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CONNECTION COEFFICIENTS FOR A CERTAIN THIRD ORDER DIFFERENTIAL EQUATION IN THE NEIGHBOURHOOD OF AN IRREGULAR SINGULAR POINT

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This paper is devoted to a two point connection problem for the differential equation

$$z^3 \frac{d^3y}{dz^3} + z^2 (b_0 + b_1 z) \frac{d^2y}{dz^2} + z(c_0 + c_1 z + c_2 z^2) \frac{dy}{dz} + (d_0 + d_1 z + d_2 z^2) y = 0. \quad \ldots (1)$$

Here, the variable z and the constants b_i $(i = 0, 1)$, c_j, d_j $(j = 0, 1, 2)$ are complex with $c_2 \neq 0$ and $b_1^2 - 4c_2 \neq 0$. Then, the differential equation (1) will have a regular singular point at $z = 0$ and an irregular singular point at $z = \infty$. The indicial equation about $z = 0$ is found to be

$$h(h-1)(h-2) + b_0 h (h-1) + c_0 h + d_0 = 0. \quad \ldots (2)$$

We shall also assume that the difference of no two roots of (2) is congruent to zero modulo 1.

The central question in a two point connection problem is to obtain a linear relationship between two sets of fundamental solutions defined in respective neighbourhoods of two different singular points. This has been successfully carried out in this paper between the fundamental solutions at the regular singular point $z = 0$ and the irregular singular point $z = \infty$.

1. INTRODUCTION

In order to introduce the investigations of this paper, let us take for consideration the linear homogeneous differential equation of third order

$$z^3 \frac{d^3y}{dz^3} + z^2 (b_0 + b_1 z) \frac{d^2y}{dz^2} + z(c_0 + c_1 z + c_2 z^2) \frac{dy}{dz} + (d_0 + d_1 z + d_2 z^2) y = 0. \quad \ldots (1.1)$$

Here, the variable z and the constants b_i $(i = 0, 1)$, c_j, d_j $(j = 0, 1, 2)$ are complex with $c_2 \neq 0$ and $b_1^2 - 4c_2 \neq 0$. Then, in the language of Fuch's theory, the differential equation (1.1) will have a regular singular point at $z = 0$ and an irregular singular point at $z = \infty$. The indicial equation about the regular singular point $z = 0$ is found to be

$$h(h-1)(h-2) + b_0 h(h-1) + c_0 h + d_0 = 0. \qquad \text{... (1.2)}$$

We shall also assume that the roots h_i $(i = 1, 2, 3)$ of (1.2) are such that the difference of no two of them is congruent to zero modulo 1. Then, the existing theory assumes that (1.1) will have three fundamental solutions $y_i(z)$, $i = 1, 2, 3$ of the forms

$$y_i(z) = z^{h_i} \sum_{n=0}^{\infty} g_i(n) z^n$$

with $\qquad g_i(0) = 0, \quad i = 1, 2, 3. \qquad$ ... (1.3)

Here, $g_i(n)$, $n = 1, 2, 3, \ldots$ are determinate functions of n and we know that each of these series solutions will converge in the whole open plane of the complex z-plane. The behaviour of these solutions in the neighbourhood of $z = \infty$ is not available so far as Fuch's theory is concerned. However, we know from the established theory of linear differential equations that there exist three linearly independent solutions $\tilde{y}_i(z)$ $(i = 1, 2, 3)$ with asymptotic representations valid in some neighbourhood of $z = \infty$. It is also well known that threre exist a set of connection coefficients c_{ij} such that

$$y_i(z) = \sum_{j=1}^{3} c_{ij} \tilde{y}_j \qquad (i = 1, 2, 3). \qquad \text{... (1.4)}$$

These connection coefficients are called Stokes multipliers. Until some method of computing these connection coefficients is devised, the solutions of the problem in the large cannot be regarded as solved. It is to the solution of this problem in the large that this present paper has been addressed.

