Indian J. pure appl. Math., **20** (11) : 1080–1082, November 1989

AN ABSTRACT FIXED POINT THEOREM FOR MULTI-VALUED MAPPINGS

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(*Received* 22 *November* 1988, *after revision* 12 *April* 1989; *accepted* 17 *July* 1989)

In this paper we have proved a fixed point theorem for multi-valued non-expansive mappings in a metric space endowed with some convexity structure.

INTRODUCTION

Kirk[1] has proved a fixed point theorem for nonexpansive mappings over a closed convex subset of a reflexive Banach space satisfying normal structure. Penot[3] and Kirk[2], under different approaches, established an abstract version of the result of Kirk[1] in a bounded metric space. In this paper, modifying slightly the proofs of Kirk[2], we extend result for multi-valued mappings, generalizing also fixed point Theorem 3.1 of Samanta[4].

SOME NOTATIONS AND DEFINITIONS

In this paper we shall assume (X, d) to be a metric space, M be a bounded subset of X and $\mathscr{F}(X)$ be the collection of all nonempty subsets of X. If $x \in X$ and $r > 0$, then $B(x, r)$ denotes the closed ball with centre at x and radius r. Following Kirk[2], we put for any nonempty subset D of M:

$$\delta(D) = \text{Sup}\,\{d(u, v) : u, v \in D\}$$

$$r_u(D) = \text{Sup}\,\{d(u, v), v \in D\}, u \in D$$

$$r(D) = \text{Inf}\,\{r_u(D) : u \in D\}$$

$$h(D) = \begin{cases} r(D)/\delta(D), \text{ if } \delta(D) > 0 \\ 1, \text{ if } \delta(D) = 0. \end{cases}$$

Further for any two subsets A, B of M, the Hausdorff distance $H(A, B)$ between A and B is defined by

$$H(A, B) = \text{Inf}\,\{t > 0 \text{ such that for } a \in A,\ b \in B, \text{ there exists } a' \in A,\ b' \in B \text{ such that } d(a, b') < t \text{ and } d(a', b) < t\}.$$

Following Kirk[2], a class $\mathscr{S}$ of subsets of X is said to be normal if for each $D \in \mathscr{S}$, $\delta(D) > 0$ implies $h(D) \in (0, 1)$. The class $\mathscr{S}$ is said to be (countably) compact if each (countable) subfamily of $\mathscr{S}$ which has finite intersection property has nonvoid intersection.

For the purpose of our theorem we shall take $\mathscr{S}$ to be a class of subsets of M which is countably compact, stable under intersections, normal and contains the closed balls of M.

Theorem—If $T: M \to \mathscr{F}(X)$ be a mapping such that

(a) $T(x) \cap M \neq \phi, \forall \; x \in M$

(b) $\forall \; x \in M, \; T(x) \cap \mathscr{S} = \{T(x) \cap D : D \in \mathscr{S}\}$

is a compact class of which each nonempty member is a compact subset of X,

(c) for any $G \in \mathscr{S}$ satisfying $T(\xi) \cap G \neq \phi, \; \forall \; \xi \in G$,

$$H(T(x) \cap G, \; T(y) \cap G) \leqslant d(x, y), \; \forall \; x, y \in G$$

then T has a fixed point in M.

PROOF : Modifying slightly the proof of Lemma of Kirk[2], we show that for each $\epsilon > 0$, there exists a nonempty set $M(\epsilon) \in \mathscr{S}$ such that $T(x) \cap M(\epsilon) \neq \phi, \forall \; x \in M(\epsilon)$ and for which $\delta(M(\epsilon)) \leqslant (h(M) + \epsilon)\, \delta(M)$. For this take $M(\epsilon) = M$ if $\delta(M) = 0$. Otherwise, construct $M(\epsilon)$ as follows : Let $\rho = (h(M) + \epsilon)\, \delta(M)$.

