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COVERING AND CONNECTEDNESS PROPERTIES OF A TOPOLOGICAL SPACE AND ITS ASSOCIATED TOPOLOGY OF α-SUBSETS

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Feebly open sets, also known in the literature as α-sets, have been investigated recently in both topology and bitopology. In this paper their rôle in covering and connectedness properties is further considered, in particular with respect to compactness and connectedness properties of the space $(X, \mathcal{T}, \mathcal{T}^\alpha)$.

1. INTRODUCTION

Njåstad[20] defined a subset B to be an α-set in a topological space $(X, \mathcal{T})$ if $B \subset \mathcal{T}$ int ($\mathcal{T}$ cl ($\mathcal{T}$ int B)), and proved that the family $\mathcal{T}^\alpha$ of all α-sets in $(X, \mathcal{T})$ is a topology on X larger than $\mathcal{T}$.

Maheshwari and Tapi[19] introduced the concept of feebly open sets. Janković and Reilly[12] (Proposition 1) proved the equivalence of these two notions.

Having two related topologies on a set, it seems quite natural to consider whether they share some topological properties. The sharing of separation axioms by $(X, \mathcal{T})$ and $(X, \mathcal{T}^\alpha)$ has been considered by Janković and Reilly[12], Dorsett[7], Garg and Sivaraj[11], Tadros[32], as a part of their investigations. Garg and Sivaraj[11] considered a number of properties besides separation axioms, including covering and connectedness properties.

Another natural question is what can be said about the corresponding bitopological space $(X, \mathcal{T}, \mathcal{T}^\alpha)$? How are its bitopological properties related to the corresponding topological properties of the spaces $(X, \mathcal{T})$ and $(X, \mathcal{T}^\alpha)$?

In Mrševič and Reilly[17] the role of α-sets in separation axioms was considered, in particular with respect to regularity and normality properties of the space $(X, \mathcal{T}, \mathcal{T}^\alpha)$. In this paper we consider some covering and connectedness properties, for example (countable) compactness, local compactness, (local) Lindelöfness, paracompactness, (local) connectedness, total and extremal disconnectedness, zero dimensionality.

One would expect that if a topological property $\mathcal{P}$ is shared by the topological spaces $(X, \mathcal{T})$ and $(X, \mathcal{T}^\alpha)$, then the bitopological space $(X, \mathcal{T}, \mathcal{T}^\alpha)$ should have the corresponding bitopological property pairwise $\mathcal{P}$. But this anticipated situation depends very much on how the property pairwise $\mathcal{P}$ is defined. Almost all of the properties we consider behave in the expected way, while pairwise compactness in the sense of Bîrsan and pairwise paracompactness in the sense of Fletcher *et al.*[9] show that is not a rule (Theorems 2.2 and 2.5).

Out notation is standard. $\mathbb{R}$ stands for the set of all real numbers and $\mathbb{N}$ stands for the set of all positive integers.

2. Covering Properties

The notion of compactness has been generalised to bitopological spaces in different ways.

Recall that a cover of a bitopological space $(X, \mathcal{U}, \mathcal{V})$ is called pairwise open if it is a subcollection of the family $\mathcal{U} \cup \mathcal{V}$ and contains at least one non-empty element from $\mathcal{U}$ and at least one non-empty element from $\mathcal{V}$. A bitopological space $(X, \mathcal{U}, \mathcal{V})$ is pairwise compact in the sense of Fletcher *et al.*[9] (FHP-pairwise compact) if every pairwise open cover of X has a finite subcover. A space $(X, \mathcal{U}, \mathcal{V})$ is compact in the sense of Swart[28] (S-compact) if every cover $\mathcal{W} \subset \mathcal{U} \cup \mathcal{V}$ of X has a finite subcover, i.e. if $(X, \mathcal{U} \vee \mathcal{V})$ is compact, where $\mathcal{U} \vee \mathcal{V}$ is the supremum topology of $\mathcal{U}$ and $\mathcal{V}$.

