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ON TOPOLOGICAL PROJECTIVE PLANES-III

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Salzmann[4] has proved the following theorem: In a topological projective plane the connectedness of

(a) the space of points

(b) the space of lines

(c) a ray of points

(d) a ray of affine points

(e) the set of points of an affine plane

are all equivalent. In this paper we prove a similar theorem for several other topological properties.

1. Introduction

A topological projective plane is a projective plane in which the set of points $\mathscr{P}$ and the set of lines $\mathscr{L}$ are endowed with topologies $\tau_{\mathscr{P}}$ and $\tau_{\mathscr{L}}$ respectively such that the operations of joining and intersecting are continuous in both variables. If either $\tau_{\mathscr{P}}$ or $\tau_{\mathscr{L}}$ is different from the indiscrete topology, then both $\tau_{\mathscr{P}}$ and $\tau_{\mathscr{L}}$ are Hausdorff. Hence we assume throughout that all the spaces involved are Hausdorff. Salzmann[4] has proved the following theorem.

In a topological projective plane the connectedness of

(a) the space of points

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are all equivalent.

In this paper we prove a similar theorem for several other topological properties. For topological terminology we refer Dugundji[2]. The following theorems are used in the sequel.

Theorem 1.1[3]—$(\mathscr{P}, \tau_{\mathscr{P}})$ and $(\mathscr{L}, \tau_{\mathscr{L}})$ are always regular topological spaces.

Theorem 1.2[3]—In a topological projective plane each ray of points and each pencil of lines are closed.

Theorem 1.3[3]—In a topological projective plane the relative topology on a ray of points and a pencil of lines are homeomorphic.

Theorem 1.4[3]—In a topological projective plane the set of points of an affine plane is homeomorphic to the product of an affine ray with itself.

2. Main Results

Theorem 2.1—In a topological projective plane the path connectedness of

(a) the space of points

(b) the space of lines

(c) a ray of points

(d) a ray of affine points

(e) the set of points of an affine plane

are all equivalent.

Proof : The product of two topological spaces is path connected iff each is path connected. Hence by Theorem 1.4, the path connectedness of (d) and (e) are equivalent. Also union of any family of path connected spaces having nonempty intersection is path connected and hence the path connectedness of (d) implies that of (e) and (a) and the path connectedness of (c) implies that of (a). Now, to prove that the path connectedness of (a) implies that of (c), suppose that the ray of points of a line l is not path connected. Let $P \in l$. The path component of P in l is left unchanged by all homeomorphisms of the ray of points of l which fix P and through such homeomorphisms any point $\neq P$ on l can be mapped to any point $\neq P$ on l. Hence the path component of P in l is $\{P\}$. Now, let Q be any point not on l and let $\mathscr{D}$ be the path component of P in $\mathscr{P}$—$\{Q\}$. Suppose $\mathscr{D}$ contains a point R not lying on the line PQ. Since $X \rightarrow QX \cap l$ is continuous at all points $\neq Q$ and the continuous image of a path connected sapce is path connected, the image of $\mathscr{D}$ under this map is a path connected subset of l containing P and $QR \cap l$. Thus the path component of P in l contains more than one point which is a contradiction. Hence $\mathscr{D} \subseteq PQ—\{Q\}$. But the path component of P in any ray of points is $\{P\}$ and hence $\mathscr{D} = \{P\}$. Thus for any arbitrary point $Q \neq P$, the path component of P in $\mathscr{P} - \{Q\}$ is $\{P\}$ and hence P cannot be joined to any other point by a path in $\mathscr{P}$. Hence $\mathscr{P}$ is not path connected and thus the path connectedness of (a) implies that of (c). Similarly we can prove that path connectedness of (c) implies that of (d). Now, suppose that the space of points is path connected. Then a ray of points is path connected and hence by Theorem 1.2, each pencil of lines is path connected. Hence the space of lines is path connected. Thus the path connectedness of (a) implies that of (b). Similarly the path connectedness of (b) implies that of (a).

Corollary 2.2—A topological projective plane which is not path connected is totally path disconnected.

Theorem 2.3—In a topological projective plane, the local connectedness, the property of having a σ-locally finite base, second countability, separability and first countability of

(a) the space of points

(b) the space of lines

(c) a ray of points

(d) a ray of affine points

(e) the set of points of an affine plane

are all equivalent.

PROOF: We prove the theorem for locally connectedness and the proof for the remaining properties is similar. Since an open subspace of a locally connected space is locally connected, the local connectedness of (a) implies that of (e) by Theorem 1.2. Now suppose that the set of points of any affine plane is locally connected. Let OEUV be the quadrangle of reference. If $\mathscr{B}_1$, $\mathscr{B}_2$, $\mathscr{B}_3$ are bases consisting of open connected sets for the affine planes $\mathscr{P} - VU$, $\mathscr{P} - OV$ and $\mathscr{P} - OU$ respectively, then $\mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_3$ is a base consisting of open connected sets for $\mathscr{P}$. Thus the local connectedness of (a) and (e) are equivalent. By a similar argument we can prove the equivalence of the local connectedness of (c) and (d). We now prove the equivalence of the local connectedness of (a) and (b). Suppose that the space of lines $\mathscr{L}$ is locally connected. By Theorem 1.2 the set $\mathscr{L}_1$ of all lines not passing through a point V is an open set in $\mathscr{L}$ and hence is locally connected. Also $\mathscr{L}_1$ is homeomorphic to the set of points of an affine plane. Hence the set of points of an affine plane is locally connected. Hence $\mathscr{P}$ is locally connected. Thus the local connectedness of (b) implies that of (a). Similarly if $\mathscr{P}$ is locally connected, if follows that the complement of any pencil of lines in $\mathscr{L}$ is locally connected and hence $\mathscr{L}$ is locally connected. The equivalence of the local connectedness of (d) and (e) follows from Theorem 1.4.

Theorem 2.4—In a topological projective plane the metrizability of

(a) the space of points

(b) the space of lines

(c) a ray of points

(d) a ray of affine points

(e) the set of points of an affine plane

are all equivalent.

PROOF : A topological space is metrizable iff it is regular and has σ-locally finite base (Dugundji[2], page 194). Hence the result follows from Theorem 2.3.

3. CONCLUSION

No known topological projective planes have non-homeomorphic topologies on the space of points and lines. In Theorems 2.1, 2.3 and 2.4 we have proved for several topological properties P that the space of points has property P iff the space of lines has P. Further in[1] we have proved that in a topological pappian plane the space of points is homeomorphic to the space of lines. Wyler[5] has proved a similar theorem for ordered projective planes. Hence we take the risk of making the following.

Conjecture—In a topological projective plane the space of points is homeomorphic to the space of lines.

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