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A REMAINDER FOR SOME NON-LOCALLY COMPACT SPACES

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Denote by $R(X)$ the set of points of X which have no compact neighbourhoods. If X is a non-locally compact space satisfying certain hypotheses with $|R(X)| = 1$, Chandler and Tzung[2] proved that there exist a compactification αX of X with $\alpha X \setminus X$ is homeomorphic to $T_1^0 = [0, 1)$. In this paper, we generalize this theorem.

Throughout this paper all spaces are assumed to be completely regular and Hausdorff. A compactification of a space X is a compact space which contains X as a dense subspace and a remainder of X is any $\alpha X \setminus X$ where αX is a compactification of X. We denote by $K(X)$ the set of all compactifications of X. If αX and γX are two compactifications of X, then $\gamma X \leq \alpha X$ if there is a continuous function $g : \alpha X \to \gamma X$ such that $g(x) = x$ for each $x \in X$. Let βX denote the Stone-Cech compactification of X and let $\mathbb{N}$ denote the natural numbers. For a space X we let $C(X)$ (resp. $C^*(X)$) denote all real valued (resp. bounded) continuous functions on X. For a set A, $|A|$ denotes the cardinality of A.

An interesting problem: for a given space X, which spaces can be a remainder of X? Almost all results on remainders are for locally compact spaces. The remainder question for non-locally compact spaces is considerably more difficult and very little has been done. One major source for the non-locally compact case is the paper by Rayburn[4]. Chandler and Tzung[2] have obtained new results for non-locally compact spaces. In this paper, we generalize one of the theorems proved by Chandler and Tzung[2] for non-locally compact spaces.

Denote by $R(X)$ the set of points of X which have no compact neighbourhoods. For any $\alpha X \in K(X)$ we have $R(X) = X \cap cl_{\alpha X}(\alpha X \setminus X)$. Let T_n denote all points in $\mathbb{R}^n$ on the positive coordinate axes whose distance to the origin is less than or equal to 1. Let $\{e_1, e_2, ..., e_n\}$ denote the set of all end points in T_n, namely those points with one coordinate equal to 1 and all the others zero. Let $T_n^0 = T_n \setminus \{e_1, e_2, ..., e_n\}$.

Theorem 1 (Rayburn[4]) — Let $f : cl_{\beta X}(\beta X \setminus X) \to Y$ be continuous where Y is compact. If f is 1-1 on $R(X)$ and $f(\beta X \setminus X) \cap f(R(X)) = \phi$ then there is a compactification αX of X such that $\alpha X \setminus X$ homeomorphic to $f(\beta X \setminus X)$ as a subspace of Y.

Definition 2 — $F \subseteq X$ has countable character if there is a countable sequence of open sets $V_1 \supseteq V_2 \supseteq \ldots \supseteq F$ such that if V is open in X and $F \subseteq V$ then there is an n for which $V_n \subseteq V$.

Theorem 3 (Chandler and Tzung[2]) — Suppose X is realcompact, $R(X) = \{p\}$ and p is contained in a compact set of countable character. Then T_1^0 is remainder of X.

We partially generalize this result to the case $|R(X)| = n < \infty$, by making a slight change in the hypothesis.

Lemma 4 — Let L be a closed subspace of [0, 1] such that $0 \in L$. Suppose 0 is cluster point of L. Then either there exists $0 < r$ such that $[0, r] \subseteq L$ or there exists a sequence $\{A_n : n \in \mathbb{N}\}$ consisting of non-empty clopen, i.e. both closed and open, subsets of L such that

(i) $A_n \cap A_m = \phi$ for $n \neq m$ and $0 \notin A_n$ for each n.

(ii) $L = \{0\} \cup \bigcup_{i=1}^{\infty} A_n$.

(iii) If $\varepsilon > 0$ then there exists k such that for $n \geq k, A_n \subseteq [0, \varepsilon]$.