S-compactness of the space $(X, \mathcal{U}, \mathcal{V})$ is equivalent to FHP-pairwise compactness together with compactness of $(X, \mathcal{U})$ and $(X, \mathcal{V})$ (see Cooke and Reilly[5]). If one topology is weaker than the other, say if $\mathcal{U} \subset \mathcal{V}$, S-compactness is equivalent to compactness of $(X, \mathcal{V})$, but need not be equivalent to FHP-pairwise compactness. In the case that the finer topology is that of α-sets, these two notions coincide as Proposition 1 shows.

Our key lemmas are the following.

Lemma 1 — Let $G = U \setminus N$, where $U \in \mathcal{T}$ and N is nowhere dense. Then

(i) $\mathcal{T}^\alpha \mathrm{cl} G = \mathcal{T}\,\mathrm{cl} G = \mathcal{T}\,\mathrm{cl} U = \mathcal{T}^\alpha\,\mathrm{cl} U$ holds.

(ii) If H is a $\mathcal{T}^\alpha$ open set disjoint to G, then $H \cap U = \phi$.

(iii) For any $\mathcal{T}$ open set V containing G, there is a $\mathcal{T}$ open set W such that $G \subset W \subset U \cap V$ holds.

Proof : (i) This is Lemma 5 of Mršević and Reilly[17].

(ii) Follows from (i) since $G \cap H = \phi$ implies $\mathcal{T}^{\alpha}$ cl$G \cap H = \phi$ = $\mathcal{T}^{\alpha}$ cl$U \cap H$. Hence $U \cap H = \phi$.

(iii) The set $W = U \cap V$ satisfies the requirements.

Lemma 2 — Let A be a subset of $(X, \mathcal{T}, \mathcal{T}^{\alpha})$ such that $\mathcal{T}$int $A \neq \phi$. Then every $\mathcal{T}^{\alpha}$ open cover of A can be enlarged to a pairwise open cover in $(X, \mathcal{T}, \mathcal{T}^{\alpha})$ of A by adding at most one non-empty $\mathcal{T}$ open set contained in an element of the given cover.

PROOF : Let $\mathcal{W} = \{W_j \mid j \in J\}$ be a $\mathcal{T}^{\alpha}$ open cover of A. If every W_j belongs to $\mathcal{T}^{\alpha} \backslash \mathcal{T}$, each $W_j = U_j \backslash N_j$, where $U_j \in \mathcal{T}$ and N_j is nowhere dense in $(X, \mathcal{T})$. Let $x \in \mathcal{T}$ int A. There is a $j' \in J$ such that $x \in W_{j'}$. The set $V = U_{j'} \backslash \mathcal{T}$ cl$N_{j'}$ is $\mathcal{T}$ open and its intersection with A is non-empty. For $x \in G \in \mathcal{T}$ and $G \subset A$ imply $x \in H = G \cap U_{j'} \in \mathcal{T}$. Hence $\mathcal{T}$ int($\mathcal{T}$ cl$N_{j'}$) = ϕ implies $H \not\subset \mathcal{T}$ cl$N_{j'}$, thus $\phi \neq H \cap (\mathcal{T}\text{ cl}N_{j'})^c \subset A \cap V$. The cover $\mathcal{W}' = \mathcal{W} \cup \{V\}$ is a desired one. □

Lemma 2 implies the following results.

Proposition 1 — In a bitopological space $(X, \mathcal{T}, \mathcal{T}^{\alpha})$ FHP-pairwise compactness and S-compactness coincide with compactness of the space $(X, \mathcal{T}^{\alpha})$. □

Remark : From now on FHP-pairwise compactness and S-compactness of the space $(X, \mathcal{T}, \mathcal{T}^{\alpha})$ will be called simply pairwise compactness.

Theorem 2.1 — $(X, \mathcal{T}^{\alpha})$ is compact $\Leftrightarrow$ $(X, \mathcal{T}, \mathcal{T}^{\alpha})$ is pairwise compact $\Rightarrow$ $(X, \mathcal{T})$ is compact. □

That compactness of $(X, \mathcal{T})$ is not equivalent to compactness of $(X, \mathcal{T}^{\alpha})$ (and hence to pairwise compactness of $(X, \mathcal{T}, \mathcal{T}^{\alpha})$) can be seen from the following example.

