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# UNIFORM FINITE DIFFERENCE SCHEMES FOR SINGULAR PERTURBATION PROBLEMS ARISING IN GAS POROUS ELECTRODES THEORY

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This paper presents two uniformly convergent three-point exponentially fitted finite difference schemes for singularly perturbed two point boundary value problems appearing in the theory of gas porous electrodes. Necessary and sufficient conditions for uniform convergence are derived. Finally numerical results are presented.

## 1. INTRODUCTION

Consider the following two point boundary value problem (TPBVP)

$$Lu(t) \equiv -\varepsilon \ (p(t)\, u'(t))' + q(t)\, u(t) = f(t), \quad 0 < t < 1, \qquad \text{... (1.1a)}$$

$$B_0\, u(0) \equiv -u'(0) = \phi_1\,,\ B_1\, u(1) \equiv u'(1) = \phi_2 \qquad \text{... (1.1b)}$$

where $\varepsilon > 0$ is a small parameter, $\phi_1, \phi_2$ are given constants, the coefficients $p$ and $q$ are smooth and satisfy $p(t) \geq \alpha > 0$, $p'(t) \geq 0$, $q(t) \geq \beta > 0$ for all $t$ in $[0, 1]$. It can be seen that the solution of (1.1a, b) converges uniformly to the solution of the reduced problem

$$q(t)\, u_0(t) = f(t) \qquad \text{... (2)}$$

where as its derivative, generally, converges nonuniformly as $\varepsilon$ goes to zero at $t$ = 0 and 1.

A complete survey of uniformly convergent finite difference methods for linear and non-linear singularly perturbed problems is given in Doolan *et al.*[2], Miller[7], O'Reilly[9] and Selvakumar[13]. A new approach for higher order methods for stiff initial value problem is given in Selvakumar[14]. A new computational procedure for solving singularly perturbed problems is given in Selvakumar[15]. Miller[7] in his lecture notes explains how his methods in Doolan *et al.*[2] for singular perturbation problems can be applied to study the local behaviour of the solution when $\varepsilon$ is very small using uniform and variable meshes.

Finite difference schemes for singularly perturbed selfadjoint problems of the form (1.1a, b) with $p(t) \equiv 1$ and subject to the Dirichlet condition, which are uniformly in $\varepsilon$ accurate, have been examined by various authors. Miller[6] gave sufficient conditions for the uniform first-order convergence of a general three-point difference scheme. Hegarty *et al.*[4] and Nijima[8] produced uniformly convergent second-order difference schemes. Boglaev[1] examined in a finite element framework and achieved uniform first-order accuracy at the nodes. Shishkin[17] examined problem (1.1a, b) on a nonuniform mesh, which depends on $\varepsilon$, and obtained convergence results for various difference schemes. A useful discussion of uniform convergence of difference schemes for singularly perturbed TPBVPs for self-adjoint differential equations is given in Doolan *et al.*[2]. Schatz and Wahlbin[16] examined problems in both one and two dimensions using a Galerkin finite-element method. For a discussion of exponential elements, see Hemker[5], O'Riordan[10], De Groen[3] and the references therein. A discretized Green's function was first used by Stynes and O'Riordan[18] to prove a super convergence result for a nonselfadjoint singularly perturbed TPBVP. Stynes and O'Riordan[18] generated tri-diagonal difference schemes for (1.1a, b) using finite element method which is uniformly second order accurate.

In this paper uniform three-point exponentially fitted finite difference schemes are presented for the TPBVP (1.1a, b) which are unconditionally stable for all values of the parameter $\varepsilon$ and mesh size $h$. The schemes reflects the asymptotic behaviour of the solution for small values of $\varepsilon$ and model the reduced problem (1.2) exactly. Doolan *et al.*[2], discussed, in detail, the Dirichlet problem for the equation (1.1a) though they presented only numerical results for Neumann type problem. We adopt the same scheme given in Doolan *et al.*[2] for the differential equation (1.1a) as well as a new scheme. It has been shown through an example that the new scheme performs well. The method of proof given here to arrive at error estimates for numerical solution is basically boot-strapping technique Doolan *et al.*[2] and is different from that of given in p.118 of Doolan *et al.*[2], in the sense that when the present method is applied to the Dirichlet type BVPs the proof becomes simpler.