2. Recurrence Relations Satisfied by each $g_i(n)$, $i = 1, 2, 3$

Substituting $y = \Sigma_{n=0}^{\infty} g(n) z^{n+h}$ into (1.1) and equating the coefficients of z^{n+h+2}, we get

$$g(n+2) P_2(n) + g(n+1) P_1(n) + g(n) P_0(n) = 0 \qquad \text{... (2.1)}$$

where

$$P_2(n) = (n+h+2)(n+h+1)(n+h)$$
$$+ b_0(n+h+2)(n+h+1) + c_0(n+h+2) + d_0, \qquad \text{... (2.2a)}$$

$$P_1(n) = b_1(n+h+1)(n+h) + c_1(n+h+1) + d_1, \qquad \text{... (2.2b)}$$

$$P_0(n) = c_2(n+h) + d_2. \qquad \text{... (2.2c)}$$

We observe that $g(n)$, $n = 1, 2, 3, \ldots$ are completely and correctly determined by (2.1), if we arbitrarily take $g(-1) = 0$, as we do. Further, we make the assumption that when one of h_i $(i = 1, 2, 3)$ is used for h, the root α of the equation $P_2(n-2) = 0$ is not zero or a negative integer. Then it turns out that $g(-n) = 0$ for $n = 2, 3, \ldots$ We will remove this restriction later on.

Since we wish to regard n as a complex variable, to emphasize this, we replace n by s. Our first objective, then, is to find that analytic function $u(s)$ of the complex variable s, which is a solution of (2.1) and which takes on the initial values $u(0) = 1$ and $u(-1) = 0$. Such a $u(s)$ will coincide with $g_i(n)$, $n = 0, 1, 2, \ldots$ when h_i, $i = 1, 2, 3$ is used for h.

3. Fundamental Solutions of (2.1)

Let $$u(s) = \frac{v(s)}{\Gamma(s)}.$$

Then (2.1) reduces to the form

$$v(s+2)\,Q_2(s) + v(s+1)\,Q_1(s) + v(s)\,Q_0(s) = 0 \qquad \text{... (3.1)}$$

where

$$Q_2(s) = P_2(s),$$

$$Q_1(s) = (s+1)\,P_1(s),$$

and $$Q_0(s) = s(s+1)\,P_0(s).$$

Let

$$V_1(s) = v(s)$$

$$V_2(s) = v(s+1).$$

Then (3.1) can be put in the matrix form

$$G(s+1) = A(s)\;G(s) \qquad \text{... (3.2)}$$

where $G(s)$ is a 2×2 matrix which can be decomposed into two column vectors each with two components and $A(s)$ is a 2×2 matrix, representable as a Power series in $1/s$ which is convergent in some neighbourhood of $s = \infty$. Therefore, there is a finite s_0 such that the series

$$A(s) = A_0 + \frac{A_1}{s} + \frac{A_2}{s^2} + \ldots \text{ converges, if } |\,s\,| > s_0 \qquad \text{... (3.3)}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ \gamma_0 & \omega_0 \end{pmatrix}, \quad A_j = \begin{pmatrix} 0 & 0 \\ \gamma_j & \omega_j \end{pmatrix}, \; j = 1, 2, \ldots . \qquad \ldots (3.4)$$

We find that

$$\left.\begin{aligned} \gamma_0 &= -c_2 \neq 0, \\ \omega_0 &= -b_1, \\ \gamma_1 &= -c_2(2h+2+b_0) - d_2, \\ \omega_1 &= b_1(b_0+h+1) - c_1. \end{aligned}\right] \qquad \ldots (3.5)$$

By applying a theorem of Culmer and Harris[2] the two, linearly independent solutions $u_1(s)$ and $u_2(s)$ of (2.1) are given in terms of known factorial series as follows :

$$u_1(s) = \frac{s^{r_1} c^s [1 + E_1(s)]}{\Gamma(s)}, \quad u_2(s) = \frac{s^{r_2} d^s [1 + E_2(s)]}{\Gamma(s)} \qquad \ldots (3.6)$$

where

$$r_1 = \frac{1}{c(d-c)} [-\gamma_1 - \omega_1 c], \quad r_2 = \frac{1}{d(d-c)} [-\gamma_1 - \omega_1 d] \qquad \ldots (3.7)$$

with

$$c = \frac{1}{2} [-b_1 + \sqrt{b_1^2 - 4c_2}], \quad d = \frac{1}{2} [-b_1 - \sqrt{b_1^2 - 4c_2}].$$

We note that $c \neq d$, since $b_1^2 - 4c_2 \neq 0$.

