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METRIC LIKE TENSOR FIELD IN A FINSLER SPACE

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In the present paper, we have tried to find out the existence of a tensor field X_{hk}, which is such that

$$C^h_{ij} X_{hk} = C_{ijk}.$$

In case of c-reducible, semi c-reducible, S-3-like Finsler space the form of X_{hk} has been worked out. In two and three dimensional Finsler space also, we have found out the form of X_{hk}. In the last we have the theorem. The only indicatory tensor in a semi c-reducible or S_3-like or two and three dimensional Finsler space, which satisfy (1.2), is the angular metric tensor.

1. INTRODUCTION

There are three kinds of torsion tensors in Cartan's theory of Finsler spaces. Two of them are $(h)hv$-torsion tensor C_{ijk} and $(v)hv$-torsion tensor P_{ijk}, which are symmetric in all their indices (Matsumoto[5]). The contravariant components of $(h)hv$-torsion tensor is given by $C^h_{ij} = g^{hk} C_{ijk}$, which may be treated as Christoffel symbols of second kind of each tangent Riemannian space of Finsler space F^n. Here, g^{hk} is the inverse of metric tensor g_{ij} of F^n. If l_i is the normalized element of support and h_{ij} is the angular metric tensor given by $h_{ij} = g_{ij} - l_i l_j$, then

$$C^h_{ij} g_{hk} = C_{ijk} = C^h_{ij} h_{hk}.$$

If b_i are components of a concurrent vector field, then $b_{i|j} = -g_{ij}$ and $b_i|_j = 0$ Matsumoto and Eguchi[6], where $_{|}\, j$ and $|\, j$ denote the h and v-covariant derivatives with respect to Cartan's connection $C\Gamma$. From this, it follows that b_i are functions of position only, and $C^h_{ij} b_h = 0$. Thus, if we consider a tensor field given by $B_{ij} = g_{ij} + \alpha\, l_i l_j + \beta\, b_i b_j$, where α and β are scalar functions, then

$$C^h_{ij} B_{hk} = C_{ijk}. \qquad \text{... (1.1)}$$

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The purpose of the present paper is to study the existence of any symmetric covariant tensor X_{hk} which satisfies,

$$C^h_{ij} X_{hk} = C_{ijk}. \qquad \text{... (1.2)}$$

Throughout the paper we are concerned with non-Riemannian Finsler spaces having positive-definite metric tensor g_{ij}. From (1.2) it follows that,

$$C^h_{ij} X_{hk} = C^h_{jk} X_{hi} = C^h_{ki} X_{hj}, \qquad \text{... (1.3)}$$

$$C^h X_{hk} = C_k \text{ and } C^h_{ij} X_{h0} = 0, \qquad \text{... (1.4)}$$

where $C^h = C^h_{ij} g^{ij}$ and 0 denotes the contraction with l^i. From the first equation of (1.4), it follows that $(X_{hk} - g_{hk}) C^h = 0$, which gives det $\| X_{hk} - g_{hk} \| = 0$ because $C^h \neq 0$.

Next the v-cruvature tensor of Cartan's connection $C\Gamma$ is given by

$$S^m_{ihk} = C^m_{hr} C^r_{ik} - C^m_{kr} C^r_{ih}. \qquad \text{... (1.5)}$$

From (1.2) and (1.5), it follows that

$$X_{mj} S^m_{ihk} = S_{ijhk}. \qquad \text{... (1.6)}$$

2. The Existence of X_{hk} in Special Finsler Spaces

In a C-reducible Finsler space the $(h)hv$-torsion tensor C^h_{ij} is given by (Matsumoto[3, 5])

$$C^h_{ij} = \frac{1}{n+1} (C^h h_{ij} + C_i h^h_j + C_j h^h_i). \qquad \text{... (2.1)}$$

Since $h^h_j X_{hk} = X_{jk} - X_{k0} l_j$, from (1.2), (1.4), it follows that (2.1) is written as

$$C_i (X_{jk} - h_{jk}) + C_j (X_{ki} - h_{ki}) - X_{k0} (C_i l_j + C_j l_i) = 0. \qquad \text{... (2.2)}$$

Cyclic permutation of i, j, k in (2.2) gives two more equations. Subtracting third of them from addition of first two we get,

$$C_j (X_{ki} - h_{ki}) = \frac{1}{2} [X_{i0} (C_j l_k + C_k l_j) + X_{k0} (C_i l_j + C_j l_i) - X_{j0} (C_k l_i + C_i l_k)]. \qquad \text{... (2.3)}$$

Contracting this equation with C^k, $(C^k C_k \neq 0)$ we get $X_{i0} l_j = X_{j0} l_i$, which implies

$$X_{i0} = X_{00} l_i, \qquad \text{... (2.4)}$$

and (2.3) reduces to

$$X_{ij} = h_{ij} + X_{00}\, l_i\, l_j. \qquad \text{... (2.5)}$$

In a semi C-reducible Finsler space the $(h)hv$-torsion tensor C^h_{ij} is given by (Matsumoto and Shibata[7])

$$C^h_{ij} = \frac{p}{n+1}\,(C^h h_{ij} + C_i h^h_j + C_j h^h_i) + \frac{q}{C^2}\, C^h C_i C_j \qquad \text{... (2.6)}$$

where $p + q = 1$ and $C^i C_i \neq 0$.

From (1.2) and (2.6) we also get eqn. (2.2), which gives eqn. (2.5). Therefore we have

Theorem 2.1 — In a semi C-reducible Finsler space the tensor field X_{ij} satisfying (1.2) is of the form (2.5).

