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FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS

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Let X be a complete metric space and $B(X)$ the space of closed bounded subsets of X with the Hausdorff metric. Two general fixed point theorems for $T : X \to B(X)$ are proved. Then examples show that these theorems include known fixed point theorems and also yield new theorems.

Let (X, d) be a complete metric space. $B(X)$ denotes all closed bounded subsets of X with the Hausdorff metric ρ defined by

$$\rho(E, F) = \max\left[\sup_{x \in F} d(x, E), \sup_{x \in E} d(x, F)\right].$$

Also,

$$\rho(E, F) = \sup_{x \in X} | d(x, E) - d(x, F) | .$$

Theorem 1—Suppose $T : (X, d) \to (B(X), \rho)$ where X is complete and T is continuous. Then there exists p in X with $p \in Tp$ if and only if there exists a sequence $\{x_n\}$ in X with $x_{n+1} \in Tx_n$ and $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$. In this case, $p = \lim_n x_n$.

Proof : If $p \in Tp$, let $x_n = p$ for every n. Suppose the condition holds. Then $\{x_n\}$ is a Cauchy sequence, so that $\lim x_n = p$ for some p in X. T is continuous implies $Tx_n \to Tp$. If $y \in Tp$, $d(p, y) \leqslant d(p, x_{n+1}) + d(x_{n+1}, y)$ and it follows that $d(p, Tp) \leqslant d(p, x_{n+1}) + d(x_{n+1}, Tp)$. Now $d(p, x_{n+1}) \to 0$ and

$$d(x_{n+1}, Tp) \leqslant \sup\{d(y, Tp) : y \in Tx_n\} \leqslant \rho(Tx_n, Tp) \to 0 \text{ as } n \to \infty.$$

Thus, $d(p, Tp) = 0$ and $p \in \overline{Tp} = Tp$.

Theorem 2—Suppose $T : (X, d) \to (B(x), \rho)$ where X is complete, T is continuous, and Tx is compact for each x in X. Assume that $\rho(Tx, Ty) \leqslant K(d(x, y))$ where $K : [0, \infty) \to [0, \infty)$, $K(0) = 0$, and K is non-decreasing. Then there exists p in X with $p \in Tp$ if and only if there exists x_0 in X with $\sum_{n=1}^{\infty} K^n (d(x_0, Tx_0)) < \infty$. In this case, we can choose $x_{n+1} \in Tx_n$ with $x_n \to p$.

(K is not assumed to be continuous and $K^2(t) = K(K(t))$.)

PROOF : If $p \in Tp$, $d(p, Tp) = 0$, $0 = K(0) = K^2(0)$, and $\sum_{n=1}^{\infty} K^n(d(p, Tp)) = 0$.

Suppose there exists x_0 such that $\sum_{n=1}^{\infty} K^n(d(x_0, Tx_0)) < \infty$. $y \to d(x_0, y)$ is continuous on the compact set Tx_0 implies there exists $x_1 \in Tx_0$ such that $d(x_1, x_0) = \min\{d(x_0, y) : y \in Tx_0\} = d(x_0, Tx_0)$. Similarly, Tx_1 is compact so there exists $x_2 \in Tx_1$ such that $d(x_2, x_1) = d(x_1, Tx_1)$. Thus, we obtain a sequence $\{x_n\}$ such that

$$x_{n+1} \in Tx_n \text{ and } d(x_{n+1}, x_n) = d(x_n, Tx_n).$$

Now, $d(x_n, Tx_{n-1}) = 0$ since $x_n \in Tx_{n-1}$. Since K is non-decseasing,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(x_n, Tx_n) - 0 \\ &= d(x_n, Tx_n) - d(x_n, Tx_{n-1}) \\ &\leqslant \rho(Tx_n, Tx_{n-1}) \leqslant K(d(x_n, x_{n-1})) \\ &\leqslant K^2(d(x_{n-1}, x_{n-2})) \leqslant \ldots \\ &\leqslant K^n(d(x_1, x_0)) = K^n(d(x_0, Tx_0)). \end{aligned}$$

Applying Theorem 1, we obtain $p \in Tp$ where $p = \lim_n x_n$.

Remarks : K need only be defined on the range of d. If you replace your metric with an equivalent metric with $d(x, y) < 1$, then clearly Theorem 2 holds for K: $[0,1) \to [0, \infty)$. In many applications, X is a normed linear space and Tx is compact and convex for each x in X. To apply Theorem 2, one needs a non-decreasing function K and x in X with $\sum_{n=1}^{\infty} K^n(d(x, Tx)) < \infty$. The following examples satisfy these conditions and therefore illustrate the generality of Theorem 2 see Hicks[1]. For the details that are not obvious and not provided.

Example 1—Suppose $0 < \lambda < 1$. Let $K(t) = \lambda t$ for $t \geqslant 0$. Then $\rho(Tx, Ty) \leqslant K(d(x, y)) = \lambda\, d(x, y)$. $K^n(d(x, Tx)) = \lambda^n d(x, Tx)$ for any x in X. It is known that there exists p with $p \in Tp$ without assuming Tx is compact.

Example 2—Suppose T satisfies $\rho(Tx, Ty) \leqslant \phi(d(x, y))\, d(x, y)$ for all x, y in X, where $\phi : [0, \infty) \to [0, 1)$ and ϕ is non-decreasing. Then $K(t) = t\,\phi(t)$, K is non-decreasing, and K: $[0, \infty) \to [0, \infty)$. It follows by induction that

$$K^n(t) \leqslant t\,[\phi(t)]^n. \text{ Since } \phi(t) < 1, \sum_{n=1}^{\infty} K^n(t) < \infty.$$

Example 3—Consider $K(t) = t\,\phi(t)$ where $\phi : [0, \infty) \to [0, \infty)$, $\phi(t) \leqslant t$ for $t < 1$. If $t < 1$, it follows that $K^n(t) \leqslant t\,[\phi(t)]^n$. If K is non-decreasing Theorem 2 can be applied.

Example 4—$K(t) = t\,\phi(t)$ where ϕ: $[0, \infty) \rightarrow [0, \infty)$ and $\phi(\alpha t) \leqslant \alpha\,\phi(t)$ for $\alpha \in (0, 1]$. If $\phi(t) < 1$, $K^n(t) \leqslant (Kt)(\phi(t))^n$ for all $n \geqslant 2$.

Example 5—Assume K is non-decreasing, K is convex on $[0, 1)$ and $K(t) < t$ for all $0 < t < 1$. If $t < 1$, $K(t) < t$ so $K(t) = \alpha t$ for some $0 < \alpha < 1$. It can be shown that $K^n(t) \leqslant \alpha^n t$ for all n and thus $\sum_{n=1}^{\infty} K^n(t) < \infty$.

Theorem 3—Suppose $T: X \rightarrow B(X)$ where X is complete and Tx is compact for each x. Suppose $\rho(Tx, Ty) \leqslant [d(x, y)]^q$ where $q > 1$. If there exists x such that $t = d(x: Tx) < 1$, we can choose a sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$ and $x_n \rightarrow p$ with $p \in Tp$.

PROOF : Let $K(t) = t^q$ for $t \geqslant 0$. $K(0) = 0$, K is increasing, $K(t) < t$ if $t < 1$, and K is convex. If $t = d(x, Tx) < 1$, $\sum_{n=1}^{\infty} K^n(t) < \infty$ from the previous example. T is continuous so Theorem 2 applies.

REFERENCES

1. T. L. Hicks, *Math. Japonica*, (to appear).
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