Here, $u_1(s)$ and $u_2(s)$ are analytic at every point in some right half plane and in this right half plane, $E_i(s) \to 0$ $(i = 1, 2)$ as $|s| \to \infty$.

4. Asymptotic Representations of Solutions $\tilde{y}_i(z)$, $i = 1, 2, 3$ About the Irregular Singular Point $z = \infty$

The point $z = \infty$, as already noted is an irregular singular point of our differential equation (1.1). By applying Birkhoff's theorem[1], we find that the differential equation has the characteristic equation

$$\rho^3 + b_1 \rho^2 + c_2 \rho = 0. \qquad \ldots (4.1)$$

Hence, (4.1) has three distinct roots 0, c, d, where c, d, are as given in (3.7).

Then (1.1) will have three linearly independent solutions $\tilde{y}_1(z)$, $\tilde{y}_2(z)$ and $\tilde{y}_3(z)$ about $z = \infty$ for which when $|z|$ is large

$$\tilde{y}_1(z) \sim s_1(z),$$

$$\tilde{y}_2(z) \sim s_2(z),$$

$$\tilde{y}_3(z) \sim s_3(z)$$

where $s_1(z)$, $s_2(z)$ and $s_3(z)$ are given by

$$\left.\begin{aligned} s_1(z) &= z^{k_1} e^{cz}\left[1 + \frac{()}{z} + \frac{()}{z^2} + \dots\right], \\ s_2(z) &= z^{k_2} e^{dz}\left[1 + \frac{()}{z} + \frac{()}{z^2} + \dots\right], \\ s_3(z) &= z^{k_3}\left[1 + \frac{()}{z} + \frac{()}{z^2} + \dots\right], \end{aligned}\right\} \quad \dots (4.2)$$

and k_1, k_2 and k_3 are constants whose values may be determined by formal substitution of these series into (1.1) and are thus found to be

$$\left.\begin{aligned} k_1 &= \frac{b_0c^2 + cc_1 + d_2}{c(d-c)} = 1 + h + r_1, \\ k_2 &= \frac{b_0d^2 + dc_1 + d_2}{d(d-c)} = 1 + h + r_2, \\ k_3 &= -\frac{d_2}{c_2}. \end{aligned}\right\} \quad \dots (4.3)$$

5. Main Results and Main Theorem

Define $f_0(h) = h(h-1)(h-2) + b_0h(h-1) + c_0h + d_0$ ... (5.1)

$$f_1(h) = b_1h(h-1) + c_1h + d_2. \quad \dots (5.2)$$

In view of (1.2), we observe that $f_0(h) \equiv 0$.

From (2.2), we have

$$P_2(s) = f_0(s + h + 2) \quad \dots (5.3)$$

$$P_1(s) = f_1(s + h + 1) \quad \dots (5.4)$$

and $P_0(s) = c_2(s - \alpha)$, where $\alpha = -h - \dfrac{d_2}{c_2}$.

In view of (1.2), $P_2(-2) = f_0(h) = 0$. So 0 is a root of

$$P_2(s-2) = (s+h)(s+h-1)(s+h-2) + b_0(s+h)(s+h-1) + c_0(s+h) + d_0 = 0. \quad \dots (5.5)$$

Let β_1 and β_2 be other roots of $P_2(s-2) = 0$.

We have already noted that $c \neq d$, where c and d are given by (3.7). Let us suppose that $\arg c \neq \arg d$. For definiteness, let us assume that each of these arguments is chosen to lie within $[0, 2\pi]$. Let us consider the four mutually exclusive sections of the z-plane defined as follows :

$$\text{(I)}\begin{cases}-\frac{\pi}{2}\le \arg cz\le\frac{\pi}{2}\\ \frac{\pi}{2}<\arg dz<\frac{3\pi}{2}\end{cases}\qquad \text{(II)}\begin{cases}\frac{\pi}{2}\le \arg cz\le\frac{3\pi}{2}\\ \frac{\pi}{2}<\arg dz<\frac{3\pi}{2}\end{cases}$$

$$\text{(III)}\begin{cases}-\frac{\pi}{2}<\arg cz\le\frac{3\pi}{2}\\ \frac{-\pi}{2}<\arg dz<\frac{\pi}{2}\end{cases}\qquad \text{(IV)}\begin{cases}\frac{-\pi}{2}\le \arg cz\le\frac{\pi}{2}\\ \frac{-\pi}{2}<\arg dz<\frac{\pi}{2}\end{cases} \quad \ldots (5.6)$$