Example 1 — Let $(X, \mathcal{T})$ be the unit interval I = [0, 1] of the real line with the usual topology. The space is compact but not $\mathcal{T}^{\alpha}$ compact. The $\mathcal{T}^{\alpha}$ open cover consisting of the set $U = \left[0, \frac{1}{2}\right) \backslash \left\{\frac{1}{n} \mid n > 2\right\}$ and the intervals $\left(\frac{1}{3}, 1\right]$ and $\left(\frac{1}{n} - r_n, \frac{1}{n} + r_n\right)$ for $n > 2$ and $r_n = \frac{1}{2(n+1)^2}$, has no finite subcover. Thus $(X, \mathcal{T}^{\alpha})$ is not (countably) compact.

Bîrsan's[4] generalization of compactness to bitopological spaces is independent of those already mentioned. In $(X, \mathcal{U}, \mathcal{V})$ the topology $\mathcal{U}$ is B-compact with respect to $\mathcal{V}$ if each $\mathcal{U}$ open cover of X has a finite $\mathcal{V}$ open refinement which covers X. Thus if $\mathcal{U}$ is B-compact with respect to $\mathcal{V}$, then $(X, \mathcal{U})$ is compact. The space $(X, \mathcal{U}, \mathcal{V})$ is B-pairwise compact if $\mathcal{U}$ is B-compact with respect to $\mathcal{V}$ and $\mathcal{V}$ is B-compact with respect to $\mathcal{U}$. The following statement follows from the fact that $\mathcal{T} \subset \mathcal{T}^{\alpha}$.

Theorem 2.2 — $\mathcal{T}^{\alpha}$ is B-compact with respect to $\mathcal{T} \Rightarrow (X, \mathcal{T}^{\alpha})$ is compact $\Rightarrow \mathcal{T}$ is B-compact with respect to $\mathcal{T}^{\alpha} \Leftrightarrow (X, \mathcal{T})$ is compact. □

Example 1 shows that if $\mathcal{T}$ is B-compact with respect to $\mathcal{T}^{\alpha}$, that is if $(X, \mathcal{T})$ is compact, the space $(X, \mathcal{T}^{\alpha})$ need not be compact. That compactness of $(X, \mathcal{T}^{\alpha})$ does not imply that $\mathcal{T}^{\alpha}$ is B-compact with respect to $\mathcal{T}$, can be seen from the following example.

Example 2 — Let $X = \mathbb{N} \cup \{p, q\}$, $p, q \notin \mathbb{N}$, and the topology $\mathcal{T} = \{G \subset \mathbb{N}\} \cup \{H \subset X \mid p, q \in H$ and $X \setminus H$ is finite$\}$. The set $\{p, q\}$ and its subsets $\{p\}$ and $\{q\}$ are (the only) nowhere dense sets in $(X, \mathcal{T})$ and the $\mathcal{T}^{\alpha}$ open cover $\{X \setminus \{p\}, X \setminus \{q\}\}$ has no $\mathcal{T}$ open refinement which covers X.

A bitopological space $(X, \mathcal{U}, \mathcal{V})$ is pairwise countably compact if every countable pairwise open cover of X has a finite subcover[22]. The space is pairwise Lindelöf if every pairwise open cover of X has a countable subcover[26].

Similarly to Theorem 2.1 we have the following statement.

Theorem 2.3 — (i) $(X, \mathcal{T}^{\alpha})$ is countably compact $\Leftrightarrow (X, \mathcal{T}, \mathcal{T}^{\alpha})$ is pairwise countably compact $\Rightarrow (X, \mathcal{T})$ is countably compact.

