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A CLASS OF SOLUTIONS FOR EINSTEIN FIELD EQUATIONS WITH SPATIALLY VARYING COSMOLOGICAL CONSTANT IN SPHERICALLY SYMMETRIC ANISOTROPIC SOURCE

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In this work a class of interior solution for Einstein field equations corresponding to a spherically symmetric anisotropic fluid sphere has been obtained under the assumption that the cosmological constant is spatially variable. The solution obtained has the characteristics that the pressure and the cosmological parameter vanish at the centre and at the boundary with a maximum value somewhere inside the body. It has been argued that a variable Λ is as much important physically in Astrophysics as in Cosmology.

1. INTRODUCTION

The introduction of cosmological constant Λ in Einstein field equations has diverse opinions (Ellis[4], Christopher[1, 2], Curry[3]). Theoretically, with the introduction of Λ the Einstein field equations become more general than those without Λ. Observational evidences, however, show that the contribution of constant Λ is negligibly small.

A variable Λ, on the other hand, has started attracting the attention of theoreticians as well as experimentalists both (Sakharov[9], Gunn and Tinsley[5], Peebles and Ratra[6], Wampler and Burke[10], Ray and Ray[8]).

Thus the present work is motivated to study the implications of variable Λ from astrophysical point of view. We consider here an anisotropic massive static spherically symmetric fluid core having Schwarzschild exterior field outside the boundary.

The scheme is as follows. In section 2 field equations are provided. Section 3 deals with the solutions whereas some comments are made in section 4.

2. FIELD EQUATIONS

The Einstein field equations for the anisotropic fluid distribution are given by

$$R_{ij} - \frac{1}{2} g_{ij} R + g_{ij} \Lambda = -8\pi T_{ij}, \quad (G = C = 1) \qquad \text{... (1)}$$

where the matter-momentum tensor is given by T^i_j = diag (ρ, $-p_r$, $-p_\perp$, $-p_\perp$) and the related conservation law here is (Peebles and Ratra[6])

$$8\pi T^i_{j;\,i} = -\Lambda_{;\,j} \qquad \text{... (2)}$$

as the cosmological constant is assumed to be spatially varying, i.e. $\Lambda = \Lambda(r)$.

We choose the spherically symmetric and static line element which in the Schwarzschild coordinates $(t, r, \theta, \varphi) \equiv (0, 1, 2, 3)$ reads

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2). \qquad \text{... (3)}$$

Hence, the field equations (1) corresponding to the above line element (3), are given by

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda = 8\pi\rho \qquad \text{... (4)}$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda = 8\pi p_r \qquad \text{... (5)}$$

$$e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r}\right) + \Lambda = 8\pi p_\perp \qquad \text{... (6)}$$

where p_r and $p_\perp$ are respectively the radial and tangential pressures, and prime denotes derivative with respect to radial coordinate r.

The conservation equation (2) becomes

$$\frac{d}{dr}\left(p_r - \frac{\Lambda}{8\pi}\right) = -(\rho + p_r)\frac{\nu'}{2} + \frac{2}{r}(p_\perp - p_r). \qquad \text{... (7)}$$

Now, if we impose the condition

$$p_r = \frac{\Lambda}{8\pi} \qquad \text{... (8)}$$

eqn. (5) yields

$$\nu' = (e^\lambda - 1)/r. \qquad \text{... (9)}$$

Substituting (8) and (9) in (4) and (6) we have

$$(e^\lambda - 1)(\rho + p_r) = 4(p_\perp - p_r). \qquad \text{... (10)}$$

Also, from (7) by virtue of (8) we get

$$r\nu' (\rho + p_r) = 4(p_\perp - p_r). \qquad \text{... (11)}$$

Again, eqn. (4) gives

$$e^{-\lambda} = 1 - \frac{2M}{r} \qquad \text{... (12)}$$

where

$$M = 4\pi \int_0^r r^2 (\rho + p_r)\, dr \qquad \text{... (13)}$$

which is the Schwarzschild mass of a spherical body.