Some analytical results for (1.1a, b) are given in section 2. In section 3 a three-point uniform difference scheme is presented for (1.1a, b). Necessary and sufficient conditions for uniform convergence are derived. A new uniform scheme is also proposed in section 4. Numerical results are given in the last section.

Throughout this paper, $\rho = h/\sqrt{\varepsilon}$ and $C$ will be used to denote a generic constant independent of $i$, $h$ and $\varepsilon$.

## 2. Analytical Results

To analyse the schemes of the next sections more informations on the behaviour of the solutions of (1.1a, b) are required. These results are contained in the following lemmas. In fact, the stability result is given in Lemma 2.1 for the solution of (1.1a, b), Lemma 2.2 gives estimates for the derivatives of the solution of (1.1a, b) and

Lemma 2.3 shows that the solution of (1.1a, b) can be written as the sum of a boundary layer term and a function smoother than the solution of (1.1a, b).

*Lemma* 2.1 — Let $v$ be a smooth function.

(1) If $B_0 v(0) \geq 0$, $B_1 v(1) \geq 0$ and $Lv(t) \geq 0$ for all $t$ in (0, 1), then $v(t) \geq 0$ for all $t$ in [0, 1].

(2) For all $t$ in [0, 1],

$$|v(t)| \leq C [|B_0 v(0)| + |B_1 v(1)| + \max |Lv(y)|].$$

PROOF : Using the barrier function techniques described in Doolan *et al.*[2] and Protter and Weinberger[12] this lemma can be proved.

*Lemma* 2.2 (Doolan *et al.*[2]) — Let $v$ be any smooth function such that

$$|(Lv)^{(j)}(t)| \leq C \text{ for all } j \geq 0 \text{ and all } 0 \leq t \leq 1.$$

Then, for all $0 \leq t \leq 1$,

(i) $|v^{(j)}(t)| \leq C\ [1 + \varepsilon^{-(j-1)/2} \{\exp(-tA(0)) + \exp(-(1-t)A(1))\}],\ j \geq 1,$

(ii) if $(qv)'(0) - \varepsilon(p'v')'(0) - \varepsilon(p'v)''(0) = (Lv)'(0)$

$$(qv)'(1) - \varepsilon(p'v')'(1) - \varepsilon(p'v)''(1) = (Lv)'(1)$$

then

$$|v^{(j)}(t)| \leq C [1 + \varepsilon^{1-j/2} \{\exp(-tA(0)) + \exp(-(1-t)A(1))\}],\ j \geq 0$$

where $A(t) = \sqrt{(q(t)/(\varepsilon p(t)))}$.

PROOF : (i) The asymptotic expansion for the solution of (1.1a, b) for any smooth function $v(t)$ is

$$v(t) = (Lv(t)/q(t)) + v_0(t) + w_0(t) + \sqrt{\varepsilon}\, R(t) \quad \text{... (2.1)}$$

where $v_0$ and $w_0$ are defined by

$$v_0(t) = p^* \sqrt{\varepsilon} \exp(-tA(0)) \quad \text{... (2.2)}$$

and

$$w_0(t) = q^* \sqrt{\varepsilon} \exp(-(1-t)A(1)). \quad \text{... (2.3)}$$

Here $R(t)$, $p^*$ and $q^*$ are bounded functions of $\varepsilon$ not depending on $t$. The proof for the result (i) above is analogous to Lemma 6.1 of Doolan *et al.*[2] and hence it is omitted.