(ii) $(X, \mathcal{T}^{\alpha})$ is Lindelöf $\Leftrightarrow (X, \mathcal{T}, \mathcal{T}^{\alpha})$ is pairwise Lindelöf $\Rightarrow (X, \mathcal{T})$ is Lindelöf. □

Example 1 shows that countable compactness (Lindelöfness) of $(X, \mathcal{T})$ does not imply countable compactness (Lindelöfness) of $(X, \mathcal{T}^{\alpha})$. The Cantor set C is uncountable and closed in the unit interval. Since it is a discrete subspace of $(X, \mathcal{T}^{\alpha})$, $(X, \mathcal{T}^{\alpha})$ is not a Lindelöf space.

Local compactness in bitopological spaces was investigated in Reilly[24] based on FHP-pairwise compactness, and in Mršević[15] where the fundamental concept was S-compactness. In both cases the stronger form of local compactness was considered, namely the existence of neighbourhoods with (pairwise) compact closures. Lemma 2 shows that in the space $(X, \mathcal{T}, \mathcal{T}^{\alpha})$ "local FHP-compactness" and "local S-compactness" are equivalent notions. We call them pairwise local compactness. Note that we also consider the stronger forms of local compactness and local Lindelöfness for the topological spaces $(X, \mathcal{T})$ and $(X, \mathcal{T}^{\alpha})$.

The notion of a (strongly) locally Lindelöf space was introduced in Somasundaram and Balasubramanian[29] and its bitopological analogue in Somasundaram and Saraswathi[30]. In $(X, \mathcal{U}, \mathcal{V})$ the topology $\mathcal{U}$ is locally compact (locally Lindelöf) with respect to $\mathcal{V}$ if for each point $x \in X$ there is a neighbourhood $U \in \mathcal{U}$ of x such that $\mathcal{V}$ cl$U$ is pairwise compact (pairwise Lindelöf)[24, 15, 30]. The space $(X, \mathcal{U}, \mathcal{V})$ is pairwise locally compact (pairwise locally Lindelöf) if $\mathcal{U}$ is locally compact (locally Lindelöf) with respect to $\mathcal{V}$ and $\mathcal{V}$ is locally compact (locally Lindelöf) with respect to $\mathcal{U}$.

Lemmas 1 and 2 yield the next statement.

Theorem 2.4 — $(X, \mathcal{T}^\alpha)$ is locally compact (locally Lindelöf) $\Leftrightarrow$ $(X, \mathcal{T}, \mathcal{T}^\alpha)$ is pairwise locally compact (locally Lindelöf) $\Rightarrow$ $(X, \mathcal{T})$ is locally compact (locally Lindelöf). □

That the last implication is not reversible follows from Example 1. The space $([0, 1], \mathcal{T}^\alpha)$ is not locally Lindelöf. Every set of the form $[0, \varepsilon]$ contains an uncountable nowhere dense set.

There are several definitions of paracompactness in a bitopological space. We shall consider some of them not requiring a bitopological space to be pairwise Hausdorff. Following Konstadilaki-Savvopoulou and Reilly[14], in a bitopological space $(X, \mathcal{U}, \mathcal{V})$ the topology $\mathcal{U}$ is FHP-paracompact with respect to $\mathcal{V}$ if every $\mathcal{U}$ open cover of X has a $\mathcal{V}$ open $\mathcal{V}$ locally finite refinement. It is FHP-pairwise paracompact if $\mathcal{U}$ is FHP-paracompact with respect to $\mathcal{V}$ and $\mathcal{V}$ is FHP-paracompact with respect to $\mathcal{U}$.

In $(X, \mathcal{U}, \mathcal{V})$ the topology $\mathcal{U}$ is RR-paracompact with respect to $\mathcal{V}$ if each $\mathcal{U}$ open cover of X has a $\mathcal{U}$ open refinement which is $\mathcal{V}$ locally finite[22].