By the definition of h, the set $\mathscr{C} = \{z \in M: M \subset B(z, \rho)\}$ is nonempty. Let $\mathscr{F} = \{D \in \mathscr{S};\; \mathscr{C} \subset D, \; T(x) \cap D \neq \phi, \; \forall \; x \in D\}$. Order the family $\mathscr{F}$ by set-inclusion relation. Let $\tau = \{D_\iota\}_{i \in \Delta}$ be a decreasing chain in $\mathscr{F}$. Let $D_0 = \bigcap_{i \in L} D_\iota$. Then $D_0 \in \mathscr{S}$, and $C \subset D_0$. Further, since $\{D_\iota\}_{i \in \Delta}$ is decreasing, it follows that for each $x \in D_0$, the family $\{T(x) \cap D_\iota\}_{i \in \Delta}$ has finite intersection property. So, by hypothesis (b), $T(x) \cap (\bigcap_{i \in \Delta} D_\iota) \neq \phi$. i. e., $T(x) \cap D_0, \neq \phi$. Thus every decreasing chain in $\mathscr{F}$ has a lower bound. Therefore, by Zorn's Lemma, $\mathscr{F}$ has a minimal element L (say). Let $A = \mathscr{C} \cup T(L)$, where $T(L) = \bigcup_{x \in \Delta} (T(x) \cap L)$. Then $T(L) \subset L$. So, $\text{Cov}(A) = \cap \{D \in \mathscr{S};\; A \subset D\} \subset L$. Also for $x \in \text{Cov}(A)$, $T(x) \cap L \neq \phi$ and $T(x) \cap L \subset T(L)$. So, $T(x) \cap A \neq \phi$. Hence $T(x) \cap \text{Cov}(A) \neq \phi$. Thus $\text{Cov}(A) \in \mathscr{F}$. Since L is a minimal member of $\mathscr{F}$ and $\text{Cov}(A) \subset L$, it follows that $\text{Cov}(A) = L$. Let $M(\epsilon) = \{x \in L;\; L \subset B(x, \rho)\}$. Then $M(\epsilon) \neq \phi$ since $M(\epsilon) \supset C$. Let $x \in M(\epsilon)$. Then $T(x) \cap L \neq \phi$. Take $x' \in T(x) \cap L$. Let $O = L \cap B(x', \rho)$. Then $C \subset O$. Next, take $\eta \in O$. Then $\eta \in L$ and $d(x', \eta) \leqslant \rho$. Now, by (c),

$$H(T(x) \cap L, \; T(\eta) \cap L) \leqslant d(x, \eta)$$
$$\leqslant \rho \; (\because x \in M(\epsilon), \eta \in L).$$

Since $T(x) \cap L$ and $T(\eta) \cap L$ are nonempty compact sets and $x' \in T(x) \cap L$, there exists $\eta' \in T(\eta) \cap L$ such that

$$d(x', \eta') \leqslant H(T(x) \cap L, T(\eta) \cap L) < \rho.$$

So $\eta' \in O$, Hence $T(\eta) \cap O \neq \phi$. Thus $O \in \mathscr{F}$ and $O \subset L$. Since L is a minimal element in $\mathscr{F}$, it follows that $O = L$. So $d(x', y) \leqslant \rho$. $\forall\, y \in L$. i. e., $B(x', \rho) \supset L$. So, $x' \in M(\epsilon)$, which implies $T(x) \cap M(\epsilon) \neq \phi$.

Now let $m = \{D \in \mathscr{S}; D \neq \phi, T(x) \cap D \neq \phi, \forall\, x \in D\}$, and for each $D \in m$, Let $\delta_0(D) = \text{Inf}\,\{\delta(F); F \in m, F \subset D\}$. From now on, the proof runs similarly as in Kirk[2] and hence it is omitted.

Remark : The necessity of the condition (c) of the Theorem has been studied by Samanta[4].

Corollary 1 (Theorem 1 of Kirk[2])—Let (M, d) be a non-empty bounded metric space and suppose M contains a class $\mathscr{S}$ of subsets which is countably compact, stable under arbitrary intersections, and normal. Suppose further that $\mathscr{S}$ contains the closed balls of M. Then every nonexpansive mapping T of M into itself has a fixed point.

Corollary 2 (Theorem 3.1 of Samanta[4])—Let X be a reflexive Banach space and K, a bounded closed convex subset of X, possessing normal structure. If $\psi : K \to 2^X$ is a mapping such that

(a) $\psi(x) \cap K \neq \phi, \forall\, x \in K$

(b) for any closed convex subset G of K satisfying $\psi(\xi) \cap G \neq \phi, \forall\, \xi \in G$,

$$H(\psi(x) \cap G, \psi(y) \cap G) < \|x - y\|, \text{ whenever } x, y\,(\neq x) \in G.$$

Then ψ has a fixed point.

ACKNOWLEDGEMENT

The author is thankful to the referee for his suggestions in rewriting the paper in the present form.

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