(ii) On taking the first derivative on both sides of

$$-\varepsilon (pv')'(t) + q(t)\, v(t) = Lv(t)$$

one gets

$$-\varepsilon p(t)\, v'''(t) = (Lv)'(t) - (qv)'(t) + \varepsilon(p'v')'(t) + \varepsilon(p'v'')(t).$$

To get the second derivative of the solution of (1.1a, b) bounded, set $v'''(t) = 0$ at $t = 0$ and 1. This will follow directly by the assumptions made in (ii). The estimate in (ii) follows by adopting the analysis described in the proof of Lemma 6.1 of Doolan *et al.*[2].

*Remark* : In particular, Lemma 2.2 holds when $u$ is the solution of (1.1a, b).

If $z$ denotes the solution of the TPBVP

$$Lz(t) = f(t) - Lv_0(t) - Lw_0(t), \quad 0 < t < 1, \qquad \text{... (2.4a)}$$

$$-z'(0) = \phi_1 + v_0'(0) + w_0'(0) \qquad \text{... (2.4b)}$$

$$z'(1) = \phi_2 - v_0'(1) - w_0'(1) \qquad \text{... (2.4c)}$$

then $u(t) = z(t) + v_0(t) + w_0(t)$. ... (2.5)

*Lemma* 2.3 — Let $u$ satisfy (1.1a, b). Then

$$u(t) = v_0(t) + w_0(t) + z(t)$$

and for all $0 \le t \le 1$,

$$|z^{(j)}(t)| \le C\,[1 + \varepsilon^{1-j/2}], \quad j \ge 2. \qquad \text{... (2.6)}$$

PROOF : Set $z(t) = z_1(t) - \varepsilon\, z_2(t)$ where

$$Lz_1(t) = f(t), \quad -z_1'(0) = z'(0), \quad z_1'(1) = z'(1)$$

and

$$Lz_2(t) = (1/\varepsilon)\,[\,Lv_0(t) + Lw_0(t)\,], \quad -z_2'(0) = z_2'(1) = 0.$$

But

$$Lv_0(t) = (\sqrt{(\varepsilon\, q(0)/p(0))})\, p'(t)\, v_0(t)$$
$$+\; p(t)\,[\,(q(t)/p(t)) - (q(0)/p(0))\,]\, v_0(t).$$

Since $(t/\sqrt{\varepsilon})\exp(-t/\sqrt{\varepsilon}) \le C$ for all $t$ and $p^*$ is bounded independent of $\varepsilon$ and $t$, then $|Lv_0(t)| \le C\varepsilon$. Similarly one gets $|Lw_0(t)| \le C\varepsilon$. And hence $|Lz_2(t)| \le C$. Using Lemma 2.2 (i),

$$|z_2^{(j)}(t)| \le C\,[1 + \varepsilon^{-j/2}] \qquad \text{... (2.7)}$$

and using Lemma 2.2 in (ii),

$$|z_1^{(j)}(t)| \le C\,[1 + \varepsilon^{1-j/2}]. \qquad \text{... (2.8)}$$

From (2.7) and (2.8), the required estimate (2.6) follows.

## 3. Difference Scheme I

We propose the following uniform scheme for the BVP (1.1a, b) :

$$L^h u_i \equiv -\varepsilon\sigma_i \,\delta\,(p(t_i)\,\delta u_i) + q(t_i)\,u_i = f(t_i),\ 1 \le i \le N-1, \qquad \text{... (3.1a)}$$

$$B_0^h u_0 \equiv -D_+ u_0 = \phi_1,\ B_1^h u_N = D_- u_N = \phi_2 \qquad \text{... (3.1b)}$$

where

$$\delta(p(t_i)\,\delta u_i) = [p(t_{i-1/2})\,u_{i-1} - \{p(t_{i-1/2}) + p(t_{i+1/2})\}u_i + p(t_{i+1/2})\,u_{i+1}\,]/h^2,$$

$$\sigma_i = (1/4)\,\gamma_i\,\rho^2 \sinh^{-2}(\rho\sqrt{\gamma_i}/2)$$

and $\gamma_i = q(t_i)/p(t_i),\ \ \rho = h/\sqrt{\varepsilon}$.