Datta[6] gave two definitions of paracompactness in bitopological spaces which correspond to FHP-pairwise compactness and S-compactness respectively. A space $(X, \mathcal{U}, \mathcal{V})$ is D-pairwise paracompact if every pairwise open cover of X has a pairwise locally finite pairwise open parallel refinement. (Each point of X has a $\mathcal{U}$ open ($\mathcal{V}$ open) neighbourhood which meets only finitely many of the $\mathcal{V}$ open ($\mathcal{U}$ open) members of the refinement.) A space $(X, \mathcal{U}, \mathcal{V})$ is strongly D-pairwise paracompact if every weakly pairwise open cover of X, i.e. every cover $\mathcal{W} \subset \mathcal{U} \cup \mathcal{V}$, has a pairwise locally finite weakly pairwise open parallel refinement. (For the details see Datta[6] and Konstadilaki-Savvopoulou and Reilly[14].) In $(X, \mathcal{T}, \mathcal{T}^\alpha)$ a weakly pairwise open cover means a T^α open cover.

Pairwise paracompactness was considered also by Gutiérrez[10]. In his definition pairwise open covers are of a special kind. Nevertheless in the space $(X, \mathcal{T}, \mathcal{T}^\alpha)$ his definition is equivalent to Datta's.

The statement (ii) in Lemma 1 implies that in $(X, \mathcal{T}, \mathcal{T}^\alpha)$ the notions of a $\mathcal{T}$ and $\mathcal{T}^\alpha$ locally finite refinement of a $\mathcal{T}$ ($\mathcal{T}^\alpha$ or pairwise) open cover are the same. Using the statement (iii), from one type of the refinements mentioned in the above definitions, the other can be constructed. Hence the following result.

Theorem 2.5 — $\mathcal{T}^\alpha$ is FHP-paracompact with respect to $\mathcal{T}$ $\Rightarrow$ $(X, \mathcal{T}^\alpha)$ is paracompact $\Leftrightarrow$ $\mathcal{T}^\alpha$ is RR-paracompact with respect to $\mathcal{T}$ $\Leftrightarrow$ $(X, \mathcal{T}, \mathcal{T}^\alpha)$ is (strongly) D-pairwise paracompact $\Rightarrow$ $\mathcal{T}$ is FHP-paracompact with respect to $\mathcal{T}^\alpha$ $\Leftrightarrow$ $\mathcal{T}$ is RR-paracompact with respect to $\mathcal{T}^\alpha$ $\Leftrightarrow$ $(X, \mathcal{T})$ is paracompact. □

Again FHP-pairwise paracompactness behaves as B-pairwise compactness. That the first and fourth implications in the above theorem are not reversible, can be seen again from Examples 1 and 2.

A topological space $(X, \mathcal{T})$ is quasi H-closed (QHC) if every open cover has a finite subfamily the union of whose members is dense in $(X, \mathcal{T})$[35]. A space $(X, \mathcal{U}, \mathcal{V})$ is $\mathcal{U}\mathcal{V}'$-QHC (also called $\mathcal{U}$ almost compact with respect to $\mathcal{V}$) if every $\mathcal{U}$ open cover of X has a finite subcollection, the union of whose members is $\mathcal{V}$ dense in X. The space is pairwise QHC (pairwise almost compact) if it is $\mathcal{U}\mathcal{V}$-QHC and $\mathcal{V}\mathcal{U}$-QHC[18].

Theorem 2.6 — $(X, \mathcal{T}^\alpha)$ is QHC $\Leftrightarrow$ $(X, \mathcal{T}, \mathcal{T}^\alpha)$ is pairwise QHC $\Leftrightarrow$ $(X, \mathcal{T})$ is QHC. □

3. Connectedness Properties

Recall that a bitopological space $(X, \mathcal{U}, \mathcal{V})$ is called pairwise connected[21] if X cannot be expressed as a union of two nonempty disjoint sets $U \in \mathcal{U}$ and $V \in \mathcal{V}$. If such sets U and V exist, $X = U \mid V$ is a separation of X. Following the terminology as in Mršević[16] $(X, \mathcal{U}, \mathcal{V})$ is pairwise totally disconnected if there is no pairwise connected subset consisting of more than one point; that is, if the bitopological components in $(X, \mathcal{U}, \mathcal{V})$ are singletons[3]. $(X, \mathcal{U}, \mathcal{V})$ is weakly pairwise totally separated if for each pair of distinct points $x, y \in X$ there is a separation $X = U \mid V$ such that one of the points belongs to U and the other to V; that is, if the bitopological quasicomponents in $(X, \mathcal{U}, \mathcal{V})$ are singletons. $(X, \mathcal{U}, \mathcal{V})$ is pairwise totally separated if for each pair of distinct points $x, y \in X$ there is a separation $X = U \mid V$ such that $x \in U$ and $y \in V$.