Next we consider a S_3-like Finsler space, whose v-curvature tensor of Cartan's connection $C\Gamma$ is given by (Matsumoto[2])

$$L^2 S^m_{jhk} = S(h_{jh}\, h^m_k - h_{jk}\, h^m_h), \qquad \text{... (2.7)}$$

where S is the scalar function of x^i only. From (1.6) and (2.7) we get,

$$h_{ih}\,(X_{jk} - h_{jk}) - h_{ik}\,(X_{hj} - h_{hj}) = X_{0j}\,(h_{ih}\, l_k - h_{ik}\, l_h),$$

which after contraction with g^{ih} gives,

$$X_{jk} = h_{jk} + X_{0j}\, l_k. \qquad \text{... (2.8)}$$

Contracting (2.8) with l^j, we get $X_{0k} = X_{00}\, l_k$ and (2.8) is written as

$$X_{jk} = h_{jk} + X_{00}\, l_j\, l_k. \qquad \text{... (2.9)}$$

Theorem 2.2 — In a S_3-like Finsler space, the tensor field X_{ij} satisfying (1.2) is of the form (2.9).

3. Existence of X_{ij} in Two and Three-dimensional Finsler Space

We consider first a two-dimensional Finsler space. With reference to the Berwald frame (l_i, m_i), the angular metric tensor and $(h)hv$-torsion tensor is given by (Matsumoto[5])

$$h_{ij} = m_j\, m_j,\ \ LC_{ijk} = I\, m_i\, m_j\, m_k, \qquad \text{... (3.1)}$$

where L is the Finsler fundamental function and I is the main scalar. Let $X_{(00)}, X_{(01)}$ and $X_{(11)}$ be scalar components of X_{ij} in the Berwald frame, i.e.,

$$X_{ij} = X_{(00)}\, l_i\, l_j + X_{(01)}\,(l_i\, m_j + l_j\, m_i) + X_{(11)}\, m_i\, m_j.$$

Then from (1.2) and (3.1) we get,

$$X_{(01)} m_i m_j l_k + X_{(11)} m_i m_j m_k = m_i m_j m_k$$

which gives $X_{(01)} = 0, X_{(11)} = 1$. Thus from $X_{(00)} = X_{00}$, we have,

$$X_{ij} = h_{ij} + X_{00} l_i l_j. \qquad \text{... (3.2)}$$

Next we deal with a three-dimensional Finsler space. Moór[1] and Matsumoto[4] developed the theory of three dimensional Finsler spaces referring to the orthonormal frame $e^i_{(\alpha)}$, $\alpha = 0, 1, 2$, where $e^i_{(0)} = l^i$, $e^i_{(1)}$ is the torsion vector C^i divided by its length C and $e^i_{(2)}$ is a unit vector orthogonal to $e^i_{(0)}$ and $e^i_{(1)}$. The metric tensor g_{ij} and the $(h)hv$-torsion tensor C_{ijk} are written as,

$$g_{ij} = \delta_{\alpha\beta}\, e_{(\alpha)i}\, e_{(\beta)j} \qquad \text{... (3.3)}$$

$$LC_{ijk} = C_{\alpha\beta\gamma}\, e_{(\alpha)i}\, e_{(\beta)j}\, e_{(\gamma)k} \qquad \text{... (3.4)}$$

where the scalar components $C_{\alpha\beta\gamma}$ of LC_{ijk} are such that $C_{0\beta\gamma} = 0$, $C_{111} = H$, $C_{122} = I$, $C_{222} = -C_{112} = J$. The scalars H, I and J are called the main scalars and satisfy the equation $H + I = LC$. If $X_{\alpha\beta}$ are the scalar components of X_{ij} satisfying (1.2), that is,

$$X_{ij} = X_{\alpha\beta}\, e_{(\alpha)i}\, e_{(\beta)j}. \qquad \text{... (3.5)}$$

From (1.2), (3.4) and (3.5), we get,

$$X_{\delta\gamma} C_{\delta\alpha\beta} = C_{\alpha\beta\gamma}. \qquad \text{... (3.6)}$$

Now $\alpha = 0$ or $\beta = 0$ in (3.6) reduces to $0 = 0$. Thus (3.6) are equivalent to the following nine equations :

$$\gamma = 0: HX_{01} - JX_{02} = JX_{01} - IX_{02} = IX_{01} + JX_{02} = 0$$

$$\gamma = 1: H(X_{11} - 1) - JX_{12} = J(X_{11} - 1) - IX_{12} = I(X_{11} - 1) + JX_{12}$$

$$\gamma = 2: HX_{12} - J(X_{22} - 1) = JX_{12} - I(X_{22} - 1) = IX_{12} + J(X_{22} - 1) = 0.$$

These conclude as

$$X_{01} = X_{12} = X_{11} - 1 = 0, \; IX_{02} = JX_{02} = 0,$$

$$I(X_{22} - 1) = J(X_{22} - 1) = 0.$$

Therefore, if I or J does not vanish, then we have

$$X_{ij} = h_{ij} + X_{00} l_i l_j. \qquad \text{... (3.7)}$$

Theorem 3.1 — In a two-dimensional Finsler space, the tensor field X_{ij} satisfying (1.2) is of the form (3.7).

In a three-dimensional Finsler space, the tensor field X_{ij} satisfying (1.2) is also written as above, provided that I or J among the main scalars does not vanish.

A tensor X_{ij} is said to be indicatory tensor if $X_{0j} = X_{i0} = 0$ (Matsumoto[5]). Thus we conclude the following theorem :

Theorem 3.2 — The only indicatory tensor in a semi C-reducible or S_3-like or two and three-dimensional Finsler space, which satisfy (1.2), is the regular metric tensor.

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