the necessity.

The proof of the sufficiency follows the same lines as in the proof of the sufficiency of Theorem 2 and therefore is omitted.

Introducing the notation $(m_0)_A = \{x : Ax \in m_0\}$ we get the following.

Theorem 5 — Let B be a normal matrix and A be a row-finite matrix. Then

$$L^*(Ax) \le w^*(Bx), \text{ for all } x \in (m_0)_B,$$

if and only if (b) and (d) of Theorem 2 hold.

Theorem 6 — Let B be a normal matrix and A be any matrix. In order that, whenever Bx is bounded, Ax should exist and bounded and satisfy

$$LA \le L^* B \qquad \dots (7)$$

if and only if conditions (a), (c) of Theorem 2 hold and $C = AB^{-1}$ is strongly regular and also

(e) $\lim_n \sum_k |c_{nk}| = 1$

hold.

PROOF : *Necessity* — Since $y := Bx \in m$, Lemma 1 holds. Hence (a) and (c) follow. By (3), we have $(Ax)_n = (Cy)_n$. Observe that Cy is bounded. Now (7) implies that

$$L(Cy) \leq L^* (y).$$

Considering Theorem B we conclude that C is strongly regular and $\sum_k |c_{nk}| \to 1$ (as $n \to \infty$), this proves the necessity.

Sufficiency — Since Lemma 1 holds we get (3). Using the fact that Cy is bounded, we conclude that Ax is bounded. By Theorem B

$$L(Cy) \leq L^* (y)$$

for all bounded y. Now writing $y = Bx$, the proof follows.

Theorem 7 — Let B be a normal matrix. In order that, whenever Bx is bounded and satisfy

$$L^* (Ax) \leq L(Bx)$$

if and only if conditions (a), (c), (d) of Theorem 1 hold and $C := AB^{-1}$ is almost regular.

The above Theorem may also be proved as in Theorem 6. But core should be taken that Theorem A must be used instead of using Theorem B.

Finally we consider the inequality $L^* A \leq l^* B$. We shall show that this inequality may not hold in some cases.

Theorem 8 — Let B an F-regular matrix for which

$$\lim_n \sup_i \sum_k \left| \frac{1}{n+1} \sum_{r=i}^{i+n} b_{rk} \right| = 1. \qquad \ldots (8)$$

Then there is no matrix A such that

$$L^* A \leq l^* B. \qquad \ldots (9)$$

PROOF : Suppose that there is a matrix A such that (9) holds. Since B is F-regular and (8) holds, it follows from Theorem C that

$$L^* B \leq L^*. \qquad \ldots (10)$$

Now (9) and (10) imply that

$$L^* A \leq l^* B \leq L^* B \leq L^*. \qquad \ldots (11)$$

This last inequality and Theorem C yield that A is F-regular. Hence, there exists a bounded sequence z so that F-lim Az does not exist[6]. Thus $l^*(Az) < L^*(Az)$. Combining this with (11) we get that $l^* B \leq l^* A$. So we have obtained that

$$l^*(Bz) < L^*(Az). \qquad \ldots (12)$$

Now (9) and (12) imply that $l^*(Bz) < l^*(Bz)$ which is a contradiction.

We now give an improvement of the above result. Since it is included by Theorem 7 of Orhan[10] we just state here for the same of completeness.

We first recall that the matrix A is called almost Schur if it maps m into F.

Theorem 9 — $L^*A \leq l^*B$ if and only if B is almost Schur and

$$\lim_n \sum_{k=0}^{\infty} \frac{1}{n+1} \left| \sum_{r=i}^{i+n} (a_{rk} - b_{rk}) \right| = 0$$

uniformly in n.

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