In a bitopological space $(X, \mathcal{U}, \mathcal{V})$, $\mathcal{U}$ is zero dimensional with respect to $\mathcal{V}$ (Reilly[25]) if for each point $x \in X$ and each $\mathcal{U}$ open set U containing x there is a $\mathcal{V}$ closed $\mathcal{U}$ open set G such that $x \in G \subset U$. $(X, \mathcal{U}, \mathcal{V})$ is pairwise zero dimensional if $\mathcal{U}$ is zero dimensional with respect to $\mathcal{V}$ and $\mathcal{V}$ is zero dimensional with respect to $\mathcal{U}$.

We shall say that in a bitopological space $(X, \mathcal{U}, \mathcal{V})$ $\mathcal{U}$ is extremally disconnected with respect to $\mathcal{V}$ if the $\mathcal{V}$ closure of every $\mathcal{U}$ open set is $\mathcal{U}$ open. $(X, \mathcal{U}, \mathcal{V})$ is pairwise extremally disconnected if $\mathcal{U}$ is extremally disconnected with respect to $\mathcal{V}$ and $\mathcal{V}$ is extremally disconnected with respect to $\mathcal{U}$ (cf. Amihăesei[1] and Mršević[16]).

In a bitopological space $(X, \mathcal{U}, \mathcal{V})$, $\mathcal{U}$ is locally connected with respect to $\mathcal{V}$[3] if for each point $x \in X$ and each $\mathcal{U}$ open set U containing x there is a pairwise connected $\mathcal{U}$ open set G such that $x \in G \subset U$. $(X, \mathcal{U}, \mathcal{V})$ is pairwise locally connected if $\mathcal{U}$ is locally connected with respect to $\mathcal{V}$ and $\mathcal{V}$ is locally connected with respect to $\mathcal{U}$.

Lemma 3 — (i) A subset $W \in \mathcal{T}^\alpha$ is $\mathcal{T}^\alpha$ connected if and only if it is pairwise connected if and only if it is $\mathcal{T}$ connected.

(ii) Let $W = U \setminus N$, where $U \in \mathcal{T}$ and N is nowhere dense. Then W is $\mathcal{T}^\alpha$ connected (= $\mathcal{T}$ connected = pairwise connected) implies U is $\mathcal{T}^\alpha$ connected (= $\mathcal{T}$ connected = pairwise connected).

PROOF : (i) The implications : W is $\mathcal{T}^\alpha$ connected $\Rightarrow$ W is pairwise connected $\Rightarrow$ W is $\mathcal{T}$ connected follow from $\mathcal{T} \subset \mathcal{T}^\alpha$.

To complete the proof, suppose that W is not $\mathcal{T}^\alpha$ connected. There are non-empty disjoint subsets $G, H \in \mathcal{T}^\alpha_W$ such that $W = G \cup H$. Thus $G, H \in \mathcal{T}^\alpha$ and by Lemma 1(i) it follows that $G = W \cap \mathcal{T}^\alpha \text{ cl } G = W \cap \mathcal{T} \text{ cl } G$ and $H = W \cap \mathcal{T}^\alpha \text{ cl } H = W \cap \mathcal{T} \text{ cl } H$. Hence W is not $\mathcal{T}$ connected.

(ii) By Lemma 1 (i), $W = U \setminus N \subset U \subset \mathcal{T}^\alpha \text{cl}\, U = \mathcal{T}^\alpha \text{cl}\, W$. If W is $\mathcal{T}^\alpha$ connected, so are $\mathcal{T}^\alpha\text{cl}W$ and U. □

Remark : That the converse in (ii) in the above lemma does not hold can be seen from Example 1.