The scheme (3.1a, b) is consistent with (1.1a, b). In the following necessary and sufficient conditions for uniform convergence of the solution of the scheme (3.1a.b) are derived.

### 3.1. *Necessary Condition for Uniform Convergence*

*Theorem* 3.1 (Doolan *et al.*[2]) — Assume that the solution of (3.1a, b) converges uniformly in $\varepsilon$ to the solution of (1.1a, b). Let $\rho = h/\sqrt{\varepsilon}$ and $n$, a non negative integer, be fixed. Then

$$\lim \sigma_n = \rho^2\,\gamma_0/(4\sinh^2(\rho\sqrt{\gamma_0}/2))$$

and

$$\lim \sigma_{N-n} = \rho^2\,\gamma_1/(4\sinh^2(\rho\sqrt{\gamma_1}/2))$$

where $\gamma_0 = q(0)/p(0)$, $\gamma_1 = q(1)/p(1)$ and lim denotes the limit $h$ goes to zero.

The scheme (3.1a, b) satisfies the necessary condition for uniform convergence and so the scheme (3.1a, b) is exponentially fitted.

### 3.2. *Sufficient Conditions for Uniform Convergence*

The analytical results presented for the solution of (1.1a, b) in the previous section will enable us to prove the convergence of the scheme (3.1a, b) easily. The operator $L^h$ admits a discrete maximum principle which is stated in Lemma 3.2. The stability result is given in Corollary 3.3. The main result is contained in Theorem 3.4.

The scheme (3.1a, b) can be written in the form

$$Au = f$$

where $u = (u_0, ..., u_N)^T$, $f = (\phi_1, f(t_1), ..., f(t_{N-1}), \phi_2)^T$, $A$ is a $(N+1)$-square matrix, $u$ and $f$ are $(N+1)$-vectors and the entrices $a_{ij}$ of $A$ are given by

$$a_{00} = 1/h,\ a_{01} = -1/h,\ a_{NN} = 1/h,\ a_{NN-1} = -1/h,$$

$$a_{ii} = (2\sigma_i/\rho^2) + b(t_i),$$

$$a_{ii+1} = -\sigma_i/\rho^2 = a_{ii-1}$$

and

$$a_{ij} = 0 \text{ otherwise.}$$

The matrix $A$ is a $M$-matrix (Doolan *et al.*[2] and Varga[19]). The operator $L^h$ admits a discrete maximum principle, which is stated in Lemma 3.2. The stability result is given in Corollary 3.3.

*Lemma* 3.2 — Let $v_i$ be a mesh function.

(1) If $B_0^h v_0 \geq 0$, $B_1^h v_N \geq 0$ and $L^h v_i \geq 0$ for all $i = 1(1)N - 1$, then $v_i \geq 0$ for all $i = O(1)N$.

(2) For all $i = O(1)N$,

$$| v_i | \leq C \max \left( | B_0^h v_0 |,\ | B_1^h v_N |,\ \max\ | L^h v_j | \right).$$

PROOF : Using the procedure described in p. 167 of Doolan *et al.*[2] one can prove the result (1). Constructing a suitable discrete barrier function (Doolan *et al.*[2]) the result (2) can be derived from the result (1).

*Corollary* 3.3 — Let $u_i$ be the solution of the scheme (3.1a, b) then

$$| u_i | \leq C \max \left( | \phi_1 |,\ | \phi_2 |,\ \max | f(t_j) | \right).$$

The main result is contained in the following.