The main theorem is the following :

If the linear independent solutions $\tilde{y}_i(z)$ of (1.1) about the irregular singular point $z = \infty$ are $\sim s_i(z)$ (i = 1, 2, 3), when $|z|$ is large, where $s_i(z)$ are given by (4.2) then the fundamental solutions $y_j(z), j$ = 1, 2, 3 of (1.1) as defined in (1.3) about the regular singular point z = 0, for values of z of large modulus, permits of the following asymptotic developments according as z lies in section I, II, III or IV of (5.6) :

$$\left.\begin{aligned} y_i(z) &\sim s_1(h_j)\,s_1(z)+c_3\,(h_j)\,s_3\,(z) \text{ in sector (I)},\\ y_j(z) &\sim -z^h R(z) \text{ in sector (II)},\\ y_i(z) &\sim c_2(h_j)\,s_2(z)+c_3(h_j\,s_3(z) \text{ in sector (III)}\\ y_j(z) &\sim c_1(h_j)\,s_1(z)+c_2(h_j)\,s_2(z)+c_3(h_j)\,s_3(z) \text{ in sector (IV)}\end{aligned}\right] \quad \ldots (5.7)$$

where

(a) in case $h_j+\dfrac{d_2}{c_2}$ is either not an integer or is an integer < 2, the constant $c_3(h_j)$ is given by

$$c_3(h)=\frac{\Gamma\left(2-h-\dfrac{d_2}{c_2}\right)\exp\left(\pi i\left(h+\dfrac{d_2}{c_2}\right)\right)\Gamma\,(1-\beta_1)\,\Gamma\,(1-\beta_2)}{\sqrt{b_1^2-4c_2}}\;w\left(-h-\frac{d_2}{c_2}\right) \quad \ldots (5.8)$$

with
$$w(s)=V^*\,(s+2)f_0\left(2-\frac{d_2}{c_2}\right)+V^*\,(s+1)f_1\left(1-\frac{d_2}{c_2}\right) \quad \ldots (5.9)$$

in which h_j is used for h while $c_1(h)$ and $c_2(h)$ are determined by

$$c_1(h) = \frac{-u_2\,(-1)}{D(-1)}\,c^{1+r_1}=-\frac{c_2}{\sqrt{b_1^2-4c_2}}\,\frac{\Gamma(1-\beta_1)\,\Gamma(1-\beta_2)\,c^{1+r_1}}{\Gamma\left(-1+h+\dfrac{d_2}{c_2}\right)}, \quad \ldots (5.10a)$$

$$c_2(h) = \frac{u_1(-1)}{D(-1)} d^{1+r_2} = -\frac{c_2}{\sqrt{b_1^2 - 4c_2}} \frac{\Gamma(1-\beta_1)\,\Gamma(1-\beta_2)\, d^{1+r_1}}{\Gamma\left(-1 + h + \frac{d_2}{c_2}\right)}. \quad \dots (5.10b)$$

(b) In case $h_j + \frac{d_2}{c_2}$ is an integer $n \geq 2$, the constant $c_3(h_j)$ is given by

$$c_3(h) = \frac{\left(\Gamma(2 - h - \frac{d_2}{c_2}\right)\Gamma(1-\beta_1)\,\Gamma(1-\beta_2)}{\sqrt{b_1^2 - 4c_2}\;\Gamma(n-1)} \left[\frac{d}{ds}\{w(s)\}\right]_{s=-h-\frac{d_2}{c_2}=-n} \quad \dots (5.11)$$

in which h_j is used for h, while the constants $c_1(h_j)$ and $c_2(h_j)$ continues to be determined from (5.10) as in case (a).

Proof of the theorem will be given in Sections 6, 7 and 8.

