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SOME PROPERTIES OF M_N SUBSETS AND ALMOST CLOSED MAPPINGS

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The aim of the present paper is to study some properties of M_N subsets and almost closed mappings.

1. INTRODUCTION

No separation properties are assumed for spaces unless explicitly stated.

A subset A of a space X is regular open (regular closed) iff $Int\ Cl\ A = A$($Cl\ Int\ A = A$) (Singal and Singal[10]).

A subset A of a space X is α-paracompact (α-nearly paracompact) with respect to a subset B iff for every open (regular open) cover $\mathcal{U} = \{U_i : i \in I\}$ of A there is an open family $\mathcal{V} = \{V_j : j \in J\}$ such that :

(1) $\mathcal{V}$ refines $\mathcal{U}$; (2) $A \subset \bigcup \{V_j : j \in J\}$,

(3) $\mathcal{V}$ is locally finite at each point $x \in B$.

Subsets A and B of a space X are mutually α-paracompact (α-nearly paracompact), iff the subset A is α-paracompact (α-nearly paracompact) with respect to the subset B and B is α-paracompact (α-nearly paracompact) with respect to the subset A (Kovačević[5]).

A subset A of a space X is α-Hausdorff iff for any two points a, b of a space X, where $a \in A$ and $b \in X \setminus A$, there are disjoint open sets U and V containing a and b respectively. A subset A of a space X is α-regular (α-almost regular) iff for any point $a \in A$ and any open (regular open) set U containing a, there is an open set V such that $a \in V \subset Cl\ V \subset U$ (Kovačević[6]).

A nonempty proper subset A of a space X is a M_N subset iff :

(a) A is an α-Hausdorff α-nearly paracompact with respect to $X \setminus A$

(b) Any two distinct points of A cannot be strongly separated by open neighbourhoods (Kovačević[3]).

A mapping $f : X \to Y$ is almost closed iff for every regular closed set F of X the set $f(F)$ is closed (Singal and Singal[10]).

A mapping $f : X \to Y$ has a closed graph $G(f)$ iff $G(f) = \{(x, f(x)) : x \in X\}$ is closed in $X \times Y$ (Hamlet and Harrington[1]).

In this paper we shall observe multifunctions $F : X \to Y$ such that for each point $x \in X$, $F(x) \neq \phi$.

A multifunction $F : X \to Y$ is closed (almost closed) iff for any closed (regular closed) set $A \subset X$, $F(A) = \bigcup \{F(x) : x \in A\}$ is closed in Y (Popa[9]).

2. M_N Subsets

Theorem 2.1 — Let $\mathcal{U} = \{U_i : i \in I\}$ be a family of M_N subsets of a space X such that $\mathcal{U}$ is locally finite at each point $x \in X \setminus U$, where $U = \bigcup \{U_i : i \in I\}$. Then U is closed.

Proof : Let $x \in X \setminus U$ be any point. Since every U_i, $i \in I$ is M_N then by Theorem 2.4 of Kovačević[5] there are disjoint regular open sets V_i and H_i such that

$$x \in H_i \qquad U_i \subset V_i.$$

Since family $\mathcal{U}$ is locally finite at x it follows that there is a finite subset J of I and open neighbourhood W of x such that

$$W \cap U_i \neq \phi, \ i \in J, \ W \cap U_i = \phi, \ i \in I \setminus J.$$

Let $H = \bigcap \{H_i : i \in J\}$ and $V = W \cap H$.

V is an open neighbourhood of x such that $V \subset X \setminus U$, hence $X \setminus U$ is open, i.e., U is closed.

Theorem 2.2 — If a M_N subset A of a space X is α-regular, $B \subset A$ a closed subset, then $B = A$.

Proof : Let $B \neq A$, and $x \in A \setminus B$ any point. Since A is α-regular there are disjoint open sets U and V such that $x \in U$, $B \subset V$. From this fact, it follows that if $b \in B$, then the points $x, b \in A$ can be strongly separated by open neighbourhoods, hence A is not M_N.

Theorem 2.3 — $\mathcal{U} = \{U_i : i \in I\}$ be a nonempty family of M_N subsets of a topological space X. If there is two different elements $U_i, U_j \in \mathcal{U}$ such that the sets $X \setminus U_i$ and $X \setminus U_j$ are compact, then X is compact. The subsets U_i and U_j are clo-open.