PROOF : Since 0 is a cluster point of L, there exists a sequence $\{y_n : n \in \mathbb{N}\} \subseteq L$ such that $0 < y_{n+1} < y_n$ for each n and $\lim_{n \to \infty} y_n = 0$. Let us assume that $[0, y] \setminus L$ is non empty for all $0 < y$. Let $y_{n_0} = y_1$ and $t_1 \in [0, y_{n_0}] \setminus L$. Then $0 < t_1 < y_{n_0}$. Since $\lim_{n \to \infty} y_n = 0$, then there is y_{n_1} such that $0 < y_{n_1} < t_1 < y_{n_0}$. Let $A_1 = L \cap [t_1, 1] = L \cap (t_1, 1]$. Then A_1 is clopen in L and A_1 is non-empty since $y_{n_0} \in A_1$. Suppose that we have already constructed sequences $\{A_1, \ldots, A_n\}$, $\{t_1, \ldots, t_n\}$ and $\{y_{k_0}, \ldots, y_{k_n}\}$ satisfying

(a) A_i is non empty clopen subset of L for $i = 1, 2, \ldots, n$.

(b) $A_i \cap A_j = \phi$ for $1 \leq i \neq j \leq n$.

(c) $0 < y_{k_i} < t_i < y_{k_{i-1}}$ for $i = 1, 2, \ldots, n$.

(d) $t_i \notin L$ for $i = 1, 2, \ldots, n$.

(e) $\bigcup_{i=1}^{n} A_i = L \cap [t_n, 1]$ and $A_n \subset [t_j, t_{j-1}]$ for $j = 1, 2, \ldots, n$ where $t_0 = 1$.

Let $t_{n+1} \in [0, y_{k_n}] \setminus L$ and $A_{n+1} = L \cap [t_n, t_{n+1}] = L \cap (t_n, t_{n+1})$. Therefore A_{n+1} is clopen in L. Also $A_n \cap A_{n+1} = \phi$ for $i = 1, 2, \ldots, n$ by (e). Since $\lim_{n \to \infty} y_n = 0$ and $0 < t_{n+1} < y_{k_n}$, then there exists $0 < y_{k_{n+1}} < t_{n+1}$. Since $y_{k_n} \in A_{n+1}$, then A_{n+1} is non-empty. Clearly $\{A_n : n \in \mathbb{N}\}$ satisfies (i) and (ii). Let $\varepsilon > 0$. Since $\lim_{n \to \infty} t_n = 0$, then there exists m such that $t_m < \varepsilon$. By (e) $A_n \subseteq [t_n, t_{n-1}] \subseteq [0, t_m] \subseteq [0, \varepsilon]$ for each $n \geq m+1$. This completes the proof.

Lemma 5 — Let Y be a dense subspace of a compact space Z such that $Z \setminus Y$ is a zero set. Let K be a non-empty closed subset of $Z \setminus Y$ and $K \subseteq V$ where V is an open subset of Z and $V \setminus Y = K$. Suppose that each nonempty clopen subset A of Y contains a copy of $\beta\mathbb{N}$. Then there is a continuous function $g : Z \to [0, 1]$ such that

(a) g is onto

(b) $g^{-1}(0) = K$

(c) $g(Z \setminus V) \subseteq \{1\}$.

PROOF : Since $Z \setminus Y$ is a zero set in Z, there is a continuous function $h : Z \to [0, 1]$ such that $Z \setminus Y = Z(h)$. Since $V \setminus Y = K$, then K is clopen in $Z \setminus Y$. Thus $H = (Z \setminus Y) \setminus K$ is closed in $Z \setminus Y$ and $H \cap K = \phi$. Since $Z \setminus Y$ is compact, then H and K are closed disjoint sets in Z. Then there is a continuous function $k : Z \to [0, 1]$ such that $k(K) \subseteq \{0\}$ and $k(Z \setminus V) \subseteq \{1\}$ since Z is normal. Let $t = \max\{h, k\}$. Then $t : Z \to [0, 1]$ is continuous $t^{-1}(0) = K$ and $t(Z \setminus V) \subseteq \{1\}$. Now let $L = t(Z)$. If there is a $\varepsilon > 0$ such that $L \cap (0, \varepsilon) = \phi$ then $t^{-1}([0, \varepsilon)) = K$ is non-empty open in Z. Thus $K \cap Y \neq \phi$. A contradiction since $K \cap Y \subseteq (Z \setminus Y) \cap K = \phi$. Hence $L \cap (0, \varepsilon) \neq \phi$ for each $\varepsilon > 0$. Hence 0 is cluster point of L. Thus by Lemma 4, there are only two cases to be considered.