6. Solutions $y_i(z)$, $i = 1, 2, 3$ About the Regular Singular Point $z = 0$

By using well known asymptotic properties of gamma functions, the two fundamental solutions $u_1(s)$ and $u_2(s)$ of (2.1) as given by (3.6) can be written as

$$u_1(s) = c^s\, G_1(s) \quad \dots (6.1)$$

$$u_2(s) = d^s\, G_2(s) \quad \dots (6.2)$$

where

$$G_1(s) = \frac{[1 + E_1(s)]}{\Gamma(s - r_1)} \quad \dots (6.3)$$

$$G_2(s) = \frac{[1 + E_2(s)]}{\Gamma(s - r_2)} \quad \dots (6.4)$$

These two solutions $u_1(s)$ and $u_2(s)$ are analytic in some right half plane and $E_i(s) \to 0$ for $i = 1, 2$ as $|s| \to \infty$ in this right half plane.

By rendering precise meaning through certain conversions and by analytic continuation by means of the difference equations (2.1), when we write this equation in the form

$$u(s) = \frac{-[P_2(s)\, u(s+2) + P_1(s)\, u(s+1)]}{P_0(s)} \quad \dots (6.5)$$

we find that $u_1(s)$ and $u_2(s)$ are single valued and analytic throughout the finite s-plane except for poles at the point

$$s = \alpha - n, n = 0, 1, 2, \ldots, \qquad \ldots (6.6)$$

where $\alpha = -h - \frac{d_2}{c_2}$, being the root of $P_0(s) = 0$.

Now that we have obtained the two linearly independent solutions $u_1(s)$ and $u_2(s)$ of (2.1), it must be possible to determine two constants a and b such that

$$u(s) = au_1(s) + bu_2(s) \qquad \ldots (6.7)$$

will constitute the particular solution desired, that is one which will be single valued and analytic throughout the finite s-plane except for the poles (5.6) and which will be such that $u(0) = 1$ and $u(-1) = 0$ in which case if we assign the value h_i for h, we shall have $u(n) = g_i(n)$ for $n = 1, 2, 3, \ldots$ and $i = 1, 2, 3$ and $u(n) = 0$ for $n = -1, -2, \ldots$ However, in order to obtain such a function $u(s)$, we shall assume that none of the poles (5.6) is an integer. In other words, we must assume that $\alpha = -h - \frac{d_2}{c_2}$ is not an integer, this to be case when $h = h_i$, $i = 1, 2, 3$. The result of removing this restriction, we shall examine at a later point, but it must be observed that in part this restriction was made at the very outset, when we assumed that the root α of $P_0(s-2) = 0$ is not zero or a negative integer, when $h = h_i$, $i = 1, 2, 3$.

Evidently the constant a and b are to be obtained from the equations

$$\left.\begin{aligned} 0 &= au_1(-1) + bu_2(-1) \\ 1 &= au_1(0) + bu_2(0). \end{aligned}\right] \qquad \ldots (6.8)$$

If $$D(s) = \begin{vmatrix} u_1(s) & u_2(s) \\ u_1(s+1) & u_2(s+1) \end{vmatrix}$$ then

$$a = \frac{u_2(-1)}{D(-1)} \text{ and } b = \frac{u_1(-1)}{D(-1)}. \qquad \ldots (6.9)$$

Our next objective is to find the functional value of the determinant $D(s)$, known as the Casorati's determinant. By Heymann's theorem, we know that $D(s)$ satisfies

$$\frac{D(s+1)}{D(s)} = \frac{P_0(s)}{P_2(s)} = \frac{c_2(s-\alpha)}{(s-\beta_1+2)(s-\beta_2+2)(s+2)}$$

where 0, β_1, β_2 are the roots of $P_2(s-2) = 0$. Then,

$$D(s) = \frac{c_2^s\, \Gamma(s-\alpha)\sqrt{b_1^2 - 4c_2}}{\Gamma(s-\beta_1+2)\,\Gamma(s-\beta_2+2)\,\Gamma(s+2)} \qquad \ldots (6.10)$$

and

$$D(-1) = \frac{\sqrt{b_1^2 - 4c_2}}{c_2} \frac{\Gamma(-1-\alpha)}{\Gamma(1-\beta_1)\,\Gamma(1-\beta_2)}. \qquad \ldots (6.11)$$

Here, it is observed that in passing that when $h = h_i$, $i = 1, 2, 3$, neither of the expressions $1 - \beta_1$ or $1 - \beta_2$ can be zero or a negative integer.

Hence $D(-1) \neq 0$.