Proof : Since $U_i \neq U_j$, it follows that $U_i \cap U_j = \phi$. Since $U_j \subset X \setminus U_i$ is closed, then U_j is compact. Since $X \setminus U_j$ is compact then the space $X = U_j \cup \{X \setminus U_j\}$ is compact.

Since $X\setminus U_i$ is an α-Hausdorff compact subset, then $X\setminus U_i$ is closed, hence it follows that the M_N subset U_i is clo-open.

Corollary 2.1 — Let $\mathcal{U} = \{U_i : i\in I\}$ be a nonempty family of M_N subsets of a topological space X. If there are two different elements $U_i, U_j \in \mathcal{U}$ such that the sets $X\setminus U_i$ and $X\setminus U_j$ are compact, then every M_N subset of X is compact.

PROOF : X is compact. Since any M_N subset is closed, it follows that any M_N subset of a compact space X is compact.

3. ALMOST CLOSED MAPPINGS

Theorem 3.1 — If f is an almost closed continuous mapping of a regular space X onto a space Y such that for each $y\in Y f^{-1}(y)$ is α-nearly paracompact with respect to $X \setminus f^{-1}(y)$, then Y is regular.

PROOF : Since X is regular, it follows that for each $y\in Y f^{-1}(y)$ is α-paracompact with respect to $X\setminus f^{-1}(y)$. Let $y\in Y$ and V be an open set containing y. Since X is regular and $f^{-1}(y)$ is α-paracompact with respect to $X\setminus f^{-1}(y)$, then by Theorem 2.6. of Kovačević[5], there is a regular open set U in X such that

$$f^{-1}(y)\subset U\subset ClU\subset f^{-1}(V).$$

Since f is almost closed, then there is an open set W in Y such $y\in W$ and $f^{-1}(W)\subset U$. Therefore,

$$y\in W\subset f(U)\subset f(ClU)\subset V.$$

Since f is almost closed and ClU is regular closed, $f(ClU)$ is closed. Hence we have

$$y\in W\subset Int\, ClW\subset ClW\subset f(ClU)\subset V.$$

We have, that for every point $y\in Y$ and evey open neighbourhood V of y there is an open set W such that

$$y\in W\subset ClW\subset V,$$ hence Y is regular.

Corollary 3.1 — Let X be a topological space such that every nonempty proper subset F is α-Hausdorff α-paracompact with respect to $X\setminus F$. If f is an almost closed continuous mapping of a space X onto a space Y such that for each point $y\in Y f^{-1}(y)$ is α-Hausdorff, α-nearly paracompact with respect to $X\setminus f^{-1}(y)$, then Y is regular.

PROOF : It is identical with the proof of previous theorem, since the space X is regular.

Theorem 3.2 — Let $f:X\rightarrow Y$ be a closed mapping of a topological space X onto a compact space Y such that for each nonempty closed subset $A\subset X$, $f^{-1}(f(A)) = A$ and for each point $y\in Y, f^{-1}(y)$ is an α-Hausdorff α-paracompact with respect to $X\setminus f^{-1}(y)$. Then X is regular.

PROOF : The space Y is Hausdorff, hence Y is regular. f is continuous.

Let A be a closed subset of a space X and $x \notin A$ be any point. Since $f(x) \notin f(A)$ it follows that there are open sets U and V such that $f(x) \in U$, $f(A) \subset V$, $U \cap V = \phi$.

Now, $f^{-1}(U)$ and $f^{-1}(V)$ are open sets such that

$$x \in f^{-1}(U),\ A \subset f^{-1}(V),\ f^{-1}(U) \cap f^{-1}(V) = \phi,$$

hence X is regular.

Theorem 3.3 — If X is regular and multifunction $F : X \to Y$ is an almost closed surjection such that for each $y \in Y F^{-}(y) = \{x \in X : y \in F(x)\}$ is α-paracompact with respect to $X \setminus F^{-}(y)$, then F is closed.

PROOF : Let A be a closed subset of X and $y \in Y \setminus F(A)$ any point. Since $F^{-}(y) \subset X \setminus A$ is α-paracompact with respect to $X \setminus F^{-}(y)$ and X is regular, there is a regular open set U such that

$$F^{-}(y) \subset U \subset ClU \subset X \setminus A.$$

Since F is almost closed, then by Lemma 2.2 of Kovačević[4], there is an open set G in Y such that $y \in G$, and

$$F^{-}(G) = \{x \in X : F(x) \cap G \neq \phi\} \subset U \subset X \setminus A.$$

Therefore, we have $y \in G \subset Y \setminus F(A)$. Hence $Y \setminus F(A)$ is open, i.e. $F(A)$ is closed. Hence F is closed.