Case 1 : There is a $r > 0$ such that $[0, r] \subseteq L$. Define $u : [0, 1] \to [0, 1]$ by

$$u(x) = \begin{cases} x/r & \text{if } 0 \leq x \leq r \\ 1 & \text{if } r \leq x \leq 1. \end{cases}$$

Then u is continuous. Let $g : Z \to [0, 1]$ be defined by $g = u_0 t$. Then $x \in Z$ and $g(x) = 0$ if and only if $x \in K$. So $g^{-1}(0) = K$. If $x \in Z \setminus V$ then $t(x) = 1$. Hence $g(x) = u(1) = 1$. So $g(Z \setminus V) \subseteq \{1\}$. Also $g(Z) = u(t(Z)) = u(L) \supseteq u([0, r]) = [0, 1]$. So $g(Z) = [0, 1]$.

Case 2 : There exists a sequence of non-empty clopen sets $\{A_n : n \in \mathbb{N}\}$ in L such that this sequence satisfies (i)-(iii) of Lemma 4. Then there is an m such that $A_n \subseteq [0, 1/2]$ for each $n \geq m$.

Let $B_1 = A_1 \cup \dots \cup A_m$, $B_i = A_{m+i-1}$ for $i \geq 2$. Then the B_i's satisfy.

(i) $1 \notin B_i$ for all $i \geq 2$ and $0 \notin B_i$

(ii) B_i's are pairwise disjoint non-empty clopen sets in L and $L = \{0\} \bigcup_{i=1}^{\infty} B_i$

(iii) For any $\varepsilon > 0$, there is a $k \in \mathbb{N}$ such that $B_n \subseteq [0, \varepsilon]$ for each $n \geq k$. Let

$C_n = t^{-1}(B_n)$ for $n \geq 1$. The C_n's are non-empty since $B_n \neq \phi$ and $B_n \subseteq L = t(Z)$. Also the C_n's are clopen in Z. Let $n \geq 2$. Then $C_n = t^{-1}(B_n) \subseteq t^{-1}([0, 1/2))$. Hence $C_n \cap (Z \setminus V) = \phi$. Since $0 \notin B_n$, then $C_n \cap K = \phi$. Thus $C_n \subseteq Y$ for all $n \geq 2$ since $V \setminus K \subseteq Y$. By hypothesis there exist a copy S of $\beta \mathbb{N}$ in C_n. Then there exist a continuous onto map $P_n : \beta \mathbb{N} \to [1/n, 1/(n-1)]$ for each $n \geq 2$. $\beta \mathbb{N}$ is a compact subspace of the compact space C_n. So for each $n \geq 2$ P_n has a continuous extension $g_n : C_n \to [1/n, 1/(n-1)]$ which is necessarily onto. $Z = f^{-1}(L) = K \cup \bigcup_{n=1}^{\infty} C_n$.

Since $K \cap (Z \setminus V) = \phi$ and $C_n \cap (Z \setminus V) = \phi$ for all $n \geq 2$, then $Z \setminus V \subseteq C_1$. Now define $g : Z \to [0, 1]$ by

$$g(x) = \begin{cases} 1 & \text{if } x \in C_1 \\ g_n(x) & \text{if } x \in C_n \;\; n \geq 2 \\ 0 & \text{if } x \in K. \end{cases}$$