If $1 - \beta_i = -N$ for $i = 1, 2$ where $N = 1, 2, \ldots$, then $\beta_i = N$ for $i = 1, 2$, where $N = 1, 2, \ldots$

Since 0, β_1 and β_2 are the roots of, we will have $P_2(s-2) = 0$

$$(N+h)(N+h-1)(N+h-2) + b_0(N+h)(N+h-1) + c_0(N+h) + d_0 = 0,$$

which would empty that the roots of the indicial equation (1.2) differ by an integer, contrary to our hypothesis.

Again, we note that the numerator in (5.11) is necessarily finite in as much as we are supposing that α is not an integer. But

$$u(s) = \frac{1}{D(-1)}[-u_2(-1)\,u_1(s) + u_1(-1)\,u_2(s)]. \qquad \ldots (6.12)$$

In view of (5.11), we have

$$u(s) = \frac{c_2}{\sqrt{b_1^2 - 4c_2}} \frac{\Gamma(1-\beta_1)\,\Gamma(1-\beta_2)}{\Gamma(-1-\alpha)} V^*(s) \qquad \ldots (6.13)$$

where
$$V^*(s) = \begin{vmatrix} u_1(-1) & u_2(-1) \\ u_1(s) & u_2(s) \end{vmatrix}. \qquad \ldots (6.14)$$

It follows from (5.13) and (5.14) that the solution of (1.1) about the regular singular point $z = 0$ as defined by (1.3) is given by

$$y(z) = z^h \left[-\frac{u_2(-1)}{D(-1)} \sum_{n=0}^{\infty} G_1(n)\,(cz)^n + \frac{u_1(-1)}{D(-1)} \sum_{n=0}^{\infty} G_2(n)\,(dz)^n \right] \ldots (6.15)$$

where

$$G_1(n) = \frac{[1 + E_1(n)]}{\Gamma(n - r_1)}, \quad G_2(n) = \frac{[1 + E_2(n)]}{\Gamma(n - r_2)}.$$

7. Asymptotic Behaviour of the Solutions $y_i(z)$, $i = 1, 2, 3$ when $\alpha = -h - \frac{d_2}{c_2}$ is not an Integer

We apply Ford's VI theorem[3] to determine the asymptotic development of the functions defined by the two series in square brackets in (6.1). Then, we find that the fundamental solution $y(z)$ of (1.1) about the regular singular point $z = 0$, as defined by (1.3), when $|z|$ is large may be developed asymptotically in the form

$y(z) \sim c_1(h)\,s_1(z) - z^h R(z)$ in sector (I),

$y(z) \sim -z^h R(z)$ in sector (II),

$$y(z) \sim c_2(h)\, s_2(z) - z^h R(z) \text{ in sector (III)},$$

$$y(z) \sim c_1(h)\, s_1(z) + c_2(h)\, s_2(z) - z^h R(z) \text{ in sector (IV)}, \qquad \ldots (7.1)$$

where $c_1(h)$, $c_2(h)$, $s_1(z)$, $s_2(z)$ are as given by (5.10), (4.2) and $R(z)$ represents the sum of the residues of the function

$$\frac{\exp(-\pi is)\, u(s)}{\sin \pi s} \qquad \ldots (7.2)$$

at poles (6.6).

Our next problem, is to find $R(z)$. If $R_n(z)$ is the residue of (7.2) at the pole $s = \alpha - n$, then

$$R(z) = \sum_{n=0}^{\infty} R_n(z). \qquad \ldots (7.3)$$

After computing $R_n(z)$ and taking the sum, we find that $-z^h R(z) = c_3(h)\, s_3(h)$, where $c_3(h)$ and $s_3(h)$ are as given by (5.8) and (4.2).

Therefore, we have, if $|z|$ is large, then the fundamental solutions $y_j(z)$ (j = 1, 2, 3) of (1.1) about the regular singular point $z = 0$ as defined by (1.3) may be developed asymptotically as follows :

$$\left.\begin{aligned} y_j(z) &\sim c_1(h_j)\, s_1(z) + c_3(h_j)\, s_3(z) \text{ in sector (I)} \\ y_j(z) &\sim c_3(h_j)\, s_3(z) \text{ in sector (II)} \\ y_j(z) &\sim c_2(h_j)\, s_2(z) + c_3(h_j)\, s_3(z) \text{ in sector (III)} \\ y_j(z) &\sim c_1(h_j)\, s_1(z) + c_2(h_j)\, s_2(z) + c_3(z)\, s_3(z) \text{ in sector (IV).} \end{aligned}\right] \qquad \ldots (7.4)$$

8. Asymptotic Behaviour of Solutions $y_i(z)$, i = 1, 2, 3 when $\alpha = -h - \dfrac{d_2}{c_2}$ is an Integer

Next we will suppose that $h + \dfrac{d_2}{c_2}$ is an integer. Let us divide the possible integral values into two classes :

(a) $h + \dfrac{d_2}{c_2}$ is an integer < 2

(b) $h + \dfrac{d_2}{c_2}$ is an integer ≥ 2.