Theorem 3.4 — Let A be a M_N subset of a topological space X. Let $f : X \to Y$ be an almost closed mapping of a space X onto a space Y such that for each $x \in X \setminus A\ f^{-1}(f(x))$ is an α-Hausdorff, α-nearly paracompact subset with respect to $X \setminus f^{-1}(f(x))$.

If $G = \{(x, f(x)) : x \in X \setminus A\}$ then the set $H = A \times Y \cup G$ is closed.

PROOF : Let $(x, y) \notin H$. Then $(x, y) \notin A \times Y$ and $(x, y) \notin G$. Since each M_N subset is closed, then $A \times Y$ is closed. Hence there is an open set W containing (x, y) such that $W \cap (A \times Y) = \phi$. Since $(x, y) \notin A \times Y$, it follows that $x \in X \setminus A$. From this fact and $(x, y) \notin G$ it follows that $y \neq f(x)$, i.e. $x \notin f^{-1}(y)$.

If there is a point $x_1 \in X \setminus A$ such that $f(x_1) = y$, then $f^{-1}(f(x_1)) = f^{-1}(y)$ is an α-nearly paracompact subset with respect to $X \setminus f^{-1}(f(x_1))$, such that $x \notin f^{-1}(y)$. Hence, there are regular open disjoint sets U and V such that $x \in U, f^{-1}(y) \subset V$. Let $f^{-1}(y) \subset A$. Since $x \notin A$ and A is M_N, there are regular open disjoint sets U and V such that

$$x \in U,\ f^{-1}(y) \subset A \subset V.$$

In any case there are disjoint regular open sets such that

$$x \in U,\ f^{-1}(y) \subset V.$$

Since f is almost closed, there is an open neighbourhood H of y such that

$$f^{-1}(y) \subset f^{-1}(H) \subset V.$$

Thus $U_1 = U \times H$ is an open neighbourhood of (x, y) such that $U_1 \bigcap G = \phi$.

Therefore, $W \bigcap U_1$ is an open neighbourhood of a point (x, y) such that $(W \bigcap U_1) \bigcap H = \phi$. Hence H is closed.

From this theorem it does not follows that the mapping f has a closed graph, what we can see from next example.

Example 3.1 — Let $X = \{a, b, c, d\}$ and

$$\tau = \{\{a, b\}, \{c\}, \{d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, X\}.$$

Let $Y = \{1, 2, 3, 4\}$ be endowed by discrete topology. Let $f: X \to Y$ be mapping defined by $f(a) = 1$, $f(b) = 2$, $f(c) = 3$, $f(d) = 4$.

$A = \{a, b\}$ is a M_N subset of the space X. The mapping f is almost closed. For each point $y \in Y f^{-1}(y)$ is compact. The set $G(f) = \{(a, 1), (b, 2), (c, 3), (d, 4)\}$ is not closed, because there is not an open set U containing $(a, 2) \notin G(f)$ such that $U \bigcap G(f) = \phi$.

Theorem 3.5 — Let A be an α-regular M_N subset of a topological space X and let $f : X \to Y$ be an almost closed mapping of a space X onto a space Y such that :

(i) for each point $x \in X \setminus A\, f^{-1}(f(x))$ is α-Hausdorff α-nearly paracompact with respect to $X \setminus f^{-1}(f(x))$,

(ii) there is a point $x \in A$ such that $f^{-1}(f(x))$ is closed, then f has a closed graph.

PROOF : By assumption, there exists a point $x \in A$ such that $f^{-1}(f(x))$ is closed. Then $f^{-1}(f(x)) \bigcap A$ is a nonempty closed set. From Theorem 2.2, $f^{-1}(f(x)) \bigcap A = A$. It follows that $A \subset f^{-1}(f(x))$, i.e., f_A is constant.

From this fact it follows that for each point $y \in Y f^{-1}(y)$ is α-Hausdorff α-nearly paracompact with respect to $X \setminus f^{-1}(y)$, hence by Theorem 2.1 of Kovačević[3] f has closed graph.