*Theorem* 3.4 — Let $u(t)$ and $u_i$ be the solutions of (1.1a, b) and (3.1a, b) respectively. Then, for all $i = O(1)N$,

$$| u(t_i) - u_i | \leq Ch. \qquad \text{... (3.2)}$$

PROOF : The function error $\tau_i$ for the scheme (3.1a, b) with respect to the TPBVP (1.1a, b) is,

for $i = 0$, $\tau_0 = B_0^h\ [u(0) - u_0]$

$$= B_0^h u(0) - \phi_1 = B_0^h u(0) - B_0 u(0)$$

$$= -\ [D_+ u(0) - u'(0)], \qquad \text{... (3.3)}$$

for $i = N$, $\tau_N = B_1^h [u(1) - u_N]$

$$= B_1^h u(1) - \phi_2 = B_1^h u(1) - B_1 u(1)$$

$$= D_- u(1) - u'(1), \qquad \text{... (3 4)}$$

and for $i = 1(1)\ N - 1$,

$$\tau_i = L^h [u(t_i) - u_i] = L^h u(t_i) - L^h u_i$$
$$= L^h u(t_i) - f(t_i) = L^h u(t_i) - Lu(t_i). \quad \text{... (3.5)}$$

Using Lemma 2.2 (ii)

$$|\tau_0| \le Ch\, |u^{(2)}(\theta_1)| \le Ch \quad \text{... (3.6)}$$

and

$$|\tau_N| \le Ch\, |u^{(2)}(\theta_2)| \le Ch \quad \text{... (3.7)}$$

where $1 < \theta_1 < h$ and $1 - h < \theta_2 < 1$. Again using Lemma 2.3, for $i = 1(1)\ N - 1$,

$$\tau_i = [\, L^h z(t_i) - Lz(t_i)\,] + [\, L^h v_0(t_i) - Lv_0(t_i)\,]$$
$$+ \ [\, L^h w_0(t_i) - Lw_0(t_i)\,], \quad \text{... (3.8)}$$

where

$$|L^h z(t_i) - Lz(t_i)| = |\varepsilon\, \sigma_i\, \delta(p(t_i)\, \delta z(t_i)) - \varepsilon(p(t_i)\, z'(t_i))'|$$
$$\le \varepsilon\, |\sigma_i - 1|\, |\delta(p(t_i)\, \delta z(t_i))|$$
$$+ \ \varepsilon\, |\delta(p(t_i)\, \delta z(t_i)) - (p(t_i) z'(t_i))'|$$
$$\le Ch^2, \quad \text{... (3.9)}$$

$$|L^h v_0(t_i) - Lv_0(t_i)|$$
$$\le |\varepsilon\, \sigma_i\, \delta(p(t_i)\, \delta v_0(t_i)) - \varepsilon(p(t_i)\, v_0'(t_i))'|$$
$$\le \varepsilon\, |(\sigma_i - \sigma_0)\, \delta(p(t_i)\, \delta v_0(t_i))|$$
$$+ \ \varepsilon\, |(p(t_i)\, v_0'(t_i))' - \sigma_0\, \delta\, (p(t_i)\, \delta v_0(t_i))|$$
$$\le Ch\sqrt{\varepsilon} \le Ch, \quad \text{... (3.10)}$$

and

$$|L^h w_0(t_i) - Lw_0(t_i)| \le \varepsilon\, |(\sigma_i - \sigma_N)\, \delta(p(t_i)\, \delta w_0(t_i))|$$
$$+ \ \varepsilon\, |(p(t_i)\, w_0'(t_i))' - \sigma_N\, \delta(p(t_i)\, \delta w_0(t_i))|$$
$$\le Ch\sqrt{\varepsilon} \le Ch, \quad \text{... (3.11)}$$

since (Doolan *et al.*[2])

$$\varepsilon\, |\sigma_i - 1| \le Ch^2,$$

$$\varepsilon \mid (\sigma_i - \sigma_0)\, \delta(p(t_i)\, \delta v_0(t_i)) \mid \leq Ch\sqrt{\varepsilon},$$

$$\varepsilon \mid (\sigma_i - \sigma_N)\, \delta(p(t_i)\, \delta w_0(t_i)) \mid \leq Ch\sqrt{\varepsilon},$$

$$\varepsilon \mid (p(t_i) v_0'(t_i))' - \sigma_0\, \delta(p(t_i)\, \delta v_0(t_i)) \mid \leq Ch\sqrt{\varepsilon},$$

and

$$\varepsilon \mid (p(t_i)\, w_0'(t_i)' - \sigma_N\, \delta(p(t_i)\, \delta w_0(t_i)) \mid \leq Ch\sqrt{\varepsilon}.$$

From (3.6)-(3.11) the desired estimate (3.2) follows from the stability result (Corollary 3.3).