Case (a) : $h + \dfrac{d_2}{c_2}$ is an integer < 2

If we now form the differential equation (1.1) in which b_0 is changed to $b_0 - E$, where E is arbitrary small and positive, the resulting value of h will be of the form $n - \eta$, where η becomes infinitely small with E and therefore the solution $y(z)$ of the new differential equation will be developable asymptotically in the forms obtained from (7.1) and the resulting $c_1(h), c_2(h)$ and $c_3(h)$ determined from (5.10) and (5.11). After the expressions thus obtained have been examined and it is noted that when $h + \frac{d_2}{c_2}$ is an integer < 2, the following limit exists :

$$\lim_{E \to 0} \Gamma\left(2 - h - \frac{d_2}{c_2} + \eta\right). \qquad \ldots (8.1)$$

We conclude that so long as $h + \frac{d_2}{c_2}$ is an integer belonging to class (a) mentioned above, the relation (7.4) continues as before.

Case (b) : $h + \frac{d_2}{c_2}$ is an integer ≥ 2

First we consider the case $h + \frac{d_2}{c_2} = 2$.

Proceeding as before, we get the differential equation obtained from (1.1) by replacing b_0 by $b_0 - E$, and hence h by $h - \eta$ and after raplacing h by $h - \eta$ in (7.4) and we endeavour to examine the limit approached as $\eta \to 0$ of $c_1(h-\eta), c_2(h-\eta)$ and $c_3(h-\eta)$ as determined from (5.10) and (5.11). In doing so, there is involved, aside from the factors which approach a limit, the product

$$\Gamma(h)[f_0(h)\, V^*(\eta) + f_1(h-1)\, V^*(\eta - 1)]. \qquad \ldots (8.2)$$

We note that, as $\eta \to 0$, the factor $\Gamma(\eta)$ becomes infinite to the first order. Since $f_0(h)$ is identically zero, $f_0(h)\, V^*(\eta)$ is identically zero. For the expression $f_1(h-1)\, V_*(\eta - 1)$ we may write

$$\lim_{\eta \to 0} f_1(h-1)\, V^*(\eta - 1) = f_1(h-1)\, V^*(-1) = f_1(h-1) \cdot 0 = 0. \qquad \ldots (8.3)$$

Hence, the limit of (8.2) as $\eta \to 0$ is indeterminate.

We can, however, evaluate it immediately upon writing

$$\Gamma(\eta) = \frac{\Gamma(1+\eta)}{\eta}$$

the result being

$$f_1(h-1)\left[\frac{d}{d\eta} V^*(\eta - 1)\right]_{\eta = 0} = f_1(h-1)\frac{d}{ds}[V^*(s-1)]_{s=0}. \qquad \ldots (8.4)$$

However, we shall prefer to write it for future purposes as

$$\left[f_0\left(2-\frac{d_2}{c_2}\right)\frac{d}{ds}V^*(s+2)+f_1\left(1-\frac{d_2}{c_2}\right)\frac{d}{ds}V^*(s+1)\right]_{s=-\left(h+\frac{d_2}{c_2}\right)=-2}$$

... (8.5)

Next let us suppose that $h+\frac{d_2}{c_2}$ is an integer $n \geq 2$.