Theorem 3.6 — Let A be a M_N subset of the topological space X. If f is an almost closed mapping of a space X onto a space Y such that :

(i) for each $y \in Y f^{-1}(y)$ is an α-Hausdorff α-nearly paracompact with respect to $X \setminus f^{-1}(f(x))\ X \setminus f^{-1}(y)$

(ii) for each point $(x, y) \notin G(f)\ (f^{-1}(f(x)) \bigcup f^{-1}(y)) \bigcap (X \setminus A) \neq \phi)$.

Then f has a closed graph. If Y is compact, then f is continuous.

PROOF : Let $(x, y) \notin G(f)$ be any point. Since $y \neq f(x)$, then $x \notin f^{-1}(y)$. If $f^{-1}(y) \bigcap (X \setminus A) \neq \phi$, then there is a point $x_1 \in X \setminus A$ such that $f^{-1}(f(x_1)) = f^{-1}(y)$. Since

$f^{-1}(y)$ is α-nearly paracompact with respect to $X \setminus f^{-1}(y)$ and $x \notin f^{-1}(y)$ it follows that there are disjoint regular open sets U and V such that $x \in U$, $f^{-1}(y) \subset V$. Since f is almost closed, there is an open neighbourhood H of y such that $f^{-1}(y) \subset f^{-1}(H) \subset V$. Thus $U_1 = U \times H$ is an open neighbourhood of (x, y) such that $U_1 \bigcap G(f) = \phi$. If $f^{-1}(y) \subset A$, then $f^{-1}(f(x)) \bigcap (X \setminus A) \neq \phi$. There exists a point $x_1 \in X \setminus A$ such that $f^{-1}(f(x_1)) = f^{-1}(f(x))$. Since the point $x \in X \setminus A$, (from $x \in A$ it follows that $f(A) = f(x_1)$, i.e. $x \in f^{-1}(y)$ which is contradiction) and A is M_N, there are disjoint regular open sets U and V such that

$$x \in U f^{-1}(y) \subset A \subset V.$$

Since f is almost closed there is an open neighbourhood H of y such that

$$f^{-1}(y) \subset f^{-1}(H) \subset V.$$

Thus, $U_1 = U \times H$ is an open neighbourhood of (x, y) such that

$$U_1 \bigcap G(f) = \phi.$$

Hence $G(f)$ is closed.

Since f has a closed graph and Y is compact it follows that f is continuous.

Theorem 3.7 — Let A be a M_N subset of a space X such $X \setminus A$ is α-regular. Let $f : X \to Y$ be an almost closed mapping of a space X onto a space Y such that for each $x \in X \setminus A\, f^{-1}(f(x))$ is α-nearly paracompact with respect to A. Then $f(A)$ is closed.

PROOF : Let $y \in Y \setminus f(A)$ be any point. Since A is closed, $X \setminus A$ is α-regular $f^{-1}(y) \subset X \setminus A$ is α-paracompact with respect to A, then it follows that there is a regular open set V such that

$$f^{-1}(y) \subset V \subset ClV \subset X \setminus A.$$

Since f is almost closed, there is an open set W containing y such that

$$f^{-1}(y) \subset f^{-1}(W) \subset V.$$

Therefore, it follows that W is an open neighbourhood of y such that $W \bigcap f(A) = \phi$, hence $f(A)$ is closed.

. If $f^{-1}(f(x))$ is α-nearly paracompact with respect to A, then $f^{-1}(f(x))$ is not always closed. It can be seen from the following example.

Example 3.2 — Let $X = \{a, b, c, d\}$ and

$$\tau = \{\{a, b\}, \{c, d\}, X\}.$$

Let $Y = \{1, 2, 3, 4\}$ be endowed by discrete topology. Let $f : X \to Y$ be a mapping defined by

$$f(a) = 1,\ f(b) = 2,\ f(c) = 3,\ f(d) = 4.$$

The set $A = \{a, b\}$ is a M_N subset of a space X, such that $X \setminus A$ is α regular. f is an almost closed mapping of a space X onto a space Y such that for each $x \in X \setminus A$ $f^{-1}(f(x))$ is compact, hence $f^{-1}(f(x))$ is α-nearly paracompact with respect to A. But $f^{-1}(f(c)) = f^{-1}(3) = c$ is not a closed subset of a space X. Also, it can be concluded that $f^{-1}(f(c)) = c$ is not α-Hausdorff. (The point c and $d \notin f^{-1}(f(c))$ cannot be strongly separated by open neighbourhoods). Since $f^{-1}(y)$ is not closed for each point $y \in Y$, it follows that the mapping f has not a closed graph. The mapping f is not continuous at the point c, hence f is not continuous.

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