Thus the scheme (3.1a, b) is uniformly convergent of order one. It is to be noted that as $\varepsilon$ goes to zero the scheme (3.1a, b) solves exactly the reduced problem (1.2) showing that it is consistent even at the reduced situation. And so the scheme (3.1a, b) reflects the asymptotic behaviour of the solution of (1.1a, b) exactly as $\varepsilon$ goes to zero.

## 4. DIFFERENCE SCHEME II

The following scheme proposed for (1.1a, b) reflects not only the asymptotic properties of the solution of (1.1a, b) but also works very well compared with the scheme (3.1a, b) :

$$\begin{aligned} L^h u_i &\equiv -\varepsilon\, \sigma_i\, \delta(p(t_i)\delta u_i) + q(t_i)u_i \\ &= \gamma_{1,i} f(t_i)/q(t_i) + \gamma_{2,i} f(t_{i-1})/q(t_{i-1}) \\ &\quad + \gamma_{3,i} f(t_{i+1})/q(t_{i+1}),\ 1 \leq i \leq N-1 \end{aligned} \qquad \text{... (4.1a)}$$

$$B_0^h u_0 = -D_+ u_0 = \phi_1,\ B_1^h u_N = D_- u_N = \phi_2 \qquad \text{... (4.1b)}$$

where $\sigma_i$ is defined as in scheme (3.1a, b),

$$\gamma_{1,i} = p(t_i) + ((\sigma_i - 1)/\rho^2)\,[p(t_{i-1/2}) + p(t_{i+1/2})\,],$$

$$\gamma_{2,i} = ((1-\sigma_i)/\rho^2)\, p(t_{i-1/2})$$

and

$$\gamma_{3,i} = ((1-\sigma_i)/\rho^2)\, p(t_{i+1/2}).$$

The scheme (4.1a, b) is consistent with (1.1a, b). It is easy to verify that this scheme satisfies the necessary condition given in Theorem 3.1 and hence exponentially fitted. Further one can derive the same estimate (3.1) for the present scheme. Still this scheme performs better than the earlier schemes (3.1a, b) (Ref. Table III). The scheme solves exactly the reduced problem (1.2) of the problem (1.1a, b) for small values of $\varepsilon$.

## 5. Numerical Experiment

In this section, some numerical results are presented. Experimental orders of uniform convergence of the schemes of this paper are obtained following the method given in Doolan *et al.*[2]. We define, for $N = 8, 16, \ldots, 512$.

$$\text{EMAX} = \max |u_j^N - u_j^{2N}|, \quad 0 \le j \le N,$$

$$\text{RATE} = \{\log(E^1) - \log(E^2)\},$$

$$\text{ORDER1} = \text{average}\ (\text{RATE})_h$$

and

$$\text{ORDER2} = \text{average}\ (\text{RATE})_\varepsilon$$

where $E^1$ and $E^2$ correspond to EMAX for $h = 1/N$ and $h = 1/2N$, respectively. The experimental order of uniform convergence is taken to be the minimum value of ORDER1 over all $\varepsilon$ considered, whereas the experimental order of classical convergence is taken as the value of ORDER1 for the largest $\varepsilon$ considered. In the case of optimal convergence the experimental order is taken to be the minimum value of ORDER2 over all $h$ considered. Here ORDER1 is introduced by Doolan *et al.*[2] and ORDER2 is introduced by Selvakumar[13] for singularly perturbed linear and non-linear problems.