Then, reasoning as in the case $h+\frac{d_2}{c_2} = 2$, we replace h in (7.4) by $h - \eta$ and attempt to obtain the limit of the resulting expressions as $\eta \to 0$. No difficulties arise except in connection with the product

$$\Gamma(2-n+\eta)\left[f_0\left(2-\frac{d_2}{c_2}\right)V^*(2-n-\eta)+f_1\left(1-\frac{d_2}{c_2}\right)V^*(1-n-\eta)\right].$$

... (8.6)

But $$\Gamma(2-n+\eta)=\frac{(-1)^n\,\Gamma(1+\eta)}{(n-2-\eta)\,(n-3-\eta)\ldots(1-\eta)\,(\eta)}.$$

Since $n > 2$, we have

$$\lim_{\eta\to 0} V^*(2-n+\eta)=V^*(2-n)=0$$

and $$\lim_{\eta\to 0} V^*(1-n+\eta)=V^*(1-n)=0.$$

We find that the limit of (8.6) as $\eta \to 0$ is

$$=\frac{(-1)^n}{\Gamma(n-1)}\left[f_0\left(2-\frac{d_2}{c_2}\right)\frac{d}{ds}V^*(s+2)+f_1\left(1-\frac{d_2}{c_2}\right)\frac{d}{ds}V^*(s+1)\right]_{s=-h-\frac{d_2}{c_2}=-n}$$

... (8.7)

Finally, we observe that (8.5) may be obtained from (8.7) by putting $n = 2$ so that when $h+\frac{d_2}{c_2}$ is an integer ≥ 2, we conclude the relation (7.4) for $y(z)$ in which $c_3(h)$ is no longer to be determined by (6.8) but by the following :

$$c_3(h)=\frac{(-1)^n}{\sqrt{b_1^2-4c_2}}\frac{\Gamma(1-\beta_1)\,\Gamma(1-\beta_2)}{\Gamma(n-1)}\left[\frac{d}{ds}w(s)\right]_{s=-\left(h+\frac{d_2}{c_2}\right)=-n}$$

where $w(s)$ is as given in (5.9)

This completes proof of the main theorem in section 6.

This paper is a sequel to Puttaswamy[8-17].

REFERENCES

1. G. D. Birkhoff, *Collected Mathematical Papers*, Denver, 1968.
2. W. J. A. Culmer and W. A. Harris Jr., *Pac. Jour. Math.* **13** (1963), 1111-38.
3. W. B. Ford, *Asymptotic Developments of Functions Defined by Maclaurin Series*, Chelsa Publishing Company (1960).
4. T. K. Puttaswamy, *Proceedings of Ramanujan Birth Centenary Year International Symposium on Analysis*, Macmillan India Limited, 1989, pp. 451-65.
5. T. K. Puttaswamy, *Proceedings of the International Conference on Theory and Applications of Differential Equations*, Ohio University Press, 1989, pp. 317-21.
6. T. K. Puttaswamy, *Proceedings of the International Conference on New Trends in Geometric Function Theory and Applications*, World Scientific Publishers, 1991, pp. 90-95.
7. T. K. Puttaswamy, *Proceedings of the Third International Colloquim on Differential Equations held at Polvdiv, Bulagaria*, International Science Publishers, 1993, pp. 157-68.
8. T. K. Puttaswamy, *J. Ramanujan Math. Soc.* **8** (1993), 139-66.
9. T. K. Puttaswamy, *Proceedings of the International Symposium on Ordinary Differential Equations*, World Scientific Publishers, 1994, pp. 243-54.
10. T. K. Puttaswamy and T. V. Sastry, *Proceedings of the International Symposium on Ordinary Differential Equations*, World Scientific Publishers, 1994, pp. 271-82.
11. T. K. Puttaswamy, Solution in the large of a certain third order differential equation of W. B. Ford's type (submitted for publication).
12. T. K. Puttaswamy, Stokes multipliers for a certain third order differential equation in the vicinity of an irregular singular point (submitted for publication).
13. T. K. Puttaswamy, *Revue Roum. Math. Pures Et Appliques*, **39**, Series 7 (1994), 703-15.
14. T. K. Puttaswamy, *Revue Roum. Math. Pures Et Appliques*, **39**, Series 7 (1994), 717-37.
15. T. K. Puttaswamy and T. V. Sastry, *Revue Roum. Math. Pures Et Appliques*, **39**, Series 7 (1994), 739-48.
16. T. K. Puttaswamy and T. V. Sastry, *Proceedings of the Fourth International Colloquim on Differential Equations*, VSP, Netherlands, 1994, pp. 235-43.
17. T. K. Puttaswamy, *Proceedings of the Fourth International Colloquim on Differential Equations*, Impulse-M, 1994, pp. 79-90.