Clearly g is continuous on the open set $W = \bigcup_{i=1}^{\infty} C_i$. Now let $x \in K$ and $\varepsilon > 0$. Let $U = K \cup \bigcup_{i=N+1}^{\infty} C_i$ where $1/N < \varepsilon$. Since $Z \setminus U = C_1 \cup \ldots \cup C_N$ is closed, U is an open neighbourhood of x in Z. Also

$$g(U) = \{0\} \cup \bigcup_{i=N+1}^{\infty} [1/i, 1/(i-1)] = [0, 1/N] \subseteq [0, \varepsilon).$$

Thus g is continuous. Clearly $g(Z) = [0, 1]$, $g^{-1}(0) = K$ and $g(Z \setminus V) \subseteq g(C_1) \subseteq \{1\}$.

Lemma 6 (Gillman and Jerison[3], 9.11) — Let X be realcompact, $Z = cl_{\beta X}(\beta X \setminus X)$ and $Y = \beta X \setminus X$. Suppose X is not compact. If A is non-empty clopen subset of Z, then $A \cap Y$ contains a copy of $\beta\mathbb{N}$.

Theorem 7 — Let X be realcompact and $R(X) = \{p_1, p_2, \ldots, p_n\}$. If $R(X)$ has countable character then T_n^0 is a remainder of X.

PROOF : We first show that the hypotheses imply that $R(X)$ is a zero set in $cl_{\beta X}(\beta X \setminus X)$. Since $R(X)$ has a countable character in X, $R(X)$ has a countable base of neighbourhoods in X. Then $R(X)$ has a countable base of neighbourhoods in βX since $R(X)$ is compact (Gillman and Jerison[3], 9.7). Thus $R(X)$ is a G_δ-set in βX. Then $R(X)$ is a zero set in βX (Gillman and Jerison[3], 9.6). Since $cl_{\beta X}(\beta X \setminus X) = (\beta X \setminus X) \cup R(X)$ and $cl_{\beta X}(\beta X \setminus X)$ is a subspace of βX, then $R(X)$ is a zero set in $cl_{\beta X}(\beta X \setminus X)$. Let $Y = \beta X \setminus X$ and $Z = cl_{\beta X}(\beta X \setminus X)$. Choose open sets $U_1, U_2, \ldots, U_n$ in Z such that $U_i \cap U_j = \phi$ for $i \neq j$ and $p_i \in U_i$. Let $K = \{p_i\}$ and $V = U_i$ for each $i = 1, 2, \ldots, n$. Then all conditions of Lemma 5 are satisfied so for each $i = 1, 2, \ldots, n$ there is a continuous onto function $g_i : Z \to [0, 1]$ such that

$g_i^{-1}(0) = \{p_i\}$ and $g_i(Z \setminus U_i) \subseteq \{1\}$. Let $f_i = 1 - g_i$ and define $f: Z \to T_n$ by $f(x) = (f_i(x))$. Then

(a) f is continuous onto

(b) $f^{-1}(e_i) = \{p_i\}$ for each $i = 1, 2, ..., n$.

So by Theorem 1, T_n^0 is a remainder of X.

Corollary 8 — If X is realcompact and $R(X) = \{p_1, p_2, ..., p_n\}$ where X has a countable neighbourhood base at p_i for each $i = 1, 2, ..., n$, then T_n^0 is a remainder of X.

Example 9 — Let $X = [0, 1] \times [0, 1] \setminus \{1\} \times (0, 1)$. Then X is realcompact and $R(X) = \{(1, 0), (1, 1)\}$. Since X is second countable, $R(X)$ has countable character. Let $\alpha X = [0, 1] \times [0, 1]$. Then $\alpha X \in K(X)$ and $\alpha X \setminus X = \{1\} \times (0, 1)$ is homeomorphic to T_2^0.

REFERENCES

1. R. E. Chandler, *Hausdorff Compactifications*, Dekker, New York, 1976.
2. R. E. Chandler and F. C. Tzung, *Proc. Am. Math. Soc.* **70**(1978), 196-202.
3. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, NJ, 1960.
4. M. C. Rayburn, *Pacific J. Math.* **44** (1973), 707-14.