The experimental orders are given in Tables I and II for the schemes (3.1a, b) and (4.1a, b) respectively. Table III compares the schemes (3.1a, b) and (4.1a, b).

## Conclusions

This paper presents two uniformly convergent finite difference schemes for singular perturbation problems arising in gas porous electrodes theory. Both the schemes are new, exponentially fitted, unconditionally stable for all values of the parameter $\varepsilon$ and the mesh size $h$ and the schemes are of order one. Both the schemes reflect the asymptotic behaviour of the solution of (1.1a, b) for small values of $\varepsilon$. Using the method given in Miller[7] one can study the local behaviour of the solution of (1.1a, b) in the neighbourhood of $\varepsilon$, when $\varepsilon$ is very small using the schemes given in this paper. Using the approach given in Selvakumar[13] one can frame higher order methods for the numerical solution of (1.1a, b). Based on the computational procedure given in Selvakumar[15] one can convert the three-point scheme into single step method.

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TABLE I

*Test problem* :

$$-\varepsilon((1+t)u')' + (1+t)u = (1+t)^2 - \varepsilon, \quad 0 < t < 1,$$

$$-u'(0) = -1, \quad u'(1) = 1$$

*Scheme* : (3.1a, b)

| | Experimental order of convergence | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon$ $h \rightarrow$ | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 | 1/256 | 1/512 | ORDER1 |
| 5.00000E-01 | 2.50010E+00 | 1.61891E+00 | 3.25873E+00 | 2.41685E+00 | 5.41956E-01 | 1.30576E+00 | 4.02863E+00 | 2.23870E+00 |
| 2.50000E–01 | 2.09323E+00 | 2.42360E+00 | 2.22484E+00 | 3.27829E+00 | 6.05973E–01 | 2.95496E–01 | 3.64194E+00 | 2.08048E+00 |
| 1.25000E–01 | 1.94819E+00 | 2.90643E+00 | 1.39403E–01 | 2.60590E+00 | 2.27824E+00 | 1.23015E+00 | 3.33424E+00 | 2.06322E+00 |
| 6.25000E–02 | 1.95605E+00 | 2.50130E+00 | 9.79822E–01 | 1.47393E+00 | 4.47492E+00 | 2.60058E–01 | 3.10622E+00 | 2.10747E+00 |
| 3.12500E–02 | 1.92470E+00 | 2.12003E+00 | 2.21131E+00 | 1.18511E+00 | 5.29813E+00 | 8.41798E–01 | 1.89102E+00 | 2.21030E+00 |
| 1.56250E–02 | 1.88674E+00 | 2.03769E+00 | 2.08746E+00 | 9.18386E–01 | 2.84503E+00 | 2.61701E+00 | 1.74525E+00 | 2.01965E+00 |
| 7.81250E–03 | 1.81781E+00 | 2.01972E+00 | 1.70210E+00 | 1.33583E–01 | 2.37776E+00 | 3.49185E+00 | 8.05603E–01 | 1.76406E+00 |
| 3.90625E–03 | 1.68408E+00 | 1.92699E+00 | 1.70907E+00 | 9.89804E–01 | 2.61180E+00 | 3.59325E+00 | 1.21693E–01 | 1.80524E+00 |
| ORDER2 | 1.97636E+00 | 2.19433E+00 | 1.78909E+00 | 1.62523E+00 | 2.62923E+00 | 1.70442E+00 | 2.33433E+00 | ....... |

Classical order of convergence = 2.23870

Uniform order of convergence = 1.76406

Optimal order of convergence = 1.62523

TABLE II

*Test problem* :

$$-\varepsilon((1+t)u')' + (1+t)u = (1+t)^2 - \varepsilon, \quad 0 < t < 1,$$

$$-u'(0) = -1, \quad u'(1) = 1$$

*Scheme* : (4.7a, b)