3. A Class of Solutions

To find out solutions of the field equations we assume that

$$p_\perp = n\, p_r, \quad (n \neq 1). \qquad \text{... (14)}$$

Using (14), eqn. (10) reduces to

$$\rho = (4n - 3 - e^{\lambda})\, p_r / (e^{\lambda} - 1) \qquad \text{... (15)}$$

which in turn makes (13) to take the form

$$M = 16(n-1)\,\pi \int_0^r \frac{r^2 p_r}{e^{\lambda} - 1}\, dr. \qquad \text{... (16)}$$

To solve the above integral, let us assume that

$$p_r = k^2 (e^{\lambda} - 1)(1 - r^2/a^2). \qquad \text{... (17)}$$

This satisfies the physical conditions at the centre and at the boundary, a being the radius of the spherical distribution of an isolated astrophysical system and k^2, a positive constant.

Using (17), we get the following solution :

$$e^{-\lambda} = 1 - AR^2 (5 - 3R^2), \qquad \text{... (18)}$$

$$e^{\nu} = \frac{(1-2A)^{5/4}}{[1 - AR^2 (5 - 3R^2)]^{1/4}} \exp\left[\frac{5B}{2} \{\tan^{-1} B(6R^2 - 5) - \tan^{-1} B\} \right], \qquad \text{... (19)}$$

$$\rho = \frac{15A\,(1-R^2)}{32(n-1)\,\pi a^2} \left[\frac{4(n-1) - (4n-3)\,AR^2 (5-3R^2)}{1 - AR^2 (5 - 3R^2)} \right], \qquad \text{... (20)}$$

$$p_r = \frac{p_\perp}{n} = \frac{\Lambda}{8\pi} = \frac{15A^2 R^2 (1-R^2)(5-3R^2)}{32(n-1)\,\pi a^2\, [1 - AR^2 (5 - 3R^2)}, \qquad \text{... (21)}$$

and

$$M = AaR^3 (5 - 3R^2)/2 \qquad \text{... (22)}$$

where

$$R = r/a, \quad A = 32(n-1)\pi k^2 a^2/15, \quad B = [A/(12 - 25A)]^{1/2}. \qquad \text{... (23)}$$

Matching of these solutions with the Schwarzschild exterior solutions at the boundary $r = a$, yields

$$A = m/a, \; B = [m/(12a - 25m)]^{1/2} \qquad \text{... (24)}$$

where m is the total active gravitational mass.

The value of the constant k, occurring in eqn. (17), then can be easily obtained by the relation

$$k^2 = 15m/[32(n-1)\pi a^3] \qquad \text{... (25)}$$

which is, obviously, always positive for $n > 1$.

4. Conclusion

It may be seen from eqns. (10) and (11) that the case $n = 1$ in relation (14), i.e., when the pressure is isotropic, does not exist in the present system as it makes the underlying space-time flat. Hence, the assumption (8) leads only to anisotropic fluid. The fluid density is constant at the centre with a value $\rho(0) = 15m/(8\pi a^3)$ and vanishes at the boundary. The corresponding fluid pressure at $r = 0$ is $p_r(0) = p_\perp(0) = 0$ and at $r = a$ is $p_r(a) = p_\perp(a) = 0$. Since p_r (or $p_\perp$) is positive in the region $0 \leq r \leq a$, it should have a maximum somewhere inside the fluid. For the present case $p_r < p_\perp$ and hence the effective pressure acts outwardly. Static equilibrium comes in the model by the adjustment of this effective pressure to the gravitational pull of the matter according to Ponce de Leon[7].

It is to be noted that like the pressures the cosmological parameter vanishes at the centre as well as at the boundary with a maximum value somewhere inside the body. However, one can also have a solution by changing the condition (8) to

$$p_r = (\Lambda - \Lambda_0)/(8\pi) \qquad \text{... (26)}$$

where Λ_0 is a constant. The cosmological parameter Λ for this solution will not vanish at the boundary. The exterior Schwarzschild solution for this case will involve the erstwhile cosmological constant to which the interior solution is to be matched. In the present solution the scalar Λ takes the role of compensatory balancing pressure varying from point to point in exactly the same way as the radial and the tangential pressures.

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