Theorem 2.10 in Andrijević[2] states that the spaces $(X, \mathcal{T})$ and $(X, \mathcal{T}^\alpha)$ have the same class of open-and-closed sets, i.e. that a subset W is open-and-closed in $(X, \mathcal{T})$ if and only if W is open-and-closed in $(X, \mathcal{T}^\alpha)$. The next proposition restates that result including the bitopological space $(X, \mathcal{T}, \mathcal{T}^\alpha)$. We also give an alternative proof of the non-trivial implication (iii) $\Rightarrow$ (i) based on Lemma 1.

Proposition 2 — The following are equivalent :

(i) $X = A \mid B$ is a separation of the space $(X, \mathcal{T})$,

(ii) $X = A \mid B$ is a separation of the space $(X, \mathcal{T}, \mathcal{T}^\alpha)$,

(iii) $X = A \mid B$ is a separation of the space $(X, \mathcal{T}^\alpha)$.

PROOF : We prove only that (iii) implies (i). Let $A = U \setminus N \in \mathcal{T}^\alpha$ and $A = \mathcal{T}^\alpha \text{ cl } A = \mathcal{T} \text{ cl } U$ by Lemma 1(i). From $U \supset U \setminus N = A = \mathcal{T} \text{ cl } U$, it follows that $A = U \in \mathcal{T}$, so A is open-and-closed in $(X, \mathcal{T})$. □

Propositions 2 yields the following statements.

Theorem 3.1 — $(X, \mathcal{T}^\alpha)$ is connected $\Leftrightarrow$ $(X, \mathcal{T}, \mathcal{T}^\alpha)$ is pairwise connected $\Leftrightarrow$ $(X, \mathcal{T})$ is connected. □

Theorem 3.2 — $(X, \mathcal{T}^\alpha)$ is totally separated $\Leftrightarrow$ $(X, \mathcal{T}, \mathcal{T}^\alpha)$ is pairwise totally separated $\Leftrightarrow$ $(X, \mathcal{T}, \mathcal{T}^\alpha)$ is weakly pairwise totally separated $\Leftrightarrow$ $(X, \mathcal{T})$ is totally separated. □

Theorem 3.3 — $(X, \mathcal{T}^\alpha)$ is zero dimensional $\Leftrightarrow$ $\mathcal{T}^\alpha$ is zero dimensional with respect to $\mathcal{T}$ $\Rightarrow$ $\mathcal{T}$ is zero dimensional with respect to $\mathcal{T}^\alpha$ $\Leftrightarrow$ $(X, \mathcal{T})$ is zero dimensional. □

That the second implication is not reversible can be seen from the following example.

Example 3 — Consider the Sorgenfrey line $(\mathbb{R}, \mathcal{S})$ (Example 51 of Steen and Seebach[31]). The space is zero dimensional, but $(\mathbb{R}, \mathcal{S}^\alpha)$ is not zero dimensional. For the point 0 and its $\mathcal{S}^\alpha$ open neighbourhood $W = [0, \varepsilon) \setminus \left\{ \frac{1}{n} \mid n > 2 \right\}$, there is no $\mathcal{S}^\alpha$ open-and-closed neighbourhood of the point 0 contained in W.

Theorem 3.4 — $(X, \mathcal{T}^\alpha)$ is extremally disconnected $\Leftrightarrow$ $(X, \mathcal{T}, \mathcal{T}^\alpha)$ is pairwise extremally disconnected $\Leftrightarrow$ $(X, \mathcal{T})$ is extremally disconnected.

PROOF : Lemma 1 (i) implies that $(X, \mathcal{T}^\alpha)$ is extremally disconnected is equivalent to $\mathcal{T}^\alpha$ is extremally disconnected with respect to $\mathcal{T}$, and that $(X, \mathcal{T})$ is extremally disconnected is equivalent to $\mathcal{T}$ is extremally disconnected with respect to $\mathcal{T}^\alpha$. Njåstad[20], (Proposition 7) has proved that $(X, \mathcal{T})$ is extremally disconnected if and only if $(X, \mathcal{T}^\alpha)$ is extremally disconnected. We give an alternative and short proof of this result based on Lemma 1.