| | Experimental order of convergence | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon$ $h \to$ | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 | 1/256 | 1/512 | ORDER1 |
| 5.00000E–01 | 3.30661E+00 | 2.54896E+00 | 1.89095E+00 | 2.41694E+00 | 5.42043E–01 | 1.30576E+00 | 4.02863E+00 | 2.29141E+00 |
| 2.50000E–01 | 2.42957E+00 | 1.30968E+00 | 1.74269E+00 | 3.27829E+00 | 6.05973E–01 | 2.95496E+00 | 3.64194E+00 | 1.90052E+00 |
| 1.25000E–01 | 1.82970E+00 | 3.62570E–01 | 1.41504E+00 | 1.76637E+00 | 2.27824E+00 | 1.23015E+00 | 3.33424E+00 | 1.74519E+00 |
| 6.25000E–02 | 2.16992E+00 | 8.14968E–01 | 1.39593E+00 | 6.78072E–01 | 4.47492E+00 | 2.60058E–01 | 3.10622E+00 | 1.84287E+00 |
| 3.12500E–02 | 1.84583E+00 | 1.01937E+00 | 1.00000E+00 | 4.22233E–01 | 5.22451E+00 | 8.41798E–01 | 1.89102E+00 | 1.74925E+00 |
| 1.56250E–02 | 2.08246E+00 | 8.24622E–02 | 2.28951E+00 | 8.62496E–01 | 3.08534E+00 | 2.61701E+00 | 1.74525E+00 | 1.82350E+00 |
| 7.81250E–03 | 2.40210E+00 | 1.58496E+00 | 1.51457E+00 | 1.12653E+00 | 2.56277E+00 | 3.47309E+00 | 8.05603E–01 | 1.92423E+00 |
| 3.90625E–03 | 1.50250E+00 | 1.22239E+00 | 1.94753E+00 | 6.88056E–01 | 2.34496E+00 | 3.56690E+00 | 1.21693E–01 | 1.62772E+00 |
| ORDER2 | 2.19609E+00 | 1.11817E+00 | 1.64953E+00 | 1.40487E+00 | 2.63984E+00 | 1.69878E+00 | 2.33433E+00 | ....... |

Classical order of convergence = 2.28141

Uniform order of convergence = 1.62772

Optimal order of convergence = 1.11817

TABLE III

*Test problem* :

$$-\varepsilon((1+t)u')' + (1+t)u = (1+t)^2 - \varepsilon, \quad 0 < t < 1,$$

$$-u'(0) = -1, \quad u'(1) = 1$$

$h = 1/8$

| $\varepsilon$ | Scheme (3.1a, b) | Scheme (4.1a, b) |
|---|---|---|
| 0.01 | 1.02949E − 03 | 7.74860E − 06 |
| 0.001 | 6.10590E − 04 | 5.96046E − 07 |
| 0.0001 | 8.89301E − 05 | 2.38419E − 07 |
| 0.00001 | 8.94070E − 06 | 0.0 |
| 0.000001 | 0.0 | 0.0 |

$h = 1/16$

| $\varepsilon$ | Scheme (3.1a, b) | Scheme (4.1a, b) |
|---|---|---|
| 0.01 | 2.82526E − 04 | 2.38419E − 06 |
| 0.001 | 2.52366E − 04 | 2.38419E − 07 |
| 0.0001 | 8.69036E − 04 | 1.19209E − 07 |
| 0.00001 | 1.88351E − 05 | 2.38419E − 07 |
| 0.000001 | 9.41753E − 06 | 2.38419E − 07 |

$h = 1/32$

| $\varepsilon$ | Scheme (3.1a, b) | Scheme (4.1a, b) |
|---|---|---|
| 0.01 | 7.30753E − 05 | 9.53674E − 07 |
| 0.001 | 7.40290E − 05 | 2.38419E − 07 |
| 0.0001 | 5.13792E − 05 | 2.38419E − 07 |
| 0.00001 | 9.77516E − 06 | 2.38419E − 07 |
| 0.000001 | 9.53674E − 07 | 1.19209E − 07 |

Above tables give the maximum of absolute errors taken over all the nodal points in the interval [0, 1].

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