Let $(X, \mathcal{T})$ be extremally disconnected and $G = U \setminus N \in \mathcal{T}^\alpha$. Then $\mathcal{T}^\alpha \text{ cl} G = \mathcal{T} \text{ cl} U \in \mathcal{T} \subset \mathcal{T}^\alpha$. For the converse, let $(X, \mathcal{T}^\alpha)$ be extremally disconnected and $U \in \mathcal{T}$. From $\mathcal{T} \text{ cl} U = \mathcal{T}^\alpha \text{ cl} U \in \mathcal{T}^\alpha$ and Proposition 2, it follows that $\mathcal{T} \text{ cl} U \in \mathcal{T}$. □

Theorem 3.5 — $(X, \mathcal{T}^\alpha)$ is locally connected $\Leftrightarrow$ $\mathcal{T}^\alpha$ is locally connected with respect to $\mathcal{T}$ $\Rightarrow$ $\mathcal{T}$ is locally connected with respect to $\mathcal{T}^\alpha$ $\Leftrightarrow$ $(X, \mathcal{T})$ is locally connected.

PROOF : The equivalences follow from Lemma 3. We prove only the remaining implication. Let $(X, \mathcal{T}^\alpha)$ be locally connected and let $x \in V \in \mathcal{T}$. There is a $\mathcal{T}^\alpha$ connected subset $G = U \setminus N \in \mathcal{T}^\alpha$ such that $x \in G \subset V$. The $\mathcal{T}$ open subset $W = U \cap V$ is $\mathcal{T}^\alpha$ connected, hence $\mathcal{T}$ connected, since $G \subset W \subset \mathcal{T}^\alpha \text{ cl} U = \mathcal{T}^\alpha \text{ cl} G$. It follows that $(X, \mathcal{T})$ is locally connected. □

Example 1 shows that the second implication in the above theorem is not reversible. Consider the point 0 and its $\mathcal{T}^\alpha$ neighbourhood $[0, 1] \setminus \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$.

Theorem 3.6 — $(X, \mathcal{T}^\alpha)$ is totally disconnected $\Leftarrow$ $(X, \mathcal{T}, \mathcal{T}^\alpha)$ is pairwise totally disconnected $\Leftarrow$ $(X, \mathcal{T})$ is totally disconnected.

PROOF : Every separation of a subset A in $(X, \mathcal{T})$ is a separation in $(X, \mathcal{T}, \mathcal{T}^\alpha)$, and every separation of A in $(X, \mathcal{T}, \mathcal{T}^\alpha)$ is a separation in $(X, \mathcal{T}^\alpha)$. □

Examples 2 and 4 show that the implications in the above theorem are not reversible.

That total disconnectedness of $(X, \mathcal{T}^\alpha)$ does not imply pairwise total disconnectedness of $(X, \mathcal{T}, \mathcal{T}^\alpha)$ can be seen from Example 2. The components in $(X, \mathcal{T}^\alpha)$ are singletons, while the component of $(X, \mathcal{T}, \mathcal{T}^\alpha)$ which contains the point

p is the set $\{p, q\}$. Every $\mathcal{T}$ open set which contains one of these points contains the other. The sets $\{p\}$ and $\{q\}$ are $\mathcal{T}^\alpha$ open in $\{p, q\}$. That is, the topology $\mathcal{T}^\alpha$ restricted to $\{p, q\}$ is discrete.

Example 4 — This is a modification of Example 2. Let $X = \mathbb{N} \cup \{p, q\}$, $p, q \notin \mathbb{N}$, and the topology be $\mathcal{T} = \{G \subset \mathbb{N}\} \cup \{H \subset X \mid p \in H$ and $X \setminus H$ is finite$\}$. Every $\mathcal{T}$ open set which contains q contains p. The set $\{q\}$ is $\mathcal{T}$ closed and the set $\{p\}$ is $\mathcal{T}^\alpha$ closed. The components in $(X, \mathcal{T}, \mathcal{T}^\alpha)$ are singletons, while the component of $(X, \mathcal{T})$ which contains the point q is the set $\{p